Kingdom of Saudi Arabia
Al-Imam Mohammad Ibn Saud Islamic University
Faculty of Science

المملكة العربية السعودية
جامعة الإمام محمد بن سعود الإسلامية
كلية العلوم

Department of Mathematics and Statistics

قسم الرياضيات والإحصاء

Application of spectral method in fluid dynamics

Research Project by:

Abeer Fahad Abdullah Al-Sohaim

Supervisor:

Prof. Fahir Talay Akyildiz

Academic Year: 1438-1439

Summary

In this report, consideration is given to the spectral method on elliptic boundary value problems, in order to develop weak solution of given problem, first some basic mathematical concepts are given . Then orthogonal polynomials, in particular, Jacobi and related Chebyshev and Legendre polynomials are defined and related approximation (error) analyses explored. We first study description of the spectral method on the two point boundary problems. Several order Legendre orthogonal polynomials used to show the how approximate solution approach to the exact analytical solution. An application of spectral method to the real world problem was undertaken in the final section where the flow in an eccentric microannulus problem considered, which arise from the micro electromechanical systems and oil industry. It is shown that Debye − Hückel approximation can be used for governing semilinear Possion-Boltzmann elliptic partial differential equation for low zeta function. Finally, Fourier Legendre pseudo spectral method is used to obtain the approximate analytical solution for Debye − Hückel approximation and reduced momentum equation solved analytically. The effects of the parameters involved in our flow problem are demonstrated graphically.

Introduction

Education in mathematical methods and principles usually begin with introduction discrete systems and this description develops toward to the continuous systems. For example at first, numbers that a child counts in a sing-song manner are just a sequence of words (positive integers). Then all of a sudden the words become useful as the child learns to match them to an amount by counting fingers. Later child develop skill and study decimals and finally real numbers which begin understanding of the location of object in (x, y) space for all points. Students first consider the slope of the curve in this space, they first try to calculate slope of the curve by using discrete expression $\frac{\Delta y}{\Delta x}$. Understanding of the slope completed when they are able to calculate the slope of any point on arbitrary curve using continuous expression $\frac{dy}{dx}$.Mathematical model in science and engineering presented by algebraic equation, like $F = ma$, but this expression in fact represent Navier Stokes equation as

$$\rho \left(\frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^*.\nabla^* \mathbf{u}^* \right) = -\nabla p^* + \mu \nabla^{*2} \mathbf{u}^* + \rho_e E^* \left(t \right),$$

Since, the fluid flows are of engineering and science significance due to the variety of applications that depend on understanding its behaviours. Because of this, this subject has been extensively studied both physically and mathematically. Numerical methods, namely, replace the differential expression with discrete one and solve this reduced equation to represent the solution of the differential system. But this approximation for differential expression should be in a tolerable error bound. A Fundamental idea for Numerical method is the reduction of differential equation to the approximation algebraic system. This reduction replace a continuous differential equation, whose solution space is generally infinite dimensional, with a finite set of algebraic equations whose solution space is finite dimensional. The most commonly used numerical methods in commercial available software are the Finite Volume Method (FVM), the Finite Differences Method

(FDM) and the Finite Element Method (FEM), they uses the same approach but has a different mathematical foundation. The previous methods can be applicable in a range of different flow geometries without any difficulty, but require careful consideration when designing the computational mesh domain in order to obtain results that successfully model the flow field. In this thesis, we use pseudo spectral method (PSM), which is characterised by its high accuracy for space variables, indeed this main advantage to use this approach. Its main disadvantages of PSM are the need of a regularly distributed mesh, which prevent its use in complicated flow geometries, and, like all methods previously mentioned, a high physical memory requirement to be capable of modelling the smaller scales of motion present in any fluid flow. Sometimes this difficulty can be avoided by using different coordinates system. The organization of the thesis is as follows: In chapter 1 devoted to the basic mathematical tools which are needed follow up chapter, we examine the properties of classical orthogonal polynomials. Chapter 2 deals with the properties of spectral methods. Chapter 3 focuses on the application of Pseudo Spectral method to flow problems and numerical results and in chapter 4 we discuss the numerical results of the method and effect of parameters involved in our problems on electric potential and flow field. Because of regulation in writing the project which require number of page must be less than 40(forty), many mathematical detail such as existence of weak solution, uniqueness of the solution and error analysis in chapter 3 were omitted.

Contents

Chapter 1

Terminologies ans Concepts

A sequence $\{x_n\}$ in a metric space $(X;d)$, is called a Cauchy sequence, if for every positive real number $\epsilon > 0$ there is a positive integer N such that for all positive $m,n > N$, the distance

$$d(x_m, x_n) = |x_m - x_n| < \epsilon,$$

or equivalently

$$d(x_m, x_n) \to 0 \quad as \quad m,n \ \to \infty,$$

If every Cauchy sequence in a metric space X converges to an element in X, then X is said to be a complete space.

1.1 Banach Spaces

Definition : Let X be a real vector space, a function $\|\cdot\| : X \to \mathbb{R}$ defines a norm on X such that

1. $\|u+v\| \leq \|u\| + \|v\| \ , \quad \forall u,v \in X;$

2. $\|\alpha u\| = |\alpha| \|u\| \ , \quad \forall u \in X$ and $\forall \alpha \in \mathbb{R};$

3. $\|u\| \geq 0 \ , \quad \forall u \in X;$

4. $\|u\| = 0$ if and only if $u = 0$.

If the first three conditions holds, then a function $|\cdot| : X \to \mathbb{R}$ defines a semi-norm on X.

The space $(X, \|\cdot\|)$ is called a normed vector space. A Banach space is a complete normed vector space with respect to the metric :

$$d(u,v) = \|u - v\|, \quad \forall u, v \in X.$$

1.2 Hilbert Spaces

Definition 1.2.1. An inner product on a vector space X is a function
$(u, v) : X \times X \to \mathbb{R}$ such that

1. $(u, v) = (v, u), \quad \forall u, v \in X;$
2. $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w), \quad \forall u, v, w \in X$ and $\forall \alpha, \beta \in \mathbb{R};$
3. $(u, u) \geq 0, \quad \forall u \in X;$
4. $(u, u) = 0$ if and only if $u = 0$.

A Banach space with an inner product defined on it, is called Hilbert space.

Definition 1.2.2. In an inner product space X, $u, v \in X$ are said to be orthogonal,If

$$(u, v) = 0,$$

The inner product $(\cdot, \cdot)$ induces a norm on X,defined by

$$\|u\| = \sqrt{(u, u)}, \quad \forall u \in X. \tag{1.2.1}$$

Or the metric on X can be defined by $d(u, v) = \|u - v\|$.
A Hilbert space is a complete inner product space.

Lemma 1.2.1. In a Hilbert space, Cauchy-Schwarz inequality holds

$$\left|(u, v)\right| \leq \|u\|\|v\|, \quad \forall u, v \in X. \tag{1.2.2}$$

Proof : Can be proven easily as in [3]
The standard inner product on $\mathbb{R}^n$ is given by

$$(x, y) = \sum_{j=1}^{n} x_j y_j$$

where $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$ with $x_i, y_i \in \mathbb{R}$.
This space is complete hence it is finite dimensional Hilbert space.
And $L^2[0, 1]$, $L^2[a, b]$, and $L^2(\mathbb{R})$ are all Hilbert spaces with respect to the inner product.

1.3 Lax-Milgram Lemma

Definition 1.3.1. Let X be a Hilbert space with norm $\|\cdot\|$. If for any $u, v, w \in X$ and $\alpha, \beta \in \mathbb{R}$ a functional $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ defines a bilinear form as follow,

$$a(\alpha u + \beta v, w) = \alpha a(u,w) + \beta a(v,w), \tag{1.3.1}$$

$$a(u, \alpha v + \beta w) = \alpha a(u,w) + \beta a(v,w). \tag{1.3.2}$$

That mean , for any fixed u, both the functionals $a(u,\cdot) : X \to \mathbb{R}$ and $a(\cdot,u) : X \to \mathbb{R}$ are linear.
If $a(u,v) = a(v,u)$ for any $u, v \in X$. the bilinear form is called as symmetric.

Definition 1.3.2. A bilinear form $a(\cdot,\cdot)$ on a Hilbert space X is called continuous, if there exists a constant $C > 0$ such that

$$\left|a(u,v)\right| \leq C\|v\|\|u\|, \quad \forall u, v \in X \tag{1.3.3}$$

and coercive on X, if there exists a constant $\alpha > 0$ such that

$$a(u,u) \geq \alpha \|u\|^2, \quad \forall u \in X. \tag{1.3.4}$$

Theorem 1.3.1. (Lax-Milgram lemma) Let X be a Hilbert space, and $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ be a continuous and coercive bilinear form, and assume that $F : X \to \mathbb{R}$ be a linear functional in X'. Then the variational problem:

$$\begin{cases} \text{Find } u \in X \text{ such that} \\ a(u,v) = F(v), \quad \forall v \in X, \end{cases} \tag{1.3.5}$$

has a unique solution.Furthermore, we have

$$\|u\| \leq \frac{1}{\alpha} \|F\|_{X'} \tag{1.3.6}$$

1.4 L_p-Space

Assume Ω is a Lebesgue-measurable subset of $\mathbb{R}^d$ $(d = 1, 2, 3)$ with non-empty interior, and let u be a Lebesgue measurable function on Ω. The following integrations are in Lebesgue sense.

Definition 1.4.1. For $1 \le p \le \infty$, we define the following set

$$L^p(\Omega) := \{u : u \text{ is measurble on } \Omega \text{ and } \|u\|_{L^p(\Omega)} < \infty\} \tag{1.4.1}$$

where for $1 \le p \le \infty$,

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}} \tag{1.4.2}$$

and

$$\|u\|_{L^\infty(\Omega)} = \text{ess} \sup_{x \in \Omega} |u(x)| \tag{1.4.3}$$

The space $L^p(\Omega)$ where $L^p(\Omega)$ endowed with $\|\cdot\|_{L^p(\Omega)}$ is the Banach space. In particular, the space $L^2(\Omega)$ endowed with the inner product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(x) \; v(x) \; dx, \qquad \forall u, v \in L^2(\Omega). \tag{1.4.4}$$

is the Hilbert space

Definition 1.4.2. Let p and q are positive real numbers , if

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then p and q are called conjugate exponents.1 and ∞ are also regarded as conjugate exponents.

