بسم الله الرحمن الرحيم

SEMESTER: SECOND	المملكة العربية السعودية
YEAR: 1427/1428	وزارة التعليم العالي
COURSE: Math 301	جامعة الإمام محمد بن سعود الإسلامية
DATE: 20/05/1428	كلية علوم الحاسب و المعلومات
DURATION: 2 HOURS	

Solutions - Final Exam

Instructors: Dr. A. Grine and Dr. A. Ben Ghorbal

Sections: 170, 171, 172, 173

	Total Marks	Score
Exercise 1	**7**	
Exercise 2	**6**	
Exercise 3	**6**	
Exercise 4	**6**	
Exercise 5	**6**	
Exercise 6	**9**	
Total	**40**	

Exercise 1 (7 Marks)

For the ABO blood typing system, each person has exactly one of 4 blood types, A, B, O or AB. Two random people attending a blood donor clinic have their blood type recorded.

1. *Explain why this is an example of a random experiment and give the sample space of the experiment.*

   Solution: [2 Marks] *This is a random experiment because in any running of the experiment a different set of 2 people from the population could be selected and so the resulting blood types will be different. We do know, however, that the 2 blood types will each be one of the four possible types and so there is a well defined sample space.*

   In defining the sample space we should assume that the two people are distinguishable so that an outcome is an ordered pair of blood types.

$$\begin{aligned} S &= \{(A,A),(A,B),(A,O),(A,AB),(B,A),(B,B),(B,O),(B,AB),\\ &\quad (O,A),(O,B),(O,O),(O,AB),(AB,A),(AB,B),(AB,O),(AB,AB)\} \end{aligned}$$

2. *Let C be the event that the two people have the same blood type. List the outcomes in C.*

   Solution: [1.5 Marks] *We have*

$$C = \{(A,A),(B,B),(O,O),(AB,AB)\}$$

3. *Let D be the event that at least one of the people has blood type AB. List the outcomes in D.*

   Solution: [1.5 Marks] *We have*

$$D = \{(A,AB),(B,AB),(AB,AB),(O,AB),(AB,A),(AB,B),(AB,O)\}$$

4. *List the outcomes in $C \cup D$ and hence give the outcomes in the events $(C \cup D)^c$ and $C^c \cap D^c$.*

   Solution: [2 Marks] *Since*

$$\begin{aligned} C \cup D &= \{(A,A),(A,AB),(B,B),(B,AB),(O,O),(O,AB),\\ &\quad (AB,A),(AB,B),(AB,AB),(AB,O)\} \end{aligned}$$

   Thus,

$$(C \cup D)^c = \{(A,B),(A,O),(B,A),(B,O),(O,A),(O,B)\}$$

   Now by the De Morgan's Law we obtain

$$C^c \cap D^c = (C \cup D)^c = \{(A,B),(A,O),(B,A),(B,O),(O,A),(O,B)\}$$

Exercise 2 (6 Marks)

Three married couples have purchased 6 adjacent seats in a row at the theater. The seats numbers are A10, A11, A12, A13, A14 and A15. Suppose that they take their seats in a random fashion.

1. *What is the probability that Faisal and his wife Nada sit next to each other in seats A10 and A11?*

   Solution: [1.5 Marks] *There are* 6 *people so there are a total of* $6! = 720$ *different seating arrangements that can be made.*

   There are 2 *ways that Faisal and Nada can sit such that they are in seats* $A10$ *and* $A11$ *There are* $4! = 24$ *ways the other four can be arranged in the remaining* 4 *seats.*

$$P\left(\textit{Faisal and Nada in } A10,\ A11\right) = \frac{2 \times 24}{6!} = \frac{48}{720} = \frac{1}{15} \approx 0.067$$

2. *What is the probability that Faisal and Nada sit next to each other?*

   Solution: [1.5 Marks] *There are* 5 *possible pairs of seats in which Faisal and Nada can sit and be next to each other. They are* $(A10, A11)$, $(A11, A12)$, $(A12, A13)$, $(A13, A14)$ *and* $(A14, A15)$. *For any such pair of seats, the probability that John and Paula sit there is* 0.067 *from part (1). They are clearly mutually exclusive since Faisal and Nada can only sit in one pair of seats. Hence*

$$P\left(\textit{Faisal and Nada sit together}\right) = 5 \times \frac{1}{15} = \frac{1}{3} \approx 0.333$$

3. *What is the probability that every husband sits next to his wife?*

SOLUTION: [1.5 Marks] *If all three couples are sitting together then one couple must be in $(A10, A11)$, one must be in $(A12, A13)$ and one must be in $(A14, A15)$. There are $3! = 6$ ways we can assign the couples to their pair of seats. Once the seats have been decided there are 2 ways to arrange each couple so there are $2^3 = 8$ arrangements for each assignment of seats to couples. Hence*

$$P\left(\text{Every couple sits together}\right) = \frac{6 \times 8}{6!} = \frac{48}{720} = \frac{1}{15} \approx 0.067$$

4. *What is the probability that at least one of the wives sits next to her husband?*

SOLUTION: [1.5 Marks] *Let us number the couples 1, 2, 3 and define B_i to be the event that couple i sits together. The event that we are interested in is the union of these three events. We can the following property of the probability*

$$\begin{aligned} P\left(B_1 \cup B_2 \cup B_3\right) &= P\left(B_1\right) + P\left(B_2\right) + P\left(B_3\right) - P\left(B_1 \cap B_2\right) \\ &\quad - P\left(B_1 \cap B_3\right) - P\left(B_2 \cap B_3\right) + P\left(B_1 \cap B_2 \cap B_3\right) \end{aligned}$$

From part (2) we see that

$$P\left(B_i\right) = \frac{1}{3}, \quad \text{for } i = 1, 2, 3.$$

In part (3) we found

$$P\left(B_1 \cap B_2 \cap B_3\right) = \frac{1}{15}$$

Also it is clear that

$$P\left(B_1 \cap B_2\right) = P\left(B_1 \cap B_3\right) = P\left(B_2 \cap B_3\right)$$

so all that remains for us to do is find $P\left(B_1 \cap B_2\right)$, the probability that couple 1 and couple 2 both sit together.

For couple 1 and 2 to both sit together there are 6 sets of seats that they can occupy: $[(A_{10}, A_{11}), (A_{12}, A_{13})]$, $[(A_{10}, A_{11}), (A_{13}, A_{14})]$, $[(A_{10}, A_{11}), (A_{14}, A_{15})]$, $[(A_{11}, A_{12}), (A_{13}, A_{14})]$, $[(A_{11}, A_{12}), (A_{14}, A_{15})]$ and $[(A_{12}, A_{13}), (A_{14}, A_{15})]$.

For any such set, there are 2 ways to decide which couple goes in the pair with the lower numbers. Once that has been decided each couple can rearrange themselves in 2 ways so there are $23 = 8$ ways to arrange the people when we have decided what seats are for which couple. This means there are $2 \times 8 = 16$ possible seating arrangements for each of the six sets of seats listed above and so there are $6 \times 16 = 96$ possible ways that couple 1 and couple 2 can both sit together.

$$\begin{aligned} P\left(B_1 \cup B_2 \cup B_3\right) &= 3P\left(B_1\right) - 3P\left(B_1 \cap B_2\right) + P\left(B_1 \cap B_2 \cap B_3\right) \\ &= 1 - 3 \times \frac{2}{15} + \frac{1}{15} = \frac{2}{3} \approx 0.667 \end{aligned}$$

Exercise 3 (6 Marks)

A worker has asked his supervisor for a letter of recommendation for a new job. He estimates that there is an $80\% = 0.80$ *chance that he will get the job if he receives a strong recommendation, a* $40\% = 0.40$ *chance if he receives a moderately good recommendation, and a* $10\% = 0.10$ *chance if he receives a weak recommendation. He further estimates that the probabilities that the recommendation will be strong, moderate, or weak are* 0.7, 0.2 *and* 0.1, *respectively.*

1. *How certain is he that he will receive the new job offer?*

   Solution: [1.5 Marks] *Define the events A that she gets the new job, B_1 that she gets a strong reference, B_2 that she gets a moderate reference and B_3 that she gets a weak reference. From the question we can write*

$$P(A|B_1) = 0.8, \quad P(A|B_2) = 0.4, \quad P(A|B_3) = 0.1$$
$$P(B_1) = 0.7, \quad P(B_2) = 0.2, \quad P(B_3) = 0.1$$

   From the Law of Total Probability we have

$$\begin{aligned} P(A) &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) \\ &= 0.8 \times 0.7 + 0.4 \times 0.2 + 0.1 \times 0.1 \\ &= 0.56 + 0.08 + 0.01 \\ &= 0.65 \end{aligned}$$

2. *Given that he has receive the offer, how likely he feels that the recommendation he has received is*

   Solution: [1.5 Marks] *Bayes Theorem tells us that*

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{P(A)}, \qquad \text{for } i = 1, 2, 3.$$

   - *strong;*

     Solution: [1.5 Marks] *Using Bayes formula we get*

$$\begin{aligned} P(B_1|A) &= \frac{P(A|B_1)P(B_1)}{P(A)} \\ &= \frac{0.8 \times 0.7}{0.65} \\ &= \frac{0.56}{0.65} \approx 0.861 \end{aligned}$$

- *moderate*

SOLUTION: [1.5 Marks] *Using Bayes formula we get*

$$\begin{aligned} P(B_2|A) &= \frac{P(A|B_2)\,P(B_2)}{P(A)} \\ &= \frac{0.4 \times 0.2}{0.65} \\ &= \frac{0.08}{0.65} \approx 0.123 \end{aligned}$$

- *weak?*

SOLUTION: [1.5 Marks] *Using Bayes formula we get*

$$\begin{aligned} P(B_3|A) &= \frac{P(A|B_3)\,P(B_3)}{P(A)} \\ &= \frac{0.1 \times 0.1}{0.65} \\ &= \frac{0.01}{0.65} \approx 0.015 \end{aligned}$$

Exercise 4 (6 Marks)

A discrete random variable, Y, has probability mass function

$$p_Y(y) = c(y-3)^2, \quad y = -2, -1, 0, 1, 2.$$

1. *Find the value of the constant c.*

Solution: [2 Marks] *Writing the probability mass function as a table we have*

y	-2	-1	0	1	2
$p_Y(y)$	$25c$	$16c$	$9c$	$4c$	c

We need to choose c such that the sum of the probability mass function is equal to 1. The sum of the probability mass function is

$$\sum_{y=-2}^{2} p_Y(y) = 55c$$

and so we have

$$55c = 1 \Longrightarrow c = \frac{1}{55} \approx 0.018$$

2. *Give the cumulative distribution function of Y.*

Solution: [2 Marks] *First we will write the probability mass function with the appropriate value of c.*

y	-2	-1	0	1	2
$p_Y(y)$	$\frac{25}{55}$	$\frac{16}{55}$	$\frac{9}{55}$	$\frac{4}{55}$	$\frac{1}{55}$

Thus we get the cumulative distribution function

$$F_Y(y) = \begin{cases} 0 & \text{for } y < -2 \\ \frac{25}{55} & \text{for } -2 \leqslant y < -1 \\ \frac{41}{55} & \text{for } -1 \leqslant y < 0 \\ \frac{50}{55} & \text{for } 0 \leqslant y < 1 \\ \frac{54}{55} & \text{for } 1 \leqslant y < 2 \\ 1 & \text{for } y \geqslant 2 \end{cases}$$

3. *Find the mean and variance of Y.*

Solution: [1.5 Marks] *The mean (or expected value) of Y is*

$$\begin{aligned} E[Y] &= \sum_{y=-2}^{2} y \times p_Y(y) \\ &= (-2) \times \frac{25}{55} + (-1) \times \frac{16}{55} + 0 \times \frac{9}{55} + 1 \times \frac{4}{55} + 2 \times \frac{1}{55} \\ &= \frac{-50 - 16 + 0 + 4 + 2}{55} \\ &= -\frac{60}{55} \approx -1.091 \end{aligned}$$

The variance is best calculated as $\mathrm{Var}[Y] = E\left[Y^2\right] - (E[Y])^2$ *where*

$$\begin{aligned} E\left[Y^2\right] &= \sum_{y=-2}^{2} y^2 \times p_Y(y) \\ &= (-2)^2 \times \frac{25}{55} + (-1)^2 \times \frac{16}{55} + 0^2 \times \frac{9}{55} + 1^2 \times \frac{4}{55} + 2^2 \times \frac{1}{55} \\ &= 4 \times \frac{25}{55} + 1 \times \frac{16}{25} + 0 \times \frac{9}{55} + 1 \times \frac{4}{55} + 4 \times \frac{1}{55} \\ &= \frac{100 + 16 + 0 + 4 + 4}{55} \\ &= \frac{124}{55} \approx 2.2545 \end{aligned}$$

and so

$$\begin{aligned} \mathrm{Var}[Y] &= \frac{124}{55} - \left(-\frac{60}{55}\right)^2 \\ &= \frac{124}{55} - \frac{144}{121} \\ &= \frac{1364 - 720}{605} \\ &= \frac{644}{605} \approx 1.064 \end{aligned}$$

Exercise 5 (6 Marks)

Let X posses a density function

$$f_X(x) = \begin{cases} dx^2(1-x) & 0 \leqslant x \leqslant 1 \\ 0 & \textit{elsewhere} \end{cases}$$

1. *Find the value of the constant d.*

Solution: (3 Marks) *Since f_X is the probability density function of a given random variable then we need to choose d such that the sum of probability distribution function is equal to 1, i.e.*

$$\int_{-\infty}^{+\infty} f_X(x)dx = 1$$

The integral of the probability distribution function is

$$\begin{aligned} \int_{-\infty}^{+\infty} f_X(x)dx &= \int_0^1 d \cdot x^2(1-x)\,dx \\ &= d \cdot \int_0^1 \left(x^2 - x^3\right) dx \\ &= d \cdot \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 \\ &= d \cdot \left(\frac{1}{3} - \frac{1}{4}\right) \\ &= d \cdot \left(\frac{4-3}{12}\right) \\ &= \frac{d}{12} \end{aligned}$$

and so we have

$$\frac{d}{12} = 1 \Longrightarrow d = 12.$$

Therefore, the continuous random variable X has the probability density function

$$f_X(x) = \begin{cases} 12x^2(1-x) & 0 \leqslant x \leqslant 1 \\ 0 & \textit{elsewhere} \end{cases}$$

2. *Give the cumulative distribution function of X and $P\left(-\frac{1}{2} \leqslant X \leqslant \frac{1}{2}\right)$.*

Solution: (3 Marks) *By definition the cumulative distribution function is given by*

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

Then we distinguish the following three situations:

- *1st case:* $x \leqslant 0$, *so* $t \leqslant x \leqslant 0$. *Thus we have* $f_X(t) = 0$ *and so*

$$F_X(x) = 0$$

- *2nd case:* $0 < x < 1$, *then*

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(t)dt \\ &= \int_{-\infty}^{0} f_X(t)dt + \int_{0}^{x} f_X(t)dt \\ &= 0 + \int_{0}^{x} \left(12t^2 - 12t^3\right) dt \\ &= \left[4t^3 - 3t^4\right]_0^x = 4x^3 - 3x^4 \end{aligned}$$

- *3rd case:* $x \geqslant 1$, *we have*

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(t)dt \\ &= \int_{-\infty}^{0} f_X(t)dt + \int_{0}^{1} f_X(t)dt + \int_{1}^{x} f_X(t)dt \\ &= 0 + \int_{0}^{1} \left(12t^2 - 12t^3\right) dt + 0 \\ &= 1 \end{aligned}$$