Theorem 1.4.1. Holder's inequality. Assume that p and q be conjugate exponents with $1 \le p \le \infty$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$, and

$$\int_\Omega |u(x) \; v(x)| \; dx \le \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)} \tag{1.4.5}$$

For $p = 2$,Holders inequality reduces to well known Cauchy-Schwarz inequality (1.2.2).

Theorem 1.4.2. Minkowski's inequality. If $u, v \in L^p(\Omega)$ with $1 \le p \le \infty$, then $u + v \in L^p(\Omega)$ and

$$\|u + v\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \tag{1.4.6}$$

Remark. Let us assume that $\omega(x)$, which is almost everywhere positive and Lebesgue integrable on Ω, $\omega(x) \; dx$ also defines a Lebesgue measure on Ω. Replacing dx in (1.4.2) by $\omega(x) \; dx$, we define the norm $\|\cdot\|_{L^p_\omega(\Omega)}$ and the weighted $L^p_\omega(\Omega)$ space with $1 \le p < \infty$, which is a Banach space. In particular, the weighted space $L^2_\omega(\Omega)$ is a Hilbert space with the following inner product and norm

$$(u, v)_\omega = \int_\Omega u\,(x)\, v\,(x)\, \omega\,(x)\, dx, \qquad \|u\|_\omega = \sqrt{(u, u)_\omega} \tag{1.4.7}$$

1.5 Sobolev Spaces

1.5.1 Weak Derivatives

A multi-index $\alpha = (\alpha_1, ..., \alpha_m)$ is a m-tuple of non-negative integers $\{\alpha_i\}$. Denote $|\alpha| = \sum_{i=1}^{m} \alpha_i$,and define the partial derivative operator

$$D^\alpha = \frac{\partial^{\,|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}$$

Definition 1.5.1. A function $f \in L^1_{(loc)}(\Omega)$ where Ω in R^m, is weakly differentiable with respect to x_i, if there exists a function $f \in L^1_{(loc)}(\Omega)$ such that

$$\int_\Omega f\partial_i\phi dx = -\int_\Omega g_i\phi dx \quad \text{forall}\,\phi \in C_c^\infty(\Omega) \tag{1.5.1}$$

Then the function g_i is called the weak ith derivative of f, which is shown as $\partial_i f$ Therefore, we can state weak derivative as integration by parts formula

$$\int_\Omega f\partial_i\phi dx = -\int_\Omega \partial f_i\phi dx \tag{1.5.2}$$

since $C_c^\infty(\Omega)$ is dense in $L^1_{loc}\left(\Omega\right))$ this definition hold for all $\phi \in C_c^\infty(\Omega)$¿ Moreover, the weak derivative of a continuously differentiable function agrees with the pointwise derivative. Higher-order derivatives are defined in a smiler manner.

Definition 1.5.2. Suppose $\alpha = (\alpha_1, \cdots, \alpha_m)$ is a multi-index, a function $f \in L^1_{loc}(\Omega)$ has weak derivative $\partial^\alpha f \in L^1_{loc}(\Omega)$ if

$$\int_\Omega f\, D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega f\, \left(\partial^\alpha \phi\right), dx \qquad \forall \phi \in C_c^\infty(\Omega).$$

We restrict our discussions to the Hilbert spaces (i.e., with $p = 2$),

1.5.2 Sobolev space definition

Sobolev spaces consist of functions whose weak derivative belong to L^p [2]. These spaces one of the most useful settings for the analysis of partial differential equations

Definition 1.5.3. The Sobolev space $H^k(\Omega)$ with $k \in \mathbb{N}$ is the space of functions $u \in L^2(\Omega)$ such that all the distributional derivatives of order up to m can be represented by functions in $L^2(\Omega)$. That is,

$$H^k(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \text{ for } 0 \leq |\alpha| \leq k\} \tag{1.5.3}$$

equipped with the norm and semi-norm

$$\|u\|_{k,\Omega} = \left(\sum_{|\alpha|=0}^{k} \|D^\alpha u\|_{L^2(\Omega)}^2 \right)^{1/2}, |u|_{k,\Omega} = \left(\sum_{|\alpha|=0}^{k} \|D^\alpha u\|_{L^2(\Omega)}^2 \right)^{1/2} \tag{1.5.4}$$

In particular, the space $H^k(\Omega)$ is a Hilbert space endowed with the inner product

$$(u, v)_{k,\Omega} = \sum_{|\alpha|=0}^{k} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx.$$

Chapter 2

Orthogonal Polynomials and Related Approximation Results

2.1 Approximability of Orthogonal Polynomials

Theorem 2.1.1. (Weierstrass Approximation Theorem) Let $u(x) \in C[a,b]$. Then there exist a sequence of polynomials $p_n(x)$ that converge uniformly to $u(x)$ on $[a,b]$, i.e any $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and $p_n \in P_n$ such that

$$\|u - p_n\|_{L^\infty_{(a,b)}} < \varepsilon, \forall n \geq N \tag{2.1.1}$$

Proof: See for example Shen [1]

The best approximation problem facilitate as to construct of p_n(Minimax polynomial):

$$\begin{cases} \text{Given a fixed } n \in \mathbb{N} \text{ , find } p_n^* \in P_n \text{ , such that} \\ \|u - p_n^*\|_{L^\infty_{(a,b)}} = \inf\limits_{p_n \in \ P_n} \|u - p_n\|_{L^\infty_{(a,b)}} \end{cases} \tag{2.1.2}$$

For $1 \leq p \leq \infty$, $L^p(\Omega) = \{u : u \ is$ measurable on Ω and $\|u\|_{L^p(\Omega)} < \infty\}$
with $\|u\|_{L^p(\Omega)} = \left(\int \left|u(x)\right|^p dx \right)^{\frac{1}{p}}$, $\|u\|_{L^\infty(\Omega)} = ess\ sup \left|u(x)\right|$.

Theorem 2.1.2. Let I be any interval finite or an infinite. Then for any given $u \in L^2_\omega(I)$ and $n \in \mathbb{N}$, there exists a unique $q_n^* \in P_n$, such that

$$\|u - q_n^*\|_\omega = \inf_{q_n \in \ P_n} \|u - q_n\|_\omega \tag{2.1.3}$$

where,$q_n^*(x)$ linear combination of polynomials $p_k(x)$ as

$$q_n^*(x) = \sum_{k=0}^{n} \hat{u}_k p_k(x) \quad with \quad \hat{u}_k = \frac{(u, p_k)_\omega}{\|p_k\|_\omega^2} \tag{2.1.4}$$

and $\{p_k\}_{k=0}^n$ forms an L^2_ω-orthogonal basis of P_n.

The L^2_ω-orthogonal projection is defined by

$$(u - \pi_n u, \phi)_\omega = 0 \ , \ \forall \phi \in P_n. \tag{2.1.5}$$

It is clear that $\pi_n u$ is the first $n+1$-term truncation of the series $u = \sum_{k=0}^{\infty} \hat{u}_k p_k(x)$.

2.2 Jacobi Polynomials

In mathematics, Jacobi polynomials sometimes called called hypergeometric polynomials is denoted by $J_n^{\alpha,\beta}(x)$, are classical orthogonal polynomials. They are orthogonal with respect to the weight function $\omega^{\alpha,\beta}(x) : = (1-x)^\alpha (1+x)^\beta$ over $I := [-1, 1]$,basically,

$$\int_{-1}^{1} J_n^{\alpha,\beta}(x)\, J_m^{\alpha,\beta}(x)\, \omega^{\alpha,\beta}(x)\, dx = \gamma_n^{\alpha,\beta} \delta_{mn}, \tag{2.2.1}$$

where $\gamma_n^{\alpha,\beta} = \left\| J_n^{\alpha,\beta} \right\|_{\omega^{\alpha,\beta}}^2$.The weight function $\omega^{\alpha,\beta}$ belongs to $L^1(I)$ if and only if $\alpha, \beta > -1$ Furthermore, Gegenbauer, Legendre and Chebyshev polynomials are special case of the Jacobi polynomials. One of the important basic properties of Jacobi polynomials are the solution of following linear homogeneous differential equations

2.2.1 Sturm-Liouville Equation

Now, we define singular Sturm-Liouville operator as

$$\begin{aligned} \mathscr{L}_{\alpha,\beta} u : &= -(1-x)^{-\alpha}(1+x)^{-\beta} \partial_x \left((1-x)^{\alpha+1}(1+x)^{\beta+1} \partial_x u(x) \right) \\ &= (x^2-1)\partial_x^2 u(x) + \{\alpha - \beta + (\alpha+\beta+2)x\} \partial_x u(x) \end{aligned} \tag{2.2.2}$$

By (2.2.2) and weight function of Jacobi polynomial we can show that

$$\mathscr{L}_{\alpha,\beta} J_n^{\alpha,\beta} = -\omega^{-\alpha,-\beta} \partial_x \left(\omega^{\alpha+1,\beta+1} \partial_x J_n^{\alpha,\beta} \right)$$

we are now ready to prove following theorem

Theorem 2.2.1. The Jacobi polynomials are the eigen functions of the singular Sturm-Liouville problem:

$$\mathscr{L}_{\alpha,\beta} J_n^{\alpha,\beta}(x) = \lambda_n^{\alpha,\beta} J_n^{\alpha,\beta}(x). \tag{2.2.3}$$

and the corresponding eigenvalues are

$$\lambda_n^{\alpha,\beta} = n(n+\alpha+\beta+1) \tag{2.2.4}$$

Proof. For any $u \in P_n$, we have $\mathscr{L}_{\alpha,\beta} u \in P_n$. Performing integration by parts twice, we get that for any $\phi \in P_{n-1}$,

$$\begin{aligned}
\left(\mathscr{L}_{\alpha,\beta} J_n^{\alpha,\beta}, \phi\right)_{\omega^{\alpha,\beta}} &= \int_{-1}^{1} \left[-\omega^{-\alpha,-\beta} \partial_x \left(\omega^{\alpha+1,\beta+1} \partial_x J_n^{\alpha,\beta}(x)\right)\right] \phi(x)\ \omega^{\alpha,\beta} dx \\
&= \int_{-1}^{1} -\partial_x \left(\omega^{\alpha+1,\beta+1} \partial_x J_n^{\alpha,\beta}(x)\right) \phi(x)\ dx \\
&= -\omega^{\alpha+1,\beta+1} \partial_x J_n^{\alpha,\beta}(x)\phi(x)\Big|_{-1}^{1} + \int_{-1}^{1} \omega^{\alpha+1,\beta+1} \partial_x J_n^{\alpha,\beta}(x) \partial_x \phi(x)\ dx \\
&= \int_{-1}^{1} \omega^{\alpha+1,\beta+1} \partial_x J_n^{\alpha,\beta}(x) \partial_x \phi(x)\ dx \qquad \text{since } \omega^{\alpha+1,\beta+1}(\pm 1) = 0 \\
&= J_n^{\alpha,\beta}(x) \omega^{\alpha+1,\beta+1} \partial_x \phi(x)\Big|_{-1}^{1} - \int_{-1}^{1} J_n^{\alpha,\beta}(x) \partial_x \left(\omega^{\alpha+1,\beta+1} \partial_x \phi(x)\right)\ dx \\
&= -\int_{-1}^{1} J_n^{\alpha,\beta}(x) \partial_x \left(\omega^{\alpha+1,\beta+1} \partial_x \phi(x)\right)\ dx \qquad \text{since } \omega^{\alpha+1,\beta+1}(\pm 1) = 0 \\
&= \int_{-1}^{1} J_n^{\alpha,\beta}(x) \left[-\omega^{-\alpha,-\beta} \partial_x \left(\omega^{\alpha+1,\beta+1} \partial_x \phi(x)\right)\right] \omega^{\alpha,\beta}\ dx \qquad \text{since } \omega^{-\alpha,-\beta}\omega^{\alpha,\beta} = 1 \\
&= \left(J_n^{\alpha,\beta}, \mathscr{L}_{\alpha,\beta} \phi\right)_{\omega^{\alpha,\beta}}
\end{aligned}$$

Now since $\mathscr{L}_{\alpha,\beta}\phi \in P_{n-1}$, and $J_n^{\alpha,\beta}$an orthogonal polynomial we have

$$\left(\mathscr{L}_{\alpha,\beta} J_n^{\alpha,\beta}, \phi\right)_{\omega^{\alpha,\beta}} = \left(J_n^{\alpha,\beta}, \mathscr{L}_{\alpha,\beta}\phi\right)_{\omega^{\alpha,\beta}} = 0$$

$J_n^{\alpha,\beta}(x)$ orthogonal any polynomial of degree less than n. we have $\mathscr{L}_{\alpha,\beta} J_n^{\alpha,\beta} \in P_n$, and using the uniqueness of orthogonal polynomials which implies that there exists a constant $\lambda_n^{\alpha,\beta}$ such that

$$\mathscr{L}_{\alpha,\beta} J_n^{\alpha,\beta} = \lambda_n^{\alpha,\beta} J_n^{\alpha,\beta}$$

To determine $\lambda_n^{\alpha,\beta}$, comparing the coefficient of the leading term x_n on both sides,

$$\begin{aligned}
\mathscr{L}_{\alpha,\beta}x^n &= (x^2-1)\partial_x^2 x^n + \{\alpha-\beta+(\alpha+\beta+2)x\}\partial_x x^n \\
&= (x^2-1)n(n-1)x^{n-2} + \{\alpha-\beta+(\alpha+\beta+2)x\}nx^{n-1} \\
&= n(n-1)x^n - n(n-1)x^{n-2} + n(\alpha-\beta)x^{n-1} + n(\alpha+\beta+2)x^n \\
&= n(n+\alpha+\beta+1)x^n + n(\alpha-\beta)x^{n-1} - n(n-1)x^{n-2} \\
&= \lambda_n^{\alpha,\beta}x^n
\end{aligned}$$

Therefore $\lambda_n^{\alpha,\beta} = n(n+\alpha+\beta+1)$. □

Remark. It can be shown that Sturm-Liouville operator $\mathscr{L}_{\alpha,\beta}$ is self-adjoint with respect to the inner product $(\cdot,\cdot)_{\omega^{\alpha,\beta}}$ *i.e.*

$$\left(\mathscr{L}_{\alpha,\beta}\phi,\psi\right)_{\omega^{\alpha,\beta}} = \left(\phi,\mathscr{L}_{\alpha,\beta}\psi\right)_{\omega^{\alpha,\beta}} \tag{2.2.5}$$

for any ϕ , $\psi \in \{u : \mathscr{L}_{\alpha,\beta}u \in L^2_{\omega^{\alpha,\beta}}(\Omega)\}$

The proof is not difficult in fact , for any ϕ , $\psi \in L^2_{\omega^{\alpha,\beta}}(\Omega)\}$, if we apply integration by parts twice, then we obtain that for any $\phi \in P_{n-1}$,

$$\begin{aligned}
\left(\mathscr{L}_{\alpha,\beta}\phi,\psi\right)_{\omega^{\alpha,\beta}} &= \int_{-1}^{1}\left[-\omega^{-\alpha,-\beta}\partial_x\left(\omega^{\alpha+1,\beta+1}\partial_x\phi(x)\right)\right]\psi(x)\ \omega^{\alpha,\beta}dx \\
&= \int_{-1}^{1} -\partial_x\left(\omega^{\alpha+1,\beta+1}\partial_x\phi(x)\right)\psi(x)\ dx \\
&= -\omega^{\alpha+1,\beta+1}\partial_x\phi(x)\psi(x)\Big|_{-1}^{1} + \int_{-1}^{1}\omega^{\alpha+1,\beta+1}\partial_x\phi(x)\partial_x\psi(x)\ dx \\
&= \int_{-1}^{1}\omega^{\alpha+1,\beta+1}\partial_x\phi(x)\partial_x\psi(x)\ dx \qquad \text{since } \omega^{\alpha+1,\beta+1}(\pm1)=0 \\
&= \phi(x)\omega^{\alpha+1,\beta+1}\partial_x\psi(x)\Big|_{-1}^{1} - \int_{-1}^{1}\phi(x)\partial_x\left(\omega^{\alpha+1,\beta+1}\partial_x\psi(x)\right)\ dx \\
&= -\int_{-1}^{1}\phi(x)\partial_x\left(\omega^{\alpha+1,\beta+1}\partial_x\psi(x)\right)\ dx \qquad \text{since } \omega^{\alpha+1,\beta+1}(\pm1)=0 \\
&= \int_{-1}^{1}\phi(x)\left[-\omega^{-\alpha,-\beta}\partial_x\left(\omega^{\alpha+1,\beta+1}\partial_x\psi(x)\right)\right]\omega^{\alpha,\beta}\ dx \qquad \text{since } \omega^{-\alpha,-\beta}\omega^{\alpha,\beta}=1 \\
&= \left(\phi(x),\mathscr{L}_{\alpha,\beta}\psi(x)\right)_{\omega^{\alpha,\beta}}
\end{aligned}$$

Theorem 2.2.2. (Szego 1975) The differential equation

$$\mathscr{L}_{\alpha,\beta}u = \lambda u$$

has a polynomial solution not identically zero if and only if λ has the form $n(n+\alpha+\beta+1)$. This solution is $J_n^{\alpha,\beta}(x)$ (up to a constant), and no solution which is linearly independent of $J_n^{\alpha,\beta}(x)$ can be a polynomial.

We prove this by using Frobenous method ,

$$(x^2-1)u''(x)+\alpha-\beta+(\alpha+\beta+2)xu'(x)-\lambda u(x)=0 \tag{2.2.6}$$

Let $p(x)=\dfrac{\alpha-\beta+(\alpha+\beta+2)x}{x^2-1}, \quad q(x)=\dfrac{-\lambda}{x^2-1}$
here we have ± 1 are the singular points of (2.2.6).

For the singularity at $x=1$,

$$(x-1)p(x)=\frac{\alpha-\beta+(\alpha+\beta+2)x}{x+1}, \qquad (x-1)^2q(x)=\frac{-\lambda(x-1)}{x+1}$$

both are analytic at $x=1$. Hence, 1 is a regular singular point.

For $x=-1$,

$$(x+1)p(x)=\frac{\{\alpha-\beta+(\alpha+\beta+2)x\}}{x-1}, \qquad (x+1)^2q(x)=\frac{-\lambda(x+1)}{x-1}$$

both are analytic at $x=-1$. Hence, -1 is a regular singular point.

Now for the indicial equation

$$r(r-1)+p_o r+q_o=0$$

where $p_0=\lim_{x\to 1}(x-x_0)p(x),\ q_0=\lim_{x\to 1}(x-x_0)^2q(x)$

For $x=1$:

$$p_0=\lim_{x\to 1}\frac{\alpha-\beta+(\alpha+\beta+2)x}{x+1}=\alpha+1$$
$$q_0=\lim_{x\to 1}\frac{-\lambda(x-1)}{x+1}=0$$

Hence, the indicial equation is $r(r-1)+(\alpha+1)r=0$ which implies that $r=0$ or $r=-\alpha$.

For $x=-1$:

$$p_0=\lim_{x\to -1}\frac{\{\alpha-\beta+(\alpha+\beta+2)x\}}{x-1}=\beta+1$$
$$q_0=\lim_{x\to -1}\frac{-\lambda(x+1)}{x-1}=0$$

Hence, the indicial equation is $r(r-1)+(\beta+1)r=0$ which implies that $r=0$ or $r=-\beta$.
Now for $x_0=1, r=0$, let $u(x)=\sum\limits_{k=0}^{\infty} a_k(x-1)^k$, we substitute $u(x)$ in (2.2.6),

$$(x^2-1)\sum_{k=0}^{\infty} k(k-1)a_k(x-1)^{k-2}+(\alpha-\beta+(\alpha+\beta+2)x)\sum_{k=0}^{\infty} ka_k(x-1)^{k-1}-\lambda\sum_{k=0}^{\infty} a_k(x-1)^k=0$$
$$\sum_{k=0}^{\infty}\left[\left(k^2+k+(\alpha+\beta)k-\lambda\right)a_k+\left(2k^2+2k+(k+1)(2\alpha+2)\right)a_{k+1}\right](x-1)^k=0.$$

which yields the recurrence formula

$$\left(k^2+k+(\alpha+\beta)k-\lambda\right)a_k+\left(2k^2+2k+(k+1)(2\alpha+2)\right)a_{k+1}=0, \quad k=0,1,\cdots$$
$$\frac{a_{k+1}}{a_k}=\frac{\lambda-k\,(k+\alpha+\beta+1)}{2(k+1)\,(k+\alpha+1)}, \quad k=0,1,\cdots \tag{2.2.7}$$

Assuming that u is a polynomial, let us suppose that a_n is the last nonzero coefficient. Then we see from (2.2.7) that for all $k=n$ the coefficient of a_n must vanish, that is, $\lambda=n(n+\alpha+\beta+1)$
Hence, we show that

$$J_n^{\alpha,\beta}(x)=\sum_{k=0}^{n} a_k(x-1)^k$$
$$\frac{a_{k+1}^n}{a_k^n}=\frac{\lambda_n^{\alpha,\beta}-k\,(k+\alpha+\beta+1)}{2(k+1)\,(k+\alpha+1)} \tag{2.2.8}$$