Therefore, the continuous random variable X has the cumulative distribution function

$$F_X(x) = \begin{cases} 0 & \text{for } x \leqslant 0 \\ 4x^3 - 3x^4 & \text{for } 0 < x < 1 \\ 1 & \text{for } x \geqslant 1 \end{cases}$$

We have

$$\begin{aligned} P\left(-\frac{1}{2} \leqslant X \leqslant \frac{1}{2}\right) &= P\left(X \leqslant \frac{1}{2}\right) - P\left(X \leqslant -\frac{1}{2}\right) \\ &= F\left(\frac{1}{2}\right) - F\left(-\frac{1}{2}\right) \\ &= 4\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^4 - 0 \\ &= \frac{8-1}{16} = \frac{7}{16} \approx 0.4375 \end{aligned}$$

Exercise 6 (9 Marks)

Part 1

The probability that a patient recovers from a stomach disease is 0.8*. Suppose* 20 *people are known to have contracted this disease*

1. *What is the probability that exactly* 14 *recover?.*

Solution: (3 Marks) *Let X be the number of recovered patients from a stomach disease. Then the distribution of X is*

$$X \sim \text{binomial}\left(n = 20, p = 0.80\right).$$

Then the probability that exactly 14 *recover is given by*

$$\begin{aligned} P\left(X = 14\right) &= \begin{pmatrix} 20 \\ 14 \end{pmatrix} (0.8)^{14} (1 - 0.8)^{20-14} \\ &= \begin{pmatrix} 20 \\ 14 \end{pmatrix} \frac{20!}{14! \times 6!} (0.8)^{14} (0.2)^{6} \\ &= \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15}{6 \times 5 \times 4 \times 3 \times 2} (0.8)^{14} (0.2)^{6} \\ &= 19 \times 17 \times 8 \times 15 \, (0.8)^{14} (0.2)^{6} \approx 0 \end{aligned}$$

2. *What is the probability that at least* 10 *recover?*

Solution: (3 Marks) *We have*

$$\begin{aligned} P\left(X \geqslant 10\right) &= 1 - P\left(X < 10\right) \\ &= 1 - P\left(X = 0\right) - P\left(X = 2\right) - \cdots - P\left(X = 9\right) \approx 1 \end{aligned}$$

PART 2

A machine is used to automatically fill 355 ml *milk bottles. The actual amount put into each bottle is a normal random variable with mean* 360 ml *and standard deviation of* 4 ml. *What proportion of bottles are filled with less than* 355 ml *of milk?*

SOLUTION: (3 Marks) *Let X be the amount of milk in a randomly chosen bottle. Then, the question tells us that $X \sim \text{Normal}(\mu = 360, \sigma = 16)$ and*

$$\begin{aligned}
P(X < 355) &= P\left(\frac{X-\mu}{\sigma} < \frac{355-360}{4}\right) \\
&= P(Z < -1.25) \qquad [Z \sim \text{Normal}(0,1)] \\
&= P(Z > 1.25) \\
&= 1 - P(Z \leqslant 1.25) \\
&= 1 - 0.8944 = 0.1056
\end{aligned}$$

بسم الله الرحمن الرحيم

SEMESTER: FISRT	المملكة العربية السعودية
YEAR: 1428/1429	وزارة التعليم العالي
COURSE: Math 301	جامعة الإمام محمد بن سعود الإسلامية
DATE: 19/01/1428	كلية علوم الحاسب و المعلومات
DURATION: 2 HOURS	

Solutions - Final Exam

Instructors: Drs. A. Ben Ghorbal, A. Berkaoui and A. Ahmad

Sections: 171, 172, 173, 174, 175, 176

	Total Marks	Score
Exercise 1	**9**	
Exercise 2	**14**	
Exercise 3	**9**	
Exercise 4	**8**	
Total	**40**	

Exercise 1 (9 Marks)

A. *The following data represents the temperature high in degrees Centigrade of* 10 *consecutive days in April in a certain city:*

$$23, \quad 27, \quad 25, \quad 20, \quad 12, \quad 8, \quad 6, \quad 12, \quad 15, \quad 17.$$

Find the followings:

(i) *the mean* $\overline{x}$

Solution: [1.5 Marks] *Using the definition of the mean we get*

$$\begin{aligned} \overline{x} &= \frac{6+8+12+12+15+17+20+23+25+27}{10} \\ &= \frac{165}{10} = 16.5 \end{aligned}$$

(ii) *the median* $\widetilde{x}$.

Solution: [1.5 Marks] *By reordering the data given above we get*

$$6, \quad 8, \quad 12, \quad 12, \quad 15, \quad 17, \quad 20, \quad 23, \quad 25, \quad 27$$

Then, the median is given

$$\widetilde{x} = \frac{15+17}{2} = 16$$

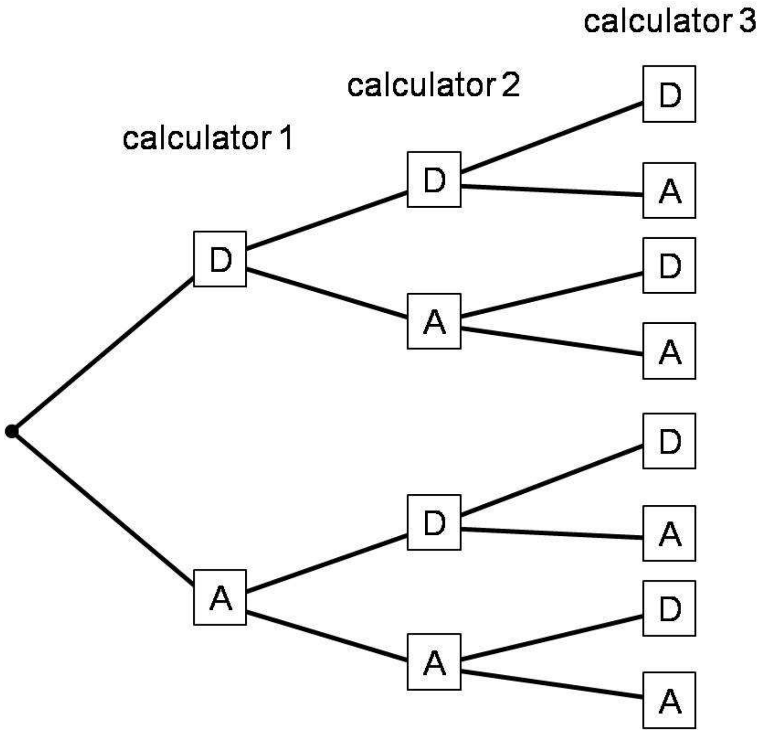
(iii) *the mode.*

Solution: [1.5 Marks] *Since the number* 12 *is repeated two times then the mode is* 12.

B. *A sample of three calculators is selected from a manufacturing line, and each calculator is classified as either defective or acceptable. Let A, B, and C denote the events that the first, second, and third calculators respectively, are defective.*

(a) *Describe the sample space for this experiment with a tree diagram.*

SOLUTION: [1.5 Marks] *Let "D" denote a defective calculator and let "A" denote an acceptable calculator. Since the experiment consist to select three calculator in ordered way so we have the following tree diagram*



Then the sample space is given by

$$S = \{DDD, DDA, DAD, DAA, ADD, ADA, AAD, AAA\}$$

Use the tree diagram to describe each of the following events:

(b) A.

SOLUTION: [1 Marks] *We have*

$$A = \{DDD, DDA, DAD, DAA\}$$

(c) B.

SOLUTION: [1 Marks] *We have*

$$B = \{DDD, DDA, ADD, ADA\}$$

(e) $A \cap B$.