There are many possible way to normalize Jacobi polynomials and each of the natural ways has advantages and disadvantages. The normalization we have used by using the Gamma function has advantage being leading coefficient can be expressed as a division of Gamma function

$$a_0^n=J_n^{\alpha,\beta}(1)=\binom{n+\alpha}{n}=\frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}, \tag{2.2.9}$$

where $\Gamma(\cdot)$ is the Gamma function. Then the leading coefficient can be obtained from (2.2.8) easily as

$$a_n^n=k_n^{\alpha,\beta}=\frac{\Gamma(2n+\alpha+\beta+1)}{2^n n!\,\Gamma(n+\alpha+\beta+1)}. \tag{2.2.10}$$

Furthermore, using (2.2.8)in above , we obtain

$$J_n^{\alpha,\beta}(x)=\frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(n+\alpha+\beta+1)}\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(n+k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)}\left(\frac{x-1}{2}\right)^k.$$

Orthogonality of $\{\partial_x J_n^{\alpha,\beta}\}$ follow from the direct consequence of this theorem

Corollary.

$$\int_{-1}^{1} \partial_x J_n^{\alpha,\beta} \partial_x J_m^{\alpha,\beta} \omega^{\alpha+1,\beta+1} dx = \lambda_n^{\alpha,\beta} \gamma_n^{\alpha,\beta} \delta_{nm} \tag{2.2.11}$$

Proof. Again, we first use integration by parts and then theorem(2.2.1), finally orthogonality of $\{J_n^{\alpha,\beta}\}$ provide us desired results as follows:

$$\begin{aligned}
\int_{-1}^{1} \partial_x J_n^{\alpha,\beta} \partial_x J_m^{\alpha,\beta} \omega^{\alpha+1,\beta+1} dx &= \left(\partial_x J_n^{\alpha,\beta}, \partial_x J_m^{\alpha,\beta}\right)_{\omega^{\alpha+1,\beta+1}} \\
&= \left(J_n^{\alpha,\beta}, \mathscr{L}_{\alpha,\beta} J_m^{\alpha,\beta}\right)_{\omega^{\alpha,\beta}} \qquad \text{Using integration by parts} \\
&= \lambda_n^{\alpha,\beta} \left\| J_n^{\alpha,\beta} \right\|^2 \delta_{nm} \qquad \text{By eq(2.2.3)} \\
&= \lambda_n^{\alpha,\beta} \gamma_n^{\alpha,\beta} \delta_{nm}
\end{aligned}$$

□

since $\{\partial_x J_n^{\alpha,\beta}\}$ is orthogonal with respect to the weight $\omega^{\alpha+1,\beta+1}$, which state that $\partial_x J_n^{\alpha,\beta}$ must be proportional to $J_{n-1}^{\alpha+1,\beta+1}$, this is

$$\partial_x J_n^{\alpha,\beta}(x) = \mu_n^{\alpha,\beta} J_{n-1}^{\alpha+1,\beta+1}(x) \tag{2.2.12}$$

Leading coefficients on both sides leads provide that there should be proportionality constant between the leading coefficient which is

$$\mu_n^{\alpha,\beta} = \frac{n k_n^{\alpha,\beta}}{k_{n-1}^{\alpha+1,\beta+1}} \overset{\text{eq(2.2.10)}}{=} \frac{1}{2}(n+\alpha+\beta+1) \tag{2.2.13}$$

This relation gives the following important derivative relation:

$$\partial_x J_n^{\alpha,\beta}(x) = \frac{1}{2}(n+\alpha+\beta+1) J_{n-1}^{\alpha+1,\beta+1}(x) \tag{2.2.14}$$

Applying this formula recursively, we can obtain the general formula for derivative relation

$$\partial_x J_n^{\alpha,\beta}(x) = d_{n,k}^{\alpha,\beta} J_{n-k}^{\alpha+k,\beta+k}(x), \quad n \geq k \tag{2.2.15}$$

where

$$d_{n,k}^{\alpha,\beta} = \frac{\Gamma(n+k+\alpha+\beta+1}{2^k \Gamma(n+\alpha+\beta+1)} \tag{2.2.16}$$

Recurrence Formulas

The Jacobi polynomials are generated by the three-term recurrence relation:

$$\begin{aligned}
&J_{n+1}^{\alpha,\beta}(x) = (a_n^{\alpha,\beta} x - b_n^{\alpha,\beta}) J_n^{\alpha,\beta}(x) - c_n^{\alpha,\beta} J_{n-1}^{\alpha,\beta}(x), \quad n \geq 1, \\
&J_0^{\alpha,\beta}(x) = 1, \quad J_1^{\alpha,\beta}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta),
\end{aligned} \tag{2.2.17}$$

where

$$a_n^{\alpha,\beta} = \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)}$$
$$b_n^{\alpha,\beta} = \frac{(\beta^2-\alpha^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}$$
$$c_n^{\alpha,\beta} = \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}$$

when we use the Jacobi differential equation, we can show the above relations which allows us to evaluate the Jacobi polynomials at any given $x \in [-1,1]$, and it is the starting point to derive other properties.

Theorem 2.2.3. Suppose that

$$J_n^{\alpha,\beta}(x) = \sum_{k=0}^{n} \hat{c}_k^n J_k^{a,b}(x), \quad a,b,\alpha,\beta > -1 \tag{2.2.18}$$

Then

$$\begin{aligned}\hat{c}_k^n &= \frac{\Gamma(n+\alpha+1)(2k+a+b+1)\Gamma(k+a+b+1)}{\Gamma(n+\alpha+\beta+1)\Gamma(k+a+1)} \\ &\times \sum_{m=0}^{n-k} \frac{(-1)^m\Gamma(n+k+m+\alpha+\beta+1)\Gamma(m+k+a+1)}{m!(n-k-m)!\Gamma(k+m+\alpha+1)\Gamma(m+2k+a+b+2)}\end{aligned} \tag{2.2.19}$$

2.3 Legendre Polynomials

In mathematics, Legendre polynomials is the solution of Legendre differential equations. Furhermore, this polynomials are the special case of the Jacobi polynomials (where $\alpha = \beta = 0$)

$$L_n(x) = J_n^{0,0}(x), \quad n \geq 0,\ x \in I = (-1,1). \tag{2.3.1}$$

The distinct feature of the Legendre polynomials is that they are mutually orthogonal with respect to the uniform weight function $\omega(x) \equiv 1$.

Most important properties of Legendre Polynomials are

- Legendre's differential equation

$$(1-x^2)u" - 2xu' + mu = [(1-x^2)u']' + mu = 0, \text{where } m = n(n=1), n = 0, 1.2 \tag{2.3.2}$$

- Three recurrence relation:

$$(n+1)L_{n+1}(x) = (2n+1)xL_n(x) - nL_{n-1}(x), \quad n \geq 1, \tag{2.3.3}$$

and the first few Legendre polynomials which are the solution of Legendre's differential equations

$$L_0(x) = 1, \qquad L_1(x) = x, \qquad L_2(x) = \frac{1}{2}(3x^2 - 1), \qquad L_3(x) = \frac{1}{2}(5x^3 - 3x)$$

- The Legendre polynomial has the expansion

$$L_n(x) = \frac{1}{2^n} \sum_{l=0}^{[n/2]} (-1)^l \frac{(2n-2l)!}{2^n l!(n-l)!(n-2l)!} x^{n-2l}, \tag{2.3.4}$$

and the leading coefficient is

$$k = \frac{(2n)!}{2^n (n!)^2} \tag{2.3.5}$$

- Sturm-Liouville problem:

$$((1-x^2)L_n'(x))' + \lambda_n L_n(x) = 0, \qquad \lambda_n = n(n+1). \tag{2.3.6}$$

Equivalently,

$$(1-x^2)L_n''(x) - 2xL_n'(x) + n(n+1)L_n(x) = 0. \tag{2.3.7}$$

- Rodrigues formula:

$$L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right], \qquad n \geq 0 \tag{2.3.8}$$

- Orthogonality:

$$\int_{-1}^{1} L_n(x) L_m(x) dx = \gamma_n \delta_{mn}, \qquad \gamma_n = \frac{2}{2n+1}, \tag{2.3.9a}$$

$$\int_{-1}^{1} L_n'(x) L_m'(x)(1-x^2) dx = \gamma_n \lambda_n \delta_{mn}. \tag{2.3.9b}$$

- Symmetric property:

$$L_n(-x) = (-1)^n L_n(x), \qquad L_n(\pm 1) = (\pm 1)^n \tag{2.3.10}$$

Hence, $L_n(x)$ is an odd (resp. even) function, if n is odd (resp. even). Moreover,we have the uniform bound

$$\left|L_n(x)\right| \leq 1, \qquad \forall x \in [-1,1], n \geq 0. \tag{2.3.11}$$

- Derivative recurrence relations:

$$(2n+1)L_n(x) = L'_{n+1}(x) - L'_{n-1}(x), \quad n \geq 1, \tag{2.3.12a}$$

$$L'_n(x) = \sum_{\substack{k=0 \\ k+n \text{ odd}}}^{n-1} (2k+1)L_k(x), \tag{2.3.12b}$$

$$L''_n(x) = \sum_{\substack{k=0 \\ k+n \text{ even}}}^{n-2} \left(k + \frac{1}{2}\right) (n(n+1) - k(k+1))L_k(x) \tag{2.3.12c}$$

$$(1-x^2)L'_n(x) = \frac{n(n+1)}{2n+1} \big(L_{n-1}(x) - L_{n+1}(x)\big) \tag{2.3.12d}$$

- The boundary values of the derivatives:

$$L'_n(\pm 1) = \frac{1}{2}(\pm 1)^{n-1} n(n+1), \tag{2.3.13a}$$

$$L''_n(\pm 1) = (\pm 1)^n (n-1)n(n+1)(n+2)/8. \tag{2.3.13b}$$

2.4 Error Estimates for Polynomial Approximation

2.4.1 Inverse Inequalities for Jacobi Polynomials

Since all norms of a function in any finite dimensional space are equivalent, we have

$$\|\partial_x \phi\| \leq C_N \|\phi\|, \qquad \forall \phi \in P_N$$

which is an example of inverse inequalities. The inverse inequalities are very useful for analyzing spectral approximations of nonlinear problems.