SOLUTION: [1 Marks] *We have*

$$A \cap B = \{DDD, DDA\}$$

Exercise 2 (14 Marks) ***PARTS A, B AND C ARE INDEPENDENT.***

A. *The sample space of a random experiment is* $\{a, b, c, d, e\}$ *with probabilities* 0.1, 0.1, 0.2, 0.4, *and* 0.2, *respectively. Let* A *denote the event* $\{a, b, c\}$, *and let* B *denote the event* $\{c, d, e\}$. *Determine the following:*

(a) $P(A)$

Solution: [0.75 Marks] *We have*

$$P(A) = P(a) + P(b) + P(c) = 0.1 + 0.1 + 0.2 = 0.4$$

(b) $P\left(A^C\right) = P(A')$

Solution: [0.75 Marks] *We have*

$$P\left(A^C\right) = 1 - P(A) = 1 - 0.4 = 0.6$$

(c) $P(B)$

Solution: [0.75 Marks] *We have*

$$P(B) = P(c) + P(d) + P(e) = 0.2 + 0.4 + 0.2 = 0.8$$

(d) $P(A \cup B)$

Solution: [1 Marks] *Since* $A \cup B = \{a, b, c, d, e\}$ *which is the sample space, then*

$$P(A \cup B) = 1$$

(e) $P(A \cap B)$

Solution: [1 Marks] *Since*

$$A \cap B = \{c\}$$

thus

$$P(A \cap B) = P(c) = 0.2$$

B. *If A, B and C are mutually exclusive events with $P(A) = 0.2$, $P(B) = 0.3$, and $P(C) = 0.4$, find the following probabilities:*

(a) $P(A \cup B \cup C)$

Solution: [1 Marks] *Since the events are mutually exclusive, therefore*

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) = 0.2 + 0.3 + 0.4 = 0.9$$

(b) $P(A \cap B \cap C)$

Solution: [1 Marks] *Since $A \cap B \cap C = \emptyset$, therefore*

$$P(A \cap B \cap C) = 0$$

(c) $P(A \cap B)$

Solution: [0.75 Marks] *Since $A \cap B = \emptyset$, therefore*

$$P(A \cap B) = 0$$

(d) $P\big((A \cup B) \cap C\big)$

Solution: [0.75 Marks] *We recall that*

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C) = \emptyset$$

Thus,

$$P\big((A \cup B) \cap C\big) = 0$$

(e) $P\left(A^C \cap B^C \cap C^C\right)$

Solution: [0.75 Marks] *We have*

$$\begin{aligned} P\left(A^C \cap B^C \cap C^C\right) &= P\left((A \cup B \cup C)^C\right) \\ &= 1 - P(A \cup B \cup C) \\ &= 1 - (0.2 + 0.3 + 0.4) = 0.1 \end{aligned}$$

C. Samples of a cast aluminum part are classified on the basis of surface finish (in microinches) and length measurements. The results of 100 *parts are summarized as follows:*

		LENGTH	
		excellent	good
SURFACE FINISH	excellent	80	2
	good	10	8

Let A denote the event that a sample has excellent surface finish, and let B denote the event that a sample has excellent length. Determine:

(a) $P(A)$

SOLUTION: [0.75 Marks] A *denote the event that a sample has excellent surface finish, therefore*

$$P(A) = 0.80 + 0.02 = 0.82$$

(b) $P(B)$

SOLUTION: [0.75 Marks] B *denote the event that a sample has excellent length, therefore*

$$P(B) = 0.80 + 0.10 = 0.90$$

(c) $P(A \mid B)$

SOLUTION: [1 Mark] *Using the definition of the conditional probability we get*

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.80}{0.90} = \frac{8}{9} = 0.889$$

(d) $P(B \mid A)$

Solution: [1 Mark] *Using the definition of the conditional probability we get*

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.80}{0.82} = \frac{80}{82} = 0.9756$$

(e) *Are the events A and B independent or not? Justify your answer*

Solution: [1 Mark] *Since we have from the previous*

$$P(A \mid B) = 0.889 \quad \text{AND} \quad P(B \mid A) = 0.0.9756.$$

Thus $P(A \mid B) \neq P(B \mid A)$ *and so the events* A *and* B *are not independent.*

(e) *If the selected part has good length, what is the probability that the surface finish is excellent?*

Solution: [1 Mark] *Let*

- C *denote the event that the selected part has good length;*
- D *denote the event that the surface finish is excellent.*

Thus,

$$P(D \mid C) = \frac{2}{10} = 0.20$$

Exercise 3 (9 Marks)

A. The following table shows the probabilities of 0 *through* 4 *companies that will be running out of business in a given year:*

x	0	1	2	3	4
$f_X(x) = P(X = x)$	0.25	0.3	0.15	0.2	0.1

1. *Verify that this is a probability distribution.*

Solution: [2 Marks] *Since*

$$0.25 + 0.3 + 0.15 + 0.2 + 0.1 = 1$$

2. *Find the mean and standard deviation of the distribution.*

Solution: [2 Marks] *We have*

x	0	1	2	3	4
$xf_X(x)$	0	0.3	0.3	0.6	0.4
$x^2 f_X(x)$	0	0.3	1.2	5.4	6.4

Then by using the previous table we get

$$\begin{aligned}
\mu &= \sum_x x f_X(x) = 0 + 0.3 + 0.3 + 0.6 + 0.4 = 1.6 \\
\sigma &= \sqrt{\text{Var}(X)} \\
&= \sqrt{\sum_x x^2 f_X(x) - \mu^2} \\
&= \sqrt{(0 + 0.3 + 0.6 + 1.8 + 1.6) - (1.6)^2} \\
&= \sqrt{4.3 - 2.56} = \sqrt{2.7} = 1.643168
\end{aligned}$$

B. Let X posses a density function

$$f_X(x) = \begin{cases} kx^2\left(1-x^3\right) & 0 \leqslant x \leqslant 1 \\ 0 & \textit{elsewhere} \end{cases}$$

1. *Find the value of the constant k.*

SOLUTION: (1.75 Marks) *Since f_X is the probability density function of a given random variable then we need to choose k such that the sum of probability distribution function is equal to 1, i.e.*

$$\int_{-\infty}^{+\infty} f_X(x)dx = 1$$

The integral of the probability distribution function is

$$\begin{aligned} \int_{-\infty}^{+\infty} f_X(x)dx &= \int_0^1 k \cdot x^2\left(1-x^3\right)dx \\ &= k \cdot \int_0^1 \left(x^2 - x^5\right)dx \\ &= k \cdot \left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 \\ &= k \cdot \left(\frac{1}{3} - \frac{1}{6}\right) \\ &= k \cdot \left(\frac{2-1}{6}\right) \\ &= \frac{k}{6} \end{aligned}$$

and so we have

$$\frac{k}{6} = 1 \Longrightarrow k = 6.$$

Therefore, the continuous random variable X has the probability density function

$$f_X(x) = \begin{cases} 6x^2\left(1-x^3\right) & 0 \leqslant x \leqslant 1 \\ 0 & \textit{elsewhere} \end{cases}$$

2. *Give the cumulative distribution function of X.*

SOLUTION: (1.75 Marks) *By definition the cumulative distribution function is given by*

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

Then we distinguish the following three situations:

- 1^{st} *case:* $x \leqslant 0$, *so* $t \leqslant x \leqslant 0$. *Thus we have* $f_X(t) = 0$ *and so*

$$F_X(x) = 0$$

- 2^{nd} *case:* $0 < x < 1$, *then*

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(t)dt \\ &= \int_{-\infty}^{0} f_X(t)dt + \int_{0}^{x} f_X(t)dt \\ &= 0 + \int_{0}^{x} \left(6t^2 - 6t^5\right) dt \\ &= \left[2t^3 - t^6\right]_0^x = 2x^3 - x^6 \end{aligned}$$