The first inverse inequality relates two norms weighted with different Jacobi weight functions. We give the following theorem which are the bases for error analysis without proof, interested reader can find the proof of theorems in the book of Shen [1]

Theorem 2.4.1. For $\alpha, \beta > -1$ and any $\phi \in P_N$, we have

$$\|\partial_x \phi\|_{\omega^{\alpha+1,\beta+1}} \leq \sqrt{\lambda_N^{\alpha,\beta}} \|\phi\|_{\omega^{\alpha,\beta}}, \tag{2.4.1}$$

where $\lambda_N^{\alpha,\beta} = N(N+\alpha+\beta+1)$

Theorem 2.4.2. For $\alpha, \beta > -1$ and any $\phi \in P_N^0$,

$$\|\partial_x \phi\|_{\omega^{\alpha,\beta}} \lesssim N \|\phi\|_{\omega^{\alpha-1,\beta-1}}, \tag{2.4.2}$$

where $\lambda_N^{\alpha,\beta} = N(N+\alpha+\beta+1)$

Theorem 2.4.3. For any $\phi \in P_N$,

$$\|\partial_x \phi\| \leq \frac{1}{2}(N+1)(N+2)\|\phi\|. \tag{2.4.3}$$

where $\lambda_N^{\alpha,\beta} = N(N+\alpha+\beta+1)$

Chapter 3

Spectral Methods and its Application

3.1 Spectral Methods for Second-Order Two-Point Boundary Value Problems

In the following we take the famous example of Prof. Shen [1]. Consider the two-point boundary value problem :

$$-\varepsilon U'' + p(x)U' + q(x)U = F \ , \ x \in (-1,1) \tag{3.1.1}$$

where $\varepsilon > 0$ and (3.1.1) with the general Robin boundary conditions

$$a_- U(-1) + b_- U'(-1) = c_- \ , \ a_+ U(1) + b_+ U'(1) = c_+ \tag{3.1.2}$$

without loss of generality , we assume that :

(i) $a_\mp \geq 0$.

(ii) $a_-^2 + b_-^2 \neq 0$. (3.1.3)

(iii) $q(x) - \dfrac{p'(x)}{2} \geq 0 \ , \ \forall x \in (-1,1)$.

(iv) $p(1) > 0 \ if \ b_+ \neq 0, p(-1) < 0 \ if \ b_- \neq 0$.

The above conditions are necessary for well-posedness of (3.1.1) − (3.1.2). The proof of well posedness can be found for example Atkinson [4]. In order to apply any spectral method we need to do boundary conditions homogeneous, in the following this is done for both two cases:

Case I :

$a_{\mp} = 0$ and $b_{\mp} \neq 0$, $\tilde{u} = \beta x^2 + \gamma x$ and β , γ are uniquely determined by asking $\tilde{u}$ to satisfy (3.1.2)

$$\begin{cases} -2b_-\beta + b_-\gamma = c_- \\ 2b_+\beta + b_+\gamma = c_+ \end{cases}$$

$$\Delta = \begin{vmatrix} -2b_- & b_- \\ 2b_+ & b_+ \end{vmatrix} = -4b_-b_+ \neq 0$$

Case II :

$a_-^2 + a_+^2 \neq 0$, $\tilde{u} = \beta x + \gamma$ and again we have :

$$\begin{cases} (-a_- + b_-)\beta + a_-\gamma = c_- \\ (a_+ + b_+)\beta + a_+\gamma = c_+ \end{cases}$$

$$\Delta = \begin{vmatrix} -a_- + b_- & a_- \\ a_+ + b_+ & a_+ \end{vmatrix} = -2a_-a_+ + a_+b_- - a_-b_+ \neq 0 \qquad \textit{where } b_- \leq 0 \textit{ and } b_+ \geq 0$$

Now set $u = U - \tilde{u}$, $f = F - (-\varepsilon\tilde{u}'' + p(x)\tilde{u}' + q(x)\tilde{u})$

substitute this into (3.1.1) we obtain

$$-\varepsilon u'' + p(x)u' + q(x)u = f \qquad x \in (-1,1) = \Omega \tag{3.1.4}$$

with the homogeneous boundary condition

$$a_-u(-1) + b_u'(-1) = 0 \qquad a_+u(1) + b_+u'(1) = 0 \tag{3.1.5}$$

Let us denote

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u(\pm 1) = 0 \; if \; b_\pm = 0\}$$

Then, a standard weak formulation of (3.1.4) with (3.1.5) is :

$$\begin{cases} Find \; u \in H_0^1(\Omega) \; such \; that \\ B(u,v) = (f,v) \; , \forall v \in H_0^1(\Omega) \end{cases}$$

$$
\begin{aligned}
B(u,v) &= \varepsilon(u',v') + (p(x)u',v) + (q(x)u,v) \\
&= \varepsilon \int u'v' \ dx + p(x) \int u'v \ dx + q(x) \int uv \ dx \\
&\leq \varepsilon \|u\|_{H^1} \|v\|_{H^1} + |p(x)| \|u\|_{H^1} \|v\|_{L^2} + |q(x)| \ \|u\|_{L^2} \|v\|_{L^2}
\end{aligned}
$$

where above, we used integration by parts for second order derivatives and homogenous boundary condition. Then using the Poincare inequality , we obtain :

$$
\leq \|u\|_{H^1} \|v\|_{H^1} + C|p(x)| \ \|u\|_{H^1} \|v\|_{H^1} + C_1 |q(x)| \ \|u\|_{H^1} \|v\|_{H^1}
$$

Since $p(x)$ and $q(x)$ are continuous on Ω , we have : $|p(x)| \leq \|p\|_\infty$, $|q(x)| \leq \|q\|_\infty$

So, $B(u,v) \leq C \|u\|_{H_0^1} \|v\|_{H_0^1}$. Therefore $B(u,v)$ bounded.

On the other hand ,

$$
\begin{aligned}
B(u,u) &= \varepsilon \int u'^2 + |p(x)| \int_\Omega u'u \ dx + |q(x)| \int u^2 \ dx \\
&= \varepsilon \int u'^2 + |p(x)| \int_\Omega \frac{1}{2} \frac{d}{dx}(u^2) + |q(x)| \int u^2 \ dx \\
&= \varepsilon \int u'^2 + |p(x)| \left(\frac{1}{2} u^2 \Big|_{-1}^{1} \right) + |q(x)| \int u^2 \ dx \\
&= \varepsilon \|u\|_{H_0^1}^2 + |q(x)| \ \|u\|_{L^2}^2 & \textit{Since} \left(\frac{1}{2} u^2 \Big|_{-1}^{1} \right) = 0 \\
&\geq \varepsilon \|u\|_{H_0^1}^2 & \textit{Since } |q(x)| \ \|u\|_{L^2}^2 \textit{ alaways positive}
\end{aligned}
$$

$B(u,u)$ is coercive. Hence, by Lax-Milgram lemma solution exist and unique.

3.2 Galerkin Method

3.2.1 Legendre-Galerkin Method

We set $f_N = I_N f$, the Legendre interpolation polynomial of f. Then Galerkin method becomes

$$
- \int_\Omega u_N'' v_N \ dx + \alpha \int_\Omega I_N f v_N \ dx, \quad \forall v_N \in X_N \tag{3.2.1}
$$

This method is called Legendre-Galerkin method.

This system depends on the choice of basis functions of X_N. Namely we look for basis function as a compact combination of Legendre polynomials, basically,

$$
\phi_k(x) = L_k(x) + a_k L_{k+1}(x) + b_k L_{k+2}(x) \tag{3.2.2}
$$

for example, for the the boundary conditions:

$$a_- u(-1) + b_- u'(-1) = 0 \ , \quad a_+ u(1) + b_+ u'(1) = 0$$

Using the properties of Legendre polynomial and their derivatives, we have $L_k(\pm 1) = (\pm 1)^k$ and $L'_k(\pm) = \frac{1}{2}(\pm 1)^{k-1} k(k+1)$ and application for above boundary conditions we obtain following system of equations for $\{a_k, b_k\}$:

$$\begin{aligned} \left(a_+ + \frac{b_+}{2}(k+1)(k+2)\right) a_k + \left(a_+ + \frac{b_+}{2}(k+2)(k+3)\right) b_k &= -a_+ - \frac{b_+}{2} k(k+1) \\ -\left(a_- - \frac{b_-}{2}(k+1)(k+2)\right) a_k + \left(a_- - \frac{b_-}{2}(k+2)(k+3)\right) b_k &= -a_- + \frac{b_-}{2} k(k+1) \end{aligned} \tag{3.2.3}$$

This system of equations can be solved easily.

First determinant of the coefficient matrix is

$$det_k = 2a_+ a_- + a_- b_+ (k+2)^2 - a_+ b_- (k+2)^2 - b_- b_+ (k+1)(k+2)^2 (k+3)/2$$

$$\begin{aligned} a_k &= \frac{(2k+3)(a_+ b_- + a_- b_+)}{det_k} \\ b_k &= \frac{-2a_- a_+ + (k+1)^2 (a_+ b_- - a_- b_+) + \frac{b_- b_+}{2} k(k+1)^2 (k+2)}{det_k} \end{aligned}$$

if $a_\pm = 1$ and $b_\pm = 0$ (Dirichlet boundary conditions), we have $a_k = 0$ and $b_k = -1$

if $a_\pm = 0$ and $b_\pm = \pm 1$ (Neumann boundary conditions), we have $a_k = 0$ and $b_k = \frac{-k(k+1)}{(k+2)(k+3)}$

It can be seen that $\{\phi_k\}$ are linearly independent. So,

$$X_N = span\{\phi_k : k = 0, 1, ..., N-2\}$$

Remark. In the very special case for example

$$-u_{xx} = f, \quad x \in (-1, 1); \quad u_x(\pm 1) = 0$$

with the condition $\int_{-1}^{1} f dx = 0$ (this is known as Hadamard stability condition). Since the solution is only determined up to a constant,

we should use

$$X_N = span\{\phi_k : k = 0, 1, ..., N-2\}$$

Lemma 3.2.1. The stiffness matrix S is diagonal matrix with

$$s_k k = -(4k+6)b_k, \quad k = 0, 1, ...$$

The mass matrix M is symmetric penta-diagonal whose nonzero elements are

$$m_{jk} = m_{kj} = \begin{cases} \dfrac{2}{2k+1} + a_k^2 \dfrac{2}{2k+3} + b_k^2 \dfrac{2}{2k+5} , & j = k, \\ a_k \dfrac{2}{2k+3} + a_{k+1} b_k \dfrac{2}{2k+5} , & j = k+1, \\ b_k \dfrac{2}{2k+5} , & j = k+2. \end{cases}$$

Proof. Integrate by parts and using the fact that ϕ_k satisfy boundary conditions , we have

$$\begin{aligned} s_{jk} &= -\int_{-1}^{1} \phi_k''(x)\phi_j(x)dx \\ &= -\phi_k'(x)\phi_j'(x)\Big|_{-1}^{1} + \int_{-1}^{1} \phi_k'(x)\phi_j'(x)dx \\ &= -\int_{-1}^{1} \phi_k(x)\phi_j''(x)dx = s_{kj} \end{aligned}$$

For Noumman boundary condition , we have $a_k = 0$, $b_k = \dfrac{-k(k+1)}{(k+2)(k+3)}$

$$\phi_k(x) = L_k(x) - \frac{k(k+1)}{(k+2)(k+3)} L_{k+2}(x)$$

from above and the definition of $\{\phi_k\}$ that S is a diagonal matrix.

$$\begin{aligned} L_{k+2}''(x) &= (k+\frac{1}{2})\big((k+2)(k+3) - k(k+1)\big) L_k(x) \\ &= (k+\frac{1}{2})(4k+6)L_k(x) \end{aligned}$$

$$\begin{aligned} s_{kk} &= \int_{-1}^{1} \big(L_k(x) - b_k L_{k+2}(x)\big)\big(L_k''(x) - b_k L_{k+2}''(x)\big)\ dx \\ &= \int_{-1}^{1} \big(L_k''(x)L_k(x) - b_k L_{k+2}''(x)L_k(x) - b_k L_k''(x)L_{k+2}(x) + b_k^2 L_{k+2}''(x)L_k(x)\big)\ dx \\ &= -b_k \int_{-1}^{1} L_{k+2}''(x)L_k(x)\ dx \\ &= -b_k \int_{-1}^{1} (k+\frac{1}{2})(4k+6)L_k(x)L_k(x)\ dx \\ &= -b_k(k+\frac{1}{2})(4k+6)\int_{-1}^{1} L_k^2(x)\ dx \\ &= -b_k(4k+6) \qquad\qquad \textit{where} \int_{-1}^{1} L_k^2(x) = \frac{1}{k+\frac{1}{2}}dx \end{aligned}$$

□

3.3 Example in 1-D BVP

Consider the following BVP:

$$\frac{d^2y(x)}{dx^2} - y(x) = x, \tag{3.3.1}$$

$$y(-1) = 0,\ y(1) = 0 \tag{3.3.2}$$

We use Galerkin method to approximate the solution.

Now assume that the solution is in the form

$$y_N(x) = \sum_{i=0}^{n} a_k \phi_k(x) \tag{3.3.3}$$

where $\phi_k(x) = L_{k+2}(x) - L_k(x)$.

for n=4 we have

$$y_N(x) = \sum_{i=0}^{4} a_k \phi_k(x) \tag{3.3.4}$$

now we substitute (3.3.4) in (3.3.1)

$$eq_i := \int_{-1}^{1} \left(\frac{d^2y_N(x)}{dx^2} - y_N(x) - x \right) \phi_i(x)\ dx, \quad i = 0, \cdots, 4. \tag{3.3.5}$$

Using matlab we get

$$\begin{aligned}
eq_0 &:= \frac{2}{5}a_3 - \frac{42}{5}a_1 \\
eq_1 &:= \frac{2}{7}a_4 - \frac{230}{21}a_2 + \frac{2}{3} \\
eq_2 &:= \frac{2}{5}a_1 - \frac{658}{45}a_3 + \frac{2}{9}a_5 \\
eq_3 &:= \frac{2}{7}a_2 - \frac{1422}{77}a_4 \\
eq_4 &:= \frac{2}{9}a_3 - \frac{2618}{117}a_5
\end{aligned}$$

Solving the linear system for $a_i,\ i = 0, \cdots 4$, we have

$$a_1 = 0, a_2 = 0.06089414, a_3 = 0, a_4 = 9.42103 \times 10^{-4}$$

Therefore

$$y_N(x) = 0.00741906\ x^5 + 0.14163669\ x^3 - 0.14905576\ x$$

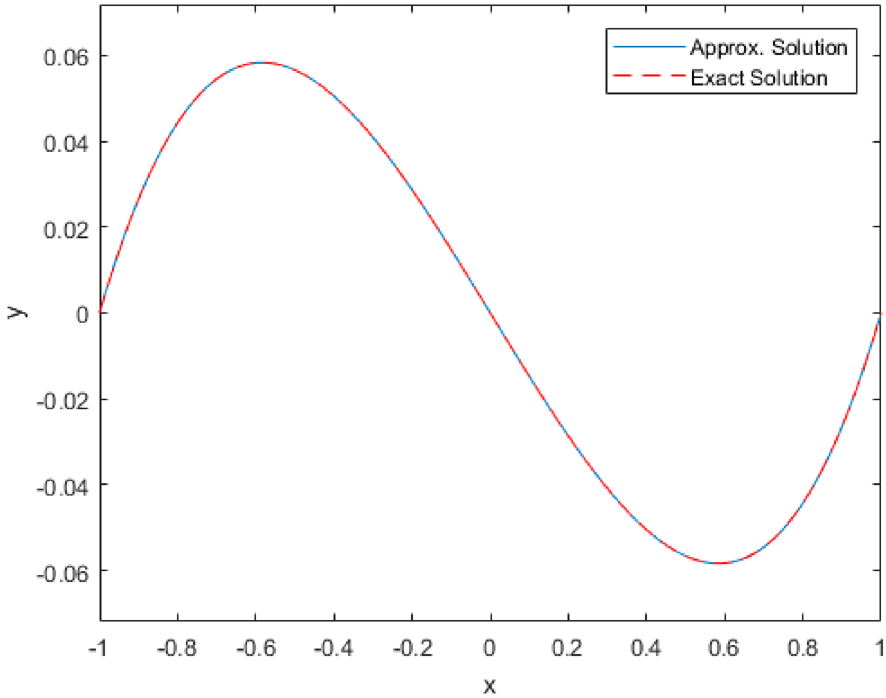


Figure 3.1: Approximated and exact solution

3.4 Electroosmotic and Pressure Driven Flow In An Eccentric Micro-Annulus

3.4.1 Introduction

Due to many applications of microfluidic devices applications in microelectromechanical systems and microbiological sensors such as laboratory on a chip.Microfluidic device have become important.One of the method to transport the fluid through microtubes or any other fluid conduit without mechanical moving part is to operate electro-osmosis (EO).The principle for electro-osmosis is as follows.Generally solid surfaces carry a negative electrostatic charge when in contact with a fluid containing dissociated salts.At the same time, the fluid acquires a positive charge near the boundary.The charged fluid can then be moved by an applied axial electric field. This has been discovered more than two century ago [5].

Because of the difficulty associated with the commercialization of electroosmosis was that it requires small length scales to take effect, it took long time to use the for electroosmosis to be used widely in practice. But after the development in microfabrication technology which lead to the invention of many different microfluidic device, considerable progresses achieved in this subject.

The concept of Electric Double Layer (EDL) was introduced by Helmholtz [6], who realized that, if a charged metal surface is immersed into an electrolyte can attract counter-ions towards the surface and repel co-ions away.Smoluchowski [7] to describe EO velocity, this is known as Helmholz- Smoluchowski velocity. Later, Debye and Hückel [8] studied on the ionic distribution in solutions of low ionic energy, where they used linearization for the Boltzmann distribution. This simplification lead the way for the development of analytical solutions for electroosmotic flow in slits and capillary tubes by Burgreen and Nakache [9]. They afterward extended their solutions to account for high surface potentials as well in [10]. Rice and Whitehead [11] investigated the fully developed electroosmotic flow in a narrow cylindrical capillary for low zeta potentials, using the Debye-Hückel linearization. Levine et al. [12] extended Rice and Whitehead's work to high zeta potentials in terms of a numerical approximation.

Recently, electroosmotic flow through an annular, Tsao [13] used DebyeHückel linearization and obtained the analytical solution. Later Huang et al [14] extended Tsao solution for high-zeta potential where he used Green function method. More recently, Sadeghi et al [15] consider the Electroosmotic Flow in Hydrophobic Microchannels of General Cross Section and they find that the flow rate is a linear increasing function of the slip length with thinner electric double layers (EDLs) providing higher slip effects. They also explored that, unlike the no-slip conditions, there is not a limit for the electroosmotic velocity when EDL extent is reduced. But There are also many work on this subject with different geometry which can be found [15][16]. We will show that this method is not suitable for our geometry.

Regarding to eccentric flow geometry, there are many studies, in this geometry, first analytical study belongs to Synder and Goldstain [17], they consider fully developed flow of Newtonian fluid in a an eccentric annulus and derive the analytical solution, later Debnath et al [18] consider the Hydromagnetic Flow between Two Rotating Eccentric Cylinders and obtain approximate solution by using the perturbation parameter. More recently , Alassar [19] investigate the slip flow case for above geometry, he obtained that for a fixed aspect ratio, fully eccentric channels sustain the maximum average velocity (flow rate) under the same pressure gradient and slip conditions.

In this work, we consider the electroosmatic and pressure driven flow of Newtonian fluid in an eccentric microannulus with DebyeHückel approximation which has not been considered before that gives us enough motivation for the work under consideration, where we first discuss the analytical solution linearized Poisson-Boltzmann equation and show that this is not possible for our boundary condition .We then use Fourier Legendre pseudo spectral method for above differential equation which is higher accurate, stable and obtain approximate analytical solution . Finally analytical solution derived for our flow problem. In section 2, we formulate the flow problem. Then we show that analytical solution is not possible of linearized Poisson-Boltzmann equation because of our boundary conditions, in section 3 and 4 Section devoted to the Fourier Legendre pseudo spectral method and analytical solution of flow problem. Results and discussions are given in the final section.

We mention that since analytical or approximate analytic solutions for EO flows are rare and useful because they not only provide structured solutions to the problem under investigation but also serve as an accuracy standard for approximate (numerical) methods, therefore our exact solution can be used as benchmark case for further studies in this subject.