- 3^{rd} *case:* $x \geqslant 1$, *we have*

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(t)dt \\ &= \int_{-\infty}^{0} f_X(t)dt + \int_{0}^{1} f_X(t)dt + \int_{1}^{x} f_X(t)dt \\ &= 0 + \int_{0}^{1} \left(6t^2 - 6t^5\right) dt + 0 \\ &= 1 \end{aligned}$$

Therefore, the continuous random variable X *has the cumulative distribution function*

$$F_X(x) = \begin{cases} 0 & \text{for } x \leqslant 0 \\ 2x^3 - x^6 & \text{for } 0 < x < 1 \\ 1 & \text{for } x \geqslant 1 \end{cases}$$

3. *Give the mean and the variance of the random variable X.*

Solution: (1.5 Marks) *Since X is a continuous random variable then the mean is given by*

$$\begin{aligned} E[X] &= \int_{-\infty}^{+\infty} t f_X(t) dt \\ &= \int_0^1 t6t^2 \left(1 - t^3\right) dt \\ &= \int_0^1 6 \left(t^3 - t^6\right) dt \\ &= 6 \left[\frac{t^4}{4} - \frac{t^7}{7}\right]_0^1 \\ &= 6 \left(\frac{1}{4} - \frac{1}{7}\right) = 6\frac{3}{28} = \frac{9}{14} \end{aligned}$$

The variance is given by

$$\begin{aligned} \mathrm{Var}[X] &= \int_{-\infty}^{+\infty} t_X^f(t) dt - (E[X])^2 \\ &= \int_0^1 t^2 6t^2 \left(1 - t^3\right) dt - (E[X])^2 \\ &= \int_0^1 6 \left(t^4 - t^7\right) dt - (E[X])^2 \\ &= 6 \left[\frac{t^5}{5} - \frac{t^8}{8}\right]_0^1 - \left(\frac{9}{14}\right)^2 \\ &= 6 \left(\frac{1}{5} - \frac{1}{8}\right) - \frac{81}{196} \\ &= 6\frac{3}{40} = \frac{9}{20} - \frac{81}{196} = \frac{36}{980} = \frac{9}{245} \end{aligned}$$

CHOOSE ONE EXERCISE FROM THE FOLLOWINGS.

Exercise 4 (8 Marks)

Imagine that, it is collected data about the weights for 500 *students from Al-Imam Muhammad Bin Saud Islamic University. It is found that the mean weight for that sample data* $\mu = 68$ *kilogram and standard deviation* $\sigma = 219$ *kilogram. After testing the data, the study shows that the weights of the students are normally distributed, find how many students weight*

1. *Between* 54 *kilogram and* 70 *kilogram.*

   Solution: (3 Marks) *Let* X *be the students weight in a randomly chosen. Then, the question tells us that* $X \sim \text{Normal}(\mu = 68, \sigma = 219)$ *and by using the normal distribution table we obtain*

$$\begin{aligned} P(54 \leqslant X \leqslant 70) &= P\left(\frac{54-68}{219} \leqslant \frac{X-68}{219} \leqslant \frac{70-68}{219}\right) \\ &= P(-0,06 \leqslant Z \leqslant 0,01) \qquad [Z \sim \text{Normal}(0,1)] \\ &= P(Z \leqslant 0,01) - P(-0,06 \leqslant Z) \\ &= P(Z \leqslant 0,01) - P(-0,06 \geqslant Z) \\ &= 0.5040 - 0.4761 = 0.0639 \end{aligned}$$

2. *More than* 84 *kilogram.*

   Solution: (3 Marks) *We have*

$$\begin{aligned} P(X \geqslant 84) &= 1 - P(X \leqslant 84) \\ &= 1 - P\left(\frac{X-68}{219} \leqslant \frac{84-68}{219}\right) \\ &= 1 - P(Z \leqslant 0,07) \qquad [Z \sim \text{Normal}(0,1)] \\ &= 1 - 0.5279 = 0.4721 \end{aligned}$$

Exercise 5 (8 Marks)

A store sells clothes for men. It has 3 different kinds of jackets, 7 different kinds of shirts, and 5 different kinds of pants. Next to the store, there is a Library which sells books, it has 8 different mystery books and 4 different history books.

Find the number of ways, a costumer can buy:

1. *One of the items from the store.*

   Solution: (3 Marks) *The number of ways is*

$$3 + 7 + 5 = 15$$

2. *One of each of the items from the store.*

   Solution: (3 Marks) *The number of ways is*

$$3 \times 7 \times 5 = 105$$

3. *3 books from the Library.*

   Solution: (3 Marks) *The number of ways is*

$$C(12, 3) = \binom{12}{3} = \frac{12!}{3! \times 9!} = \frac{12 \times 11 \times 10}{6} = 220$$

Exercise 6 (8 Marks)

The probability that a lab specimen contains high levels of contamination is 0.10. *Five samples are checked, and the samples are independent.*

1. *What is the probability that none contains high levels of contamination?*

Solution: [4 Marks] *Let H_i denote he event that the ii^{th} sample contains high levels of contamination. Thus, since the samples are independent we get*

$$\begin{aligned} P\left(H_1^C \cap H_2^C \cap H_3^C \cap H_4^C \cap H_5^C\right) &= P\left(H_1^C\right) P\left(H_2^C\right) P\left(H_3^C\right) P\left(H_4^C\right) P\left(H_5^C\right) \\ &= (0.90)^5 = 0.59 \end{aligned}$$

(ii) *What is the probability that exactly one contains high levels of contamination?*

Solution: [4 Marks] *We have five possibilities for this situation:*

- $A_1 = H_1 \cap H_2^C \cap H_3^C \cap H_4^C \cap H_5^C$;
- $A_2 = H_1^C \cap H_2 \cap H_3^C \cap H_4^C \cap H_5^C$;
- $A_3 = H_1^C \cap H_2^C \cap H_3 \cap H_4^C \cap H_5^C$;
- $A_4 = H_1^C \cap H_2^C \cap H_3^C \cap H_4 \cap H_5^C$;
- $A_5 = H_1^C \cap H_2^C \cap H_3^C \cap H_4^C \cap H_5$.

The requested probability is the probability of the union $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$ and these events are mutually exclusive. Also, by independence

$$P\left(A_i\right) = (0.90)^4(0.10) = 0.0656$$

Therefore,

$$P\left(A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5\right) = 5\big((0.90)^4(0.10)\big) = 5(0.0656) = 0.328$$

بسم الله الرحمن الرحيم

KINGDOOM OF SAUDI ARABIA
Al-Imam Muhammad Ibn Saud Islamic University
Faculty of Sciences

المملكة العربية السعودية
جامعة الإمام محمد بن سعود الإسلامية
كلية العلوم

FINAL EXAM SOLUTIONS

SEMESTER: SECOND **YEAR:** 1429/1430

COURSES: MAT 301 & MAT 141 & STA 111

DATE: 27/06/1430 (20/06/2009) **DURATION:** 2 HOURS

Instructors: Drs. A. MUSTAFA, A. BARKAOUI, B. CHOURAR & A. S. BEN GHORBAL

Name ________________________________ **Section** ______

Student ID___________ **Your Signature** ____________________

	TOTAL MARKS	*SCORE*
EXERCISE 1	**07.00**	
EXERCISE 2	**12.00**	
EXERCISE 3	**08.00**	
EXERCISE 4	**13.00**	
EXERCISE BONUS	**03.00**	
TOTAL	**43.00**	

EXERCISE 1. (7 MARKS)

The following display gives the number of Friday newspapers published in each of the 11 district of Kingdom of Saudi Arabia during 1430.

39 25 28 10 14 93 25 22 19 31 33

1) (1.75 MARKS) Find the sample mean.
2) (2.25 MARKS) Find the variance s^2 and standard deviation s .
3) (3 MARKS) Find the quartiles Q_1, Q_2.