3.4.2 Problem Formulation

Assuming that the electrolyte is a Newtonian fluid, and the flow is driven by both electroosmosis and pressure forces or one of them which fluid motion is governed by the Navier-Stokes (N-S) equation

$$\rho\left(\frac{\partial \mathbf{u}^*}{\partial t^*}+\mathbf{u}^*.\nabla^*\mathbf{u}^*\right)=-\nabla p^*+\mu\nabla^{*2}\mathbf{u}^*+\rho_e E^*\left(t\right), \tag{3.4.1}$$

where ρ is the density, $\mathbf{u}^*$ is the fluid velocity, p is the pressure, t^* is the time, ρ_e is the charge of electrolyte, and $E^*(t)=(0,0,E_0^*(t^*))$ is the externally applied longitudinal electric field. Note that "*" indicates that parameters are in a dimensional form. Hence, without star notation parameters and variables is in non dimensionless form. The flow in this study is steady and parallel to the longitudinal direction (z-axis),hence Eq.(3.4.1) with $u^*=(0,0,w^*)$ is reduced to

$$\mu\nabla^{*2}w^*-\frac{dp^*}{dz^*}+\rho_e E_0^*=0. \tag{3.4.2}$$

As shown in Fig.3.2 , annular microchannel between two eccentric circular cylinders of radii $\delta\bar{R}$ and $\bar{R}$ is considered in which an electric field is applied along the length of channel. We assume that inner and outer walls uniformly charged with zeta potential ζ_1 and ζ_0 respectively. The relation of EDL potential ψ^* and electric charge density in a symmetric electrolyte is expressed with the Poisson-Boltzmann equation,

$$\rho_e = -\varepsilon\nabla^{*2}\psi^* = -2zen_0 \sinh\left(\frac{ze\psi^*}{k_B T}\right), \tag{3.4.3}$$

where $\psi^*, z, e, n_0, \varepsilon, k_B$ and T are the electric potential, the valence, the electron charge, the bulk ion concentration, the electric permittivity of the electrolyte, the Boltzmann constant and the reference absolute temperature respectively. The above equation in two-dimensional cylindrical co-ordinates can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi^*}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi^*}{\partial\theta^2} = \frac{2zen_0}{\varepsilon}\sinh\left(\frac{ze\psi^*}{k_B T}\right), \tag{3.4.4}$$

the boundary conditions are

$$\psi^*\left(\delta\bar{R},\ \theta\right) = \zeta_1, \quad \psi^*\left(\bar{R},\ \theta\right) = \zeta_0 \text{ and } \frac{\partial\psi^*}{\partial\theta} = 0,\ at\ \theta = 0,\ \pi, \tag{3.4.5}$$

using the following dimensionless form as

$$R = \frac{r}{\bar{R}} \text{ and } \psi = \frac{ze}{k_B T}\psi^* \tag{3.4.6}$$

then we can rewrite (3.4.4) and (3.4.5) in dimensionless form as

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2} = K^2\sinh(\psi) \tag{3.4.7}$$

where $K = \kappa R$ is the length scale ratio (electrokinetic radius) and κ is the Debye-Hückel parameter which is defined as

$$\kappa = \left(\frac{2z^2e^2n_0}{\varepsilon k_B T}\right)^{1/2}, \tag{3.4.8}$$

where $\kappa = 1/\lambda_D$ and λ_D is called as Debye length.Under Debye- Hückel approximation the Eq.(3.4.7) is reduce to more simpler form

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2} = K^2\psi \tag{3.4.9}$$

and dimensionless boundary conditions

$$\psi(\delta,\theta) = Z_1, \quad \psi(1,\theta) = Z_0, \text{ and } \frac{\partial\psi}{\partial\theta} = 0, \text{ at } \theta = 0,\pi \tag{3.4.10}$$

We can use the method eigen function expansion like in [20] and [21], however end up with system of equation which is singular, Also, the method used in [22] cannot be used for our geometry, because, radius are fixed (constants) which give us singular system. Because of this problems, many numerical method derive the solve Helmholtz equation in irregular region, for example in [23], Gass used finite element method and more recently, Green function method is used in [24]

3.4.3 Fourier Legendre pseudo spectral method for Debye-Hückel approximation

The geometry of our flow problem shown in Fig.3.2 which is not suitable to use of Spectral method, in order to use the spectral method, in order to use the spectral method, we use bipolar coordinates system (η, φ, z) in Fig.3.2, the relation between the rectangular and bipolar coordinates is shown to be

$$x = \frac{C\sinh(\eta)}{\cosh(\eta) - \cos(\varphi)}, y = \frac{C\sin(\varphi)}{\cosh(\eta) - \cos(\varphi)} \text{ and } z = z, \tag{3.4.11}$$

Where, if we specified the inner and outer radii of eccentric cylinders by $\delta\bar{R}$, $\bar{R}$ and the eccentricity e, then the C is focal distance is given by

$$C = \frac{\sqrt{(e - \delta\bar{R} - \bar{R})(e - \delta\bar{R} + \bar{R})(e + \delta\bar{R} - \bar{R})(e + \delta\bar{R} + \bar{R})}}{2e}. \tag{3.4.12}$$

The surfaces of the inner and the outer cylinders are identified by $\eta = \eta_0$ and $\eta = \eta_1$, where $\eta_0 = \sinh^{-1}(C/r_0)$ and $\eta_1 = \sinh^{-1}(C/r_1)$ respectively. Hence, given the radius of each of the two cylinders ($\delta\bar{R}$ and $\bar{R}$) and the center-to-center distance (e), one can fix a particular bipolar coordinates system (i.e.η_0, η_1 and C are obtained uniquely). Equation (3.4.4) can be written in bipolar coordinates as

$$\frac{\partial^2\psi^*}{\partial\eta^2} + \frac{\partial^2\psi^*}{\partial\varphi^2} = g^* K^2 \psi^*, g^* = \frac{C^2}{(\cosh(\eta) - \cos(\varphi))^2}, \tag{3.4.13}$$

and boundary conditions

$$\psi^*(\eta_1, \varphi) = \zeta_1, \psi^*(\eta_2, \varphi) = \zeta_0 \text{ and } \frac{\partial\psi^*}{\partial\varphi} = 0, \text{ at } \varphi = 0, \pi. \tag{3.4.14}$$

Let $\psi = \frac{\psi^*}{\psi_0}$, then equation can put into dimensionless form

$$\frac{\partial^2\psi}{\partial\eta^2} + \frac{\partial^2\psi}{\partial\varphi^2} = gK^2\psi, g = \frac{1}{(\cosh(\eta) - \cos(\varphi))^2}, \tag{3.4.15}$$

and boundary condition becomes

$$\psi(\eta_1,\varphi)=\zeta_1, \psi(\eta_2,\varphi)=\zeta_0 \text{ and } \frac{\partial\psi}{\partial\varphi}=0, \text{ at } \varphi=0,\pi \tag{3.4.16}$$

In order to make boundary conditions homogeneous, we change variable as $\psi(\eta,\varphi)=\frac{-\eta_1 Z_0+Z_1\eta_2}{\eta_1-\eta_2}+\frac{Z_1-Z_0}{\eta_1-\eta_2}\eta+\psi_1(\eta,\varphi)$, and we transform the region by $x=\frac{2\eta-(\eta_2+\eta_1)}{\eta_1-\eta_2}$ from $[\eta_1,\eta_2]$ to $[-1,1]$ and substituting this into Eq.(3.4.15), we have

$$\begin{aligned}&\frac{4}{(\eta_1-\eta_2)^2}\frac{\partial^2\psi_1}{\partial x^2}+\frac{\partial^2\psi_1}{\partial\varphi^2}=\\ &g\kappa^2\left[\frac{-\eta_1 Z_0+Z_1\eta_2}{\eta_1-\eta_2}+\frac{Z_1-Z_0}{\eta_1-\eta_2}\left[\frac{(\eta_1-\eta_2)x+(\eta_1+\eta_2)}{2}\right]+\psi_1(x,\varphi)\right],\end{aligned} \tag{3.4.17}$$

and boundary conditions became

$$\psi_1(-1,\varphi)=0, \psi_1(1,\varphi)=0 \text{ and } \frac{\partial\psi_1}{\partial\varphi}=0, \text{ at } \varphi=0,\pi. \tag{3.4.18}$$

For the Galerkin spectral method, we define set

$$S_M^n=\text{span}\{\varphi_i(\rho)\omega_j(\theta): i,j=0,1,\cdots M\} \text{ and } V_M^2=\{v\in S_M^2\} \tag{3.4.19}$$

where $\varphi_i(\rho)=L_{i+2}(\rho)-L_i(\rho)$, is the difference of two Lagendre polynomial) and $\omega_j(\theta)=cos(jn\theta), j=0..M$, then classical Fourier-Legendre Galerkin method is: Find $\hat{\psi}^M\in V_M^2$ such that $\forall v\in V_M^2$

$$\begin{aligned}&\int_\Omega \hat{\psi}_x^{\,M} v_x\frac{4}{(\eta_1-\eta_2)^2}\,dxd\varphi\\ &\quad+\int_\Omega \hat{\psi}_\varphi^{\,M} v_\varphi\,dxd\varphi\\ &\quad+\int_\Omega \hat{\psi}^M K^2\left[\frac{-\eta_1 Z_0+Z_1\eta_2}{\eta_1-\eta_2}+\frac{Z_1-Z_0}{\eta_1-\eta_2}\left[\frac{(\eta_1-\eta_2)x+(\eta_1+\eta_2)}{2}\right]+v\right]\\ &\qquad\frac{1}{(\cosh(\eta)-\cos(\varphi))^2}\,dxd\varphi=0\end{aligned} \tag{3.4.20}$$

Let us denote

$$\hat{\psi}^M=\sum_{i,j=0}^{M-2,M}a_{i,j}\varphi(\rho)\omega_j(\theta) \tag{3.4.21}$$

Taking $v=\varphi_i(\rho)\omega_j(\theta)$ in (3.4.20) for $i,j=0,1,2\cdots,M-2,M$, since the integrand in (3.4.20) cannot be integrated analytically, hence, we used in the last integration Gauss quadrature method obtain the integration numerically (the results analysed carefully until we get the difference two consecutive numerical integration less than 10^{-10} and we stop) and the integrals evaluated exactly, in this manner, we obtain system of equations which we solved numerically. Note that we increase the number of the base elements until we get the difference two consecutive approximation for Eq.(3.4.21) less than 10^{-7}. In figure (2) and (3), we show the effect of Debye length on the electric potential, we note here that this has not been before.