SOLUTIONS

1. MEAN = 30.82
2. s^2 = 494.76 s = 22.24
3. $Q1$ = 19 , Q2 = 25

EXERCISE 2. (10 MARKS)

PART A. Suppose that a parking contains 24 cars where only six cars are BMW and 4 cars are Mercedes. Four cars are selected at random and without replacement for daily use.

1) (2 MARKS) What is the probability that exactly one car in the sample is a BMW one?
2) (2 MARKS) What is the probability that at least one car is BMW in the sample?
3) (2 MARKS) What is the probability that exactly one car is BMW in the sample and exactly one car is a Mercedes?

SOLUTIONS

PART B. At a certain gas station, 40% of the customers use regular gasoline 91 (event A_1), 35% use super gasoline 95 (event A_2), and 25% use diesel (event A_3). Of those customers using regular gasoline 91, only 30% fill their tanks (event B). Of those customers using super gasoline 95, 60% fill their tanks, whereas of those using diesel, 50% fill their tanks.

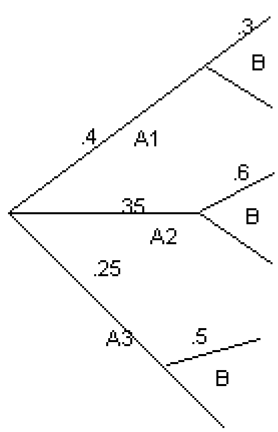
a. (1.75 MARKS) What is the probability that the next customer will request super gasoline 95 and fill the tank $(A_2 \cap B)$?.

b. (1.75 MARKS) What is the probability that the next customer fills the tank?

c. (2.50 MARKS) If the next customer fills the tank, what is the probability that regular gasoline 91 is requested? Super gasoline 95 is requested? Diesel is requested?

SOLUTIONS

SOLUTION



$.4 \times .3 = .12 = P(A_1 \cap B) = P(A_1) \bullet P(B \mid A)$

$.35 \times .6 = .21 = P(A_2 \cap B)$

$.25 \times .5 = .125 = P(A_3 \cap B)$

a. $P(A_2 \cap B) = .21$

b. $P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) = .455$

c. $P(A_1 | B) = \dfrac{P(A_1 \cap B)}{P(B)} = \dfrac{.12}{.455} = .264$

$P(A_2 | B) = \dfrac{.21}{.455} = .462$, $P(A_3 | B) = 1 - .264 - .462 = .274$

EXERCISE 3. (8 MARKS)

PART A. An oil exploration company currently has two active projects, one in KSA and the other in Kuwait. Let A be the event that the event the Saudi project is successful and B be the event that the Kuwaiti project is successful. Suppose that A and B are independent events with $P(A) = 0.4$ and $P(B) = 0.7$.

a. (2 MARKS) If the Saudi project is not successful, what is the probability that the Kuwaiti project is also not successful? Explain your answer.

b. (2 MARKS) What is the probability that at least one of the two projects will be successful?

SOLUTION

a. Since the events are independent, then A′ and B′ are independent, too. (see paragraph below equation 2.7. $P(B'|A') = .$ $P(B') = 1 - .7 = .3$

b. $P(A \cup B)=P(A) + P(B) - P(A)\cdot P(B) = .4 + .7 + (.4)(.7) = .82$

PART B. A total of 46% of the voters in a certain city classify themselves as Independent, whereas 30% classify themselves as Liberals and 24% as Conservatives. In a recent local election, 35% of the Independents, 62% of the Liberals, and 58% of the Conservatives voted. A voter is chosen at random.

a) (2 MARKS) What fraction of voters participated in the local election?

b) (2 MARKS) Given that this person voted in the local election, what is that he or she is an Independent.

SOLUTIONS

(d) $P\{\text{voted}\} = .35(.46) + .62(.3) + .58(.24) = .4862$
That is, 48.62 percent of the voters voted.

(a) $$P(\text{Ind} \mid \text{voted}) = \frac{P(\text{voted}|\text{Ind})P(\text{Ind})}{\sum P(\text{voted}|\text{type})P(\text{type})}$$
$$= \frac{.35(.46)}{.35(.46) + .62(.3) + .58(.24)} \approx 331$$

EXERCISE 4. (15 MARKS)

1) The random variable X has a binomial distribution with $n = 10$ and $p = 0.01$. Determine the following probabilities.

a. (1.25 MARKS) $P(X = 5) =$

b. (1.25 MARKS) $P(X \le 2) =$

c. (1.25 MARKS) $P(X \ge 9) =$

SOLUTION

$$\text{a) } P(X=5) = \binom{10}{5} 0.01^5 (0.99)^5 = 2.40 \times 10^{-8}$$

$$b) P(X \le 2) = \binom{10}{0} 0.01^0 (0.99)^{10} + \binom{10}{1} 0.01^1 (0.99)^9 + \binom{10}{2} 0.01^2 (0.99)^8$$

$$= 0.9999$$

$$c) P(X \ge 9) = \binom{10}{9} 0.01^9 (0.99)^1 + \binom{10}{10} 0.01^{10} (0.99)^0 = 9.91 \times 10^{-18}$$

2) Automobiles arrive at a vehicle equipment inspection station according to a Poisson process with rate $\lambda = 10$ per hour. Suppose that with probability 0.5 an arriving vehicle will have no equipment violations.

a. (1.50 MARKS) What is the probability that exactly 10 arrive during the hour?

b. (1.50 MARKS) What is the probability that exactly 10 arrive during the hour and all 10 have no violations?

SOLUTION

P(*y* arrive and exactly 10 have no violations)

= *P*(exactly 10 have no violations / *y* arrive). *P*(*y* arrive)

$$= P(10 \text{ successes in } y \text{ trials when } p = .5) \cdot e^{-10} \frac{(10)^y}{y!}$$

$$= \binom{y}{10} (.5)^{10} (.5)^{y-10} e^{-10} \frac{(10)^y}{y!} = \frac{e^{-10} (5)^y}{10!(y-10)!}$$

3) Let Z be a standard normal random variable, $Z \sim \mathcal{N}$.

a. (1.50 MARKS) Calculate the following probabilities $P(1.51 \le Z \le 2.25)$.

b. (1.50 MARKS) Determine the value of the constant c that makes the probability statement correct: $P\left(0 \le Z \le c\right) = 0.195$.

4) (3.25 MARKS) Suppose only 40% of all drivers in Florida regularly wear a seatbelt. A random sample of 500 drivers is selected. What is the probability that fewer than 175 of those in the sample regularly wear a seatbelt? (**HINT:** consider X the normal distribution with mean $\mu = np = 500 \times 0.4 = 200$ and standard deviation $\sigma = \sqrt{npq} = \sqrt{500 \times 0.4 \times 0.6} = 10.95$).

SOLUTIONS

$$n = 500, p = .4, \mu = 200, \sigma = 10.9545$$

$$P(X < 175) = P(X \le 174) = P(\text{normal} \le 174.5) = P(Z \le -2.33) = .0099$$

EXERCISE BONUS (3 MARKS)

Let X = the time between two successive arrivals at the drive-up window of a local bank. If X has an exponential distribution with $\lambda = 1$, compute the following:

a. (1 MARK) The expected time between two successive arrivals.

b. (1 MARK) The standard deviation of the time between two successive arrivals.

c. (1 MARK) $P(X \le 5)$.

SOLUTIONS

a. $E(X) = 1/\lambda = 1$

b. $\sigma = 1/\lambda = 1$

c. $P(X \le 5) = 1 - e^{-(1)(5)} = 1 - e^{-5} = 0.9933$

بسم الله الرحمن الرحيم

KINGDOM OF SAUDI ARABIA
Ministry of Higher Education
Al-Imam Muhammad Ibn Saud Islamic University
College of Science – Department of Mathematics

المملكة العربية السعودية
وزارة التعليم العالي
جامعة الإمام محمد بن سعود الإسلامية
كلية العلوم - قسم الرياضيات

FINAL EXAM - Solutions

SEMESTER: SECOND | **YEAR:** 1430/1431 (2009/2010)

COURSE: STA 111 | **LEVEL:** 3^{rd} & 4^{th} | **SECTIONS:** 174, 175, 176, 177, 191

DATE: 09/07/1431 (21/06/2010) | **DURATION:** 2 HOURS

Instructors: Drs. H. FAIRES & F. BELLALOUNA & A. BERKAOUI & A. S. BEN GHORBAL

Name ______________________ **Section** ______

Student ID ______________ **Your Signature** ______________________

	TOTAL MARKS	SCORE
EXERCISE 1	09.00	
EXERCISE 2	15.00	
EXERCISE 3	15.00	
EXERCISE 4	03.00	
TOTAL	42.00	

Solutions

Sample

INSTRUCTIONS

- Please check that your exam contain **13 Pages** total (including the first page!! & Normal Tables) and **04 EXERCISES**.
- Please check your test for completeness.
- Read instructions for each problem carefully.
- NO book, NO notes but you may use a calculator which does not graph and which is not programmable.
- Show all your work to get full credit. You must explain how you get your answers using techniques developed in this class so far. Answer with no supporting work, obtained by guess-and-check, or via other methods will result in little or no credit, even correct.
- **Place a box** around **Your Final Answer** to each question.
- Answer the equation in the space provided on the question sheets. If you need more room (space), use the backs of the pages and indicate to the reader that you have done so.
- If you are not sure what a question means, raise your hand and ask me.
- Check your work!

Good Luck!

مع دعائنا لكم بالتوفيق

EXERCISE 1. (09.00 Marks)

PART A (04.50 Marks) A multi-choice exam is composed of **10 questions**; each question consists of **4 possible answers** such that only one is true. The student chooses randomly an answer for each question. Find the probability that:

1. (01.50 Marks) He has exactly 5 correct answers.

Let X be the random variable denoting the number of the correct answers. Thus $X \sim Bin(n=10, k, p=\frac{1}{4})$

X is Binomial distribution where

$$P(X=k) = \binom{10}{k}\left(\frac{1}{4}\right)^k \times \left(1-\frac{1}{4}\right)^{10-k}$$

So $P(X=5) = \binom{10}{5} \times \left(\frac{1}{4}\right)^5 \times \left(\frac{3}{4}\right)^5 = 0.058$

2. (01.50 Marks) He has at most 9 correct answers.

We have

$$P(X \leq 9) = 1 - P(X=10)$$
$$= 1 - \left(\frac{1}{4}\right)^{10}$$
$$\simeq 0.999\ldots = 1$$

3. (01.50 Marks) He has at least 2 correct answers.

We have

$$P(X \geq 2) = 1 - P(X<2)$$
$$= 1 - P(X \leq 1)$$
$$= 1 - P(X=0) - P(X=1)$$
$$= 1 - \binom{10}{0}\left(\frac{1}{4}\right)^0 \times \left(\frac{3}{4}\right)^{10-0} - \binom{10}{1}\left(\frac{1}{4}\right) \times \left(\frac{3}{4}\right)^9$$
$$= 0.756$$

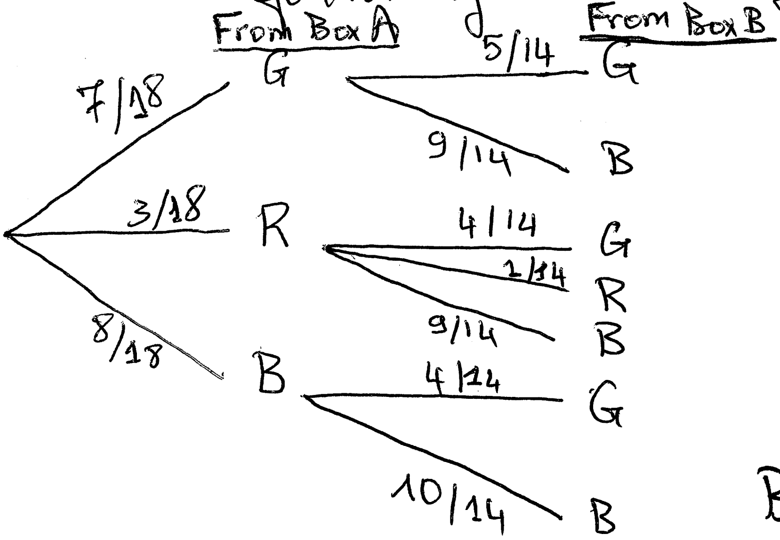
PART B (04.50 Marks) Two boxes A and B are as follows:

- Box A contains 7 green balls, 3 red balls and 8 black balls,
- Box B contains 4 green balls and 9 black balls,

One ball is taken randomly from box *A* (ball 1) and put in box B and then one ball is taken randomly from box *B* (ball 2).

(a) (00.75 Marks) Find the probability that the ball 1 is black.

We have the following tree diagram



B: Black Ball

R: Red Ball

G: Green Ball

Denote:

B_1: Ball 1 is black

B_2: Ball 2 is black

G_1: Ball 1 is Green

R_2: Ball 1 is Red

Thus $P(\text{Ball 1 is black}) = \frac{8}{18}$

(b) (01.50 Marks) Find the probability that the ball 2 is black.

We have

$$P(\text{Ball 2 is black}) = P(B_2|B_1)P(B_1) + P(B_2|G_1)P(G_1) + P(B_2|R_1)P(R_1)$$

$$= \frac{10}{14}\times\frac{8}{18} + \frac{9}{14}\times\frac{7}{18} + \frac{9}{14}\times\frac{3}{18} = \frac{80+63+27}{14\times 18}$$

$$\simeq 0.675$$

(c) (02.25 Marks) Find the probability that the ball 1 is green, given that the ball 2 is black.

We have

$$P(G_1|B_2) = \frac{P(B_2|G_1)P(G_1)}{P(B_2)}$$

$$= \frac{\frac{9}{14}\times 7/18}{\frac{170}{14\times 18}} = \frac{9\times 7}{170} \simeq 0.371$$

EXERCISE 2. (15.00 Marks)

PART A (04.50 Marks) The waiting time, in hours, between successive speeders spotted by a radar unit is a continuous random variable X with cumulative distribution function

$$F_X(x) = \begin{cases} 1-e^{-8x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

1. (02.00 Marks) Find the density distribution function f_X of the random variable X.

Since F_X is continuous function by intervals then X is a continuous random variable with probability density function F'_X where

$$F'_X(x) = f_X(x) = \begin{cases} 8e^{-8x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

2. (01.00 Marks) Find the probability that the waiting time is lesser than 2 hours, $P(X \leq 2)$.

We have

$$P(X \leq 2) = F_X(2) = 1 - e^{-16} \simeq 1$$

3. (01.50 Marks) Find the probability that the waiting time is bigger than 1 hour, $P(X \geq 1)$.

We have

$$P(X \geq 1) = 1 - P(X < 1)$$
$$= 1 - F_X(1)$$
$$= 1 - (1 - e^{-8}) = e^{-8}$$

PART B (03.00 Marks) A continuous random variable X has the following density distribution function

$$f_X(x) = \begin{cases} kx & 0 \le x \le 1 \\ 0 & elsewhere \end{cases}$$

Find the followings:

1. (02.00 Marks) The value of the parameter k.

Since f_X is the p.d.f. of the r.v. X then

$$\int_{-\infty}^{+\infty} f_X(x) = 1 \iff \int_0^1 kx\,dx = 1$$

$$\iff k\left[\frac{x^2}{2}\right]_0^1 = 1$$

$$\iff \frac{k}{2} = 1 \iff k = 2$$

2. (03.00 Marks) The cumulative distribution function F_X, of the random variable X.

We have $f_X(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$

By the definition of c.d.f. we have

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$$

Then

$$F_X(x) = \begin{cases} 0 & \text{if } x \le 0 \\ x^2 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

3. (01.00 Marks) $P(X \leq 0.5)$,

We have $P(X \leq 0.5) = F_X(0.5)$
$= (0.5)^2$
$= 0.25$

4. (01.50 Marks) $P(-0.4 \leq X \leq 0.4)$,

We have $P(-0.4 \leq X \leq 0.4) = P(X \leq 0.4) - P(X \leq -0.4)$
$= F_X(0.4) - F_X(-0.4)$
$= (0.4)^2 - 0$
$= 0.16$

5. (03.00 Marks) The mean and the variance of X.

X is a continuous r.v.

We have : the mean

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x)\, dx = \int_0^1 2x^2 dx = \frac{2}{3}$$

We have : the variance

$$V[X] = E[X^2] - (E[X])^2$$
$$= \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx - \left(\frac{2}{3}\right)^2$$
$$= \int_0^1 2x^3 dx - \frac{4}{9}$$
$$= \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$$

EXERCISE 3. (15.00 Marks)

PART A (04.50 Marks) Let Z be a standard normal random variable and calculate the following probabilities:

(a) (00.50 Mark) $P(Z \le 1.32)$

We have $P(Z \le 1.32) = 0.9066$

(b) (01.00 Mark) $P(Z \ge -1.70)$

$$P(Z \ge -1.70) = 1 - P(Z \le -1.70)$$
$$= 1 - 0.0446$$
$$= 0.9554$$

(c) (01.50 Marks) $P(1.50 \le Z \le 2.50)$

We have $P(1.50 \le Z \le 2.50) = P(Z \le 2.50) - P(Z \le 1.50)$
$$= 0.9938 - 0.9332$$
$$= 0.0606$$

(d) (01.50 Marks) $P(|Z| \le 2.40)$

We have

$$P(|Z| \le 2.40) = P(-2.40 \le Z \le 2.40)$$
$$= P(Z \le 2.40) - P(Z \le -2.40)$$
$$= 0.9918 - 0.0082$$
$$= 0.9836$$

PART B (06.00 Marks) A continuous random variable X has the normal distribution with mean $\mu = 1$ and standard deviation $\sigma = 2$. Compute the following probabilities by standardizing

a) (01.00 Marks) $P(X \le 3.5)$,

Since $X \sim \mathcal{N}(\mu = 1, \sigma^2 = 4)$ then we consider the standarized Normal Distribution $Z = \frac{X-\mu}{\sigma} = \frac{X-1}{2} \sim \mathcal{N}(1,0)$

$P(X \le 3.5) = P(Z \le \frac{3.5-1}{2}) = P(Z \le 1.25) = 0.8944$

b) (01.50 Marks) $P(-2 \le X \le 4.4)$

$P(-2 \le X \le 4.4) = P(\frac{-2-1}{2} \le Z \le \frac{4.4-1}{2}) = P(-\frac{3}{2} \le Z \le \frac{3.4}{2})$

$= P(Z \le 1.70) - P(Z \le -0.50)$

$= 0.9554 - 0.3085 = 0.6469$

c) (01.50 Marks) $P(X \ge 4.24)$

$P(X \ge 4.24) = 1 - P(X \le 4.24)$

$= 1 - P(Z \le \frac{4.24-1}{2})$

$= 1 - P(Z \le 1.62)$

$= 1 - 0.9474 = 0.0526$

d) (02.00 Marks) $P(|X - 8.75| \le 10)$

$P(|X - 8.75| \le 10) = P(-10 \le X - 8.75 \le 10)$

$= P(-1.25 \le X \le 18.75)$

$= P(\frac{-1.25-1}{2} \le Z \le \frac{18.75-1}{2})$

$= P(-1.125 \le Z \le 8.875)$

$= P(-1.13 \le Z \le 8.88)$

$= P(Z \le 8.88) - P(Z \le -1.13)$

$\simeq 1 - 0.1292$

$\simeq 0.8708$

PART C Suppose only 90% of all drivers in Florida regularly wear a seatbelt. A random sample of 100 drivers is selected. Let X be the random variable denoting the number of drivers wearing a seatbelt.

1. (02.00 Marks) What is the probability distribution of the random variable X?

Since any driver will wear a seat belt or not then X is a Binomial Distribution where

$$X \sim Bin(100, 90\%)$$ with $n = 100$ sample size, $p = 0.90$ probability

2. Approximate the probabilities that

(a) (01.50 Marks) Between 89 and 95 of the drivers in the sample regularly wear a seatbelt?

Since $X \sim Bin(100, 0.90)$ with sample size $n = 100 > 30$

then $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ where $\mu_X = E[X] = np = 90$

The Normal approximation to the Binomial

$\sigma_X^2 = npq = 100 \times 0.90 \times 0.1 = 9$

$\sigma_X = 3$

So

$$P(83 \leq X \leq 95) = P\left(\frac{83-90}{3} \leq Z \leq \frac{95-90}{3}\right)$$

$Z = \frac{X-90}{3} \sim \mathcal{N}(0,1)$

$$= P(-2.33 \leq Z \leq 1.66)$$

$$= P(Z \leq 1.66) - P(Z \leq -2.33) = 0.9515 - 0.0099 = 0.9416$$

(b) (01.00 Marks) Fewer than 93 of those in the sample regularly wear a seatbelt?

$$P(X < 93) = P\left(Z \leq \frac{93-90}{3}\right)$$

$$= P(Z \leq 1.00)$$

$$= 0.8413$$

EXERCISE 4. (03.00 Marks)

Assume that each of your calls to a popular radio station has a probability of 0.02 of connecting, that is, of not obtaining a busy signal. Assume that your calls are independent.

a) (01.50 Marks) What is the probability that your first call that connects is your **tenth call**?

Let X be the random variable denoting the number of calls until the first connection.

So $X \sim Geom(0.02)$: Geometric distribution with parameter $0 < p < 1$

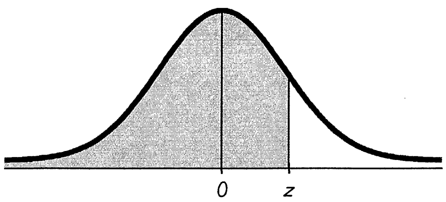
Thus $P(X = k) = (1-p)^{k-1} p$

and

$$P(X=10) = (1-0.02)^9 \times 0.02 = (0.98)^9 \times 0.02$$
$$= 0.0167$$

b) (01.50 Marks) What is the probability that it requires more than **three calls** for you to connect?

$$P(X > 3) = 1 - P(X \leq 2)$$
$$= 1 - (P(X=1) + P(X=2))$$
$$= 1 - 0.02 - 0.98 \times 0.02$$
$$= 0.9604$$



POSITIVE *z* Scores

TABLE A-2 (continued) Cumulative Area from the LEFT

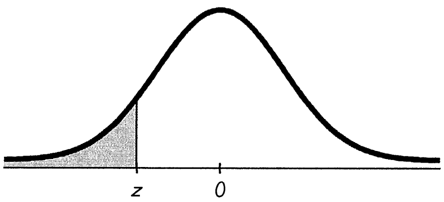
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495 *	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949 *	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.50 and up	.9999									

NOTE: For values of *z* above 3.49, use 0.9999 for the area.
*Use these common values that result from interpolation:

z score	Area
1.645	0.9500
2.575	0.9950

Common Critical Values

Confidence Level	Critical Value
0.90	1.645
0.95	1.96
0.99	2.575



NEGATIVE *z* Scores

TABLE A-2 Standard Normal (*z*) Distribution: Cumulative Area from the LEFT

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.50 and lower	.0001									
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051 *	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505 *	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

NOTE: For values of *z* below −3.49, use 0.0001 for the area.
*Use these common values that result from interpolation:

z score	Area
−1.645	0.0500
−2.575	0.0050