3.4.4 Velocity Field

The momentum conservation equation in bipolar cylindrical coordinates reads

$$\mu\left(\frac{\partial^2 w^*}{\partial\eta^2}+\frac{\partial^2 w^*}{\partial\varphi^2}\right)=\frac{C^2}{(\cosh(\eta)-\cos(\varphi))^2}\frac{dp^*}{dz^*}+\varepsilon E_0^*\left(\frac{\partial^2\psi^*}{\partial\eta^2}+\frac{\partial^2\psi^*}{\partial\varphi^2}\right), \qquad (3.4.22)$$

introducing the following dimensionless parameters

$$w=-\frac{w^*\mu}{E_0^*\varepsilon\psi_0},\ P=\frac{Cp^*}{E_0^*\varepsilon\psi_0},\ Z=\frac{z}{C}\text{ and }\Re=w+\psi \qquad (3.4.23)$$

then (3.4.22) can be written as a new operator

$$\begin{aligned}\frac{\partial^2\Re}{\partial\eta^2}+\frac{\partial^2\Re}{\partial\varphi^2}&=-\frac{1}{(\cosh(\eta)-\cos(\varphi))^2}\frac{dP}{dZ} &(3.4.24)\\ \frac{\partial\Re}{\partial\varphi}&=0\text{ at }\varphi=0\text{ and }\pi &(3.4.25)\\ \Re=Z_1\text{ at }\eta&=\eta_1\text{ and }\Re=Z_0\text{ at }\eta=\eta_2. &(3.4.26)\end{aligned}$$

A particular solution of Eq.(3.4.24) is the $\dfrac{-\cosh(\eta)\frac{dP}{dZ}}{2(\cosh(\eta)-\cos(\varphi))}$.Then the general solution of equation (3.4.24) given by

$$\begin{aligned}\Re(\eta,\varphi)&=-\frac{\cosh(\eta)\dfrac{dP}{dZ}}{2(\cosh(\eta)-\cos(\varphi))}+a+b\eta\\ &+\sum_{n=1}^{\infty}\left[A_n\sinh(n(\eta-\eta_1))+B_n\sinh(n(\eta-\eta_1))\right]\cos(n\varphi),\end{aligned} \qquad (3.4.27)$$

where $a=\frac{\eta_1 Z_0-\eta_2 Z_1}{\eta_1-\eta_2}$, $b=\frac{Z_0-Z_1}{\eta_1-\eta_2}$ which make boundary conditions to be homogeneous. The other coefficient easily obtained if someone use the orthogonality of $\cos(n\varphi)$. Therefore solution of momentum equation is given by

$$\begin{aligned}w(\eta,\varphi)&=-\frac{\cosh(\eta)\dfrac{dP}{dZ}}{2(\cosh(\eta)-\cos(\varphi))}-\psi+a+b\eta\\ &+\sum_{n=1}^{\infty}\left[A_n\sinh(n(\eta-\eta_1))+B_n\sinh(n(\eta-\eta_1))\right]\cos(n\varphi).\end{aligned} \qquad (3.4.28)$$

3.4.5 Results and Discussion

In the limiting case, if $\frac{dP}{dZ} = 0$,we note that our solution reduce to the electroosmatic flow and if $\psi = 0$ our solution reduce flow generated by constant pressure gradient case. However, In the case of high zeta function Debye-Hückel approximation is not valid and we need to solve the semi-linear Poisson-Boltzmann equation, in this case we cannot use the Galerkin spectral method, we mean approximate analytical solution may not possible, but, we can use the finite difference method as follows: First, due to strong gradients of the electrical potential near the wall, it is necessary to have smaller grid sizes in this region. Therefore, a transformation is used to cluster the grid points near the wall where more information about EDL and velocity field is required [25]. The η and φ coordinates are transformed into η^* and φ^* in bipolar coordinates:

$$\eta^* = \frac{\ln\left(\frac{\beta+\eta}{\beta-\eta}\right)}{\ln\left(\frac{\beta+1}{\beta-1}\right)}, \varphi^* = \frac{ln\left(\frac{\beta+\varphi}{\beta-\varphi}\right)}{ln\left(\frac{\beta+1}{\beta-1}\right)}, \tag{3.4.29}$$

where β is the stretching parameter that controls the degree of clustering. With this transformation, Eq.(3.4.7) in bipolar coordinates can be rewritten in terms of η^* and φ^* as

$$\begin{aligned} C_2(\eta^*)\frac{\partial\psi}{\partial\eta^*} + C_1^2(\eta^*)\frac{\partial^2\psi}{\partial\eta^{*2}} + C_2(\varphi^*)\frac{\partial\psi}{\partial\varphi^*} + C_1^2(\varphi^*)\frac{\partial^2\psi}{\partial\varphi^{*2}} \\ = \frac{K^2}{(\cosh\eta^* - \cos\varphi^*)^2}\sinh(\psi), \end{aligned} \tag{3.4.30}$$

where

$$\begin{aligned} C_1(\eta^*) &= \frac{e^{\Sigma\eta^*} + e^{-\Sigma\eta^*} + 2}{2\beta\Sigma}, \; C_2(\eta^*) = \frac{e^{2\Sigma\eta^*} + 2e^{\Sigma\eta^*} - 2e^{-2\Sigma\eta^*} - e^{-2\Sigma\eta^*}}{2\beta\Sigma} \\ C_1(\varphi^*) &= \frac{e^{\Sigma\varphi^*} + e^{-\Sigma\varphi^*} + 2}{2\beta\Sigma}, \; C_2(\varphi^*) = \frac{e^{2\Sigma\varphi^*} + 2e^{\Sigma\varphi^*} - 2e^{-2\Sigma\varphi^*} - e^{-2\Sigma\varphi^*}}{2\beta\Sigma} \end{aligned} \tag{3.4.31}$$

and Σ is defined as $\Sigma = \ln((\beta+1)/(\beta-1))$.Then, Eq.(3.4.30) is numerically solved by means of implicit finite difference method. Applying the central difference scheme, the difference equations for the inner points are obtained as

$$A(\psi_{i,j}, \eta^*, \varphi^*)\psi_{i,j} = A_1\psi_{i-1,j} + A_2\psi_{i+1,j} + A_3\psi_{i,j-1} + A_4\psi_{i,j+1} \tag{3.4.32}$$

where

$$
\begin{aligned}
A(\psi_{i,j},\eta^*,\varphi^*) &= 2C_1^2(\eta^*)\frac{\Delta\varphi^*}{\Delta\eta^*}+2C_1^2(\varphi^*)\frac{\Delta\eta^*}{\Delta\varphi^*}+\left(\frac{\sinh(\psi_{i,j})}{\psi_{i,j}}\right)^{prev}\\
&\qquad \frac{K^2}{(\cosh\eta_i^*-\cos\varphi_j^*)^2}\Delta\eta^*\Delta\varphi^*,\\
A_1 =& -\frac{1}{2}C_2(\eta^*)\Delta\varphi^*+C_1^2(\eta^*)\frac{\Delta\varphi^*}{\Delta\eta^*}, A_2=\frac{1}{2}C_2(\eta^*)\Delta\varphi^*+C_1^2(\eta^*)\frac{\Delta\varphi^*}{\Delta\eta^*},\\
A_3 =& -\frac{1}{2}C_2(\varphi^*)\Delta\eta^*+C_1^2(\varphi^*)\frac{\Delta\eta^*}{\Delta\varphi^*}, A_4=\frac{1}{2}C_2(\varphi^*)\Delta\eta^*+C_1^2(\varphi^*)\frac{\Delta\eta^*}{\Delta\varphi^*},
\end{aligned}
\tag{3.4.33}
$$

where superscripts *prev* refers to previous iteration results, the first iteration guess values provided. Then, we used SOR method to solve the Eq.(3.4.33), we iterate the solution until the required overall error. The procedure continues until the required overall relative error of 10^{-7} is achieved.

In this report, we first discuss the limitation of Debye-Hückel approximation(DHA),this is done in Figure 3.3-a-b,3.4-a-b and 3.5a-b where the first figures are represent the semi analytical solution for Debye-Hückel approximation and second figures represent the finite difference solution of semi linear poisson-Boltzmann equation , we see that relatively small for electrokinetic width and small zeta number, we can use the Debye-Hückel approximation,also, this approximation does not depend on eccentricity, this is given 3.6-a-b, 3.7-a-b and 3.8-a-b.So we can use Debye-Hückel approximation under above restriction, then we have approximate analytical solution as shown in Equation (3.4.21).We also note that higher value of electrokinetic radius for distribution of electric potential is lower bigger side of eccentric region, the lowest value of electric potential located in the canter of bigger side of eccentric region (Fig. 3.5 and 3.8). We also observed distribution of electric potential more uniform for lower value of electrokinetic radius. We now discuss the properties of the velocity field.Fig. 3.9-3.10-3.11-a-b show the effect of pressure gradient and electrokinetic radius and eccentricity on the distribution of velocity field. We first state that the value of velocity profiles in bigger side of eccentric region always bigger than smaller side of eccentric region. In order to see the effect of above parameters on the velocity distributions, same value of contour line are chosen. It is interesting to see that there is like plateau on the graph and the size of plateau depend on the parameters.

we also observed that large enough electrokinetic radius or pressure gradient, there is stagnation region where there is no flow (See Fig. 3.11-a-b) also size of this region depend on above parameters. This is also observed for other studies in this field.

The average value of velocity also important which can be calculated by following formula

$$W_m = \frac{\int\limits_0^{2\pi}\int\limits_{\eta_1}^{\eta_2} \frac{w(\eta,\varphi)}{(\cosh(\eta)-\cos(\varphi))^2} d\eta d\varphi}{\int\limits_0^{2\pi}\int\limits_{\eta_1}^{\eta_2} \frac{1}{(\cosh(\eta)-\cos(\varphi))^2} d\eta d\varphi} \tag{3.4.34}$$

The effect of the eccentricity on the average value of distribution is given in Fig.3.12 for fixed value of pressure gradient and electrokinetic radius, we see

3.4.6 Conclusions

In this study, consideration is given to the electroosmotic and pressure driven flow of Newtonian fluids in an eccentric microannulus.We used Fourier Legendre pseudo spectral method to obtain new higher accurate approximate analytical solution of linearized Poisson-Boltzmann in bipolar coordinates, we also used finite difference method to solve full semilinear Poisson-Boltzmann numerically and we show that our approximate analytical solution is valid for small zeta number, then we solved the governing momentum equation analytically and finally we obtain approximate analytical solution our flow problem which has not been given before. In a follow up our report, we shall work on the electroosmotic and pressure driven flow of non-Newtonian fluids in an eccentric microannulus.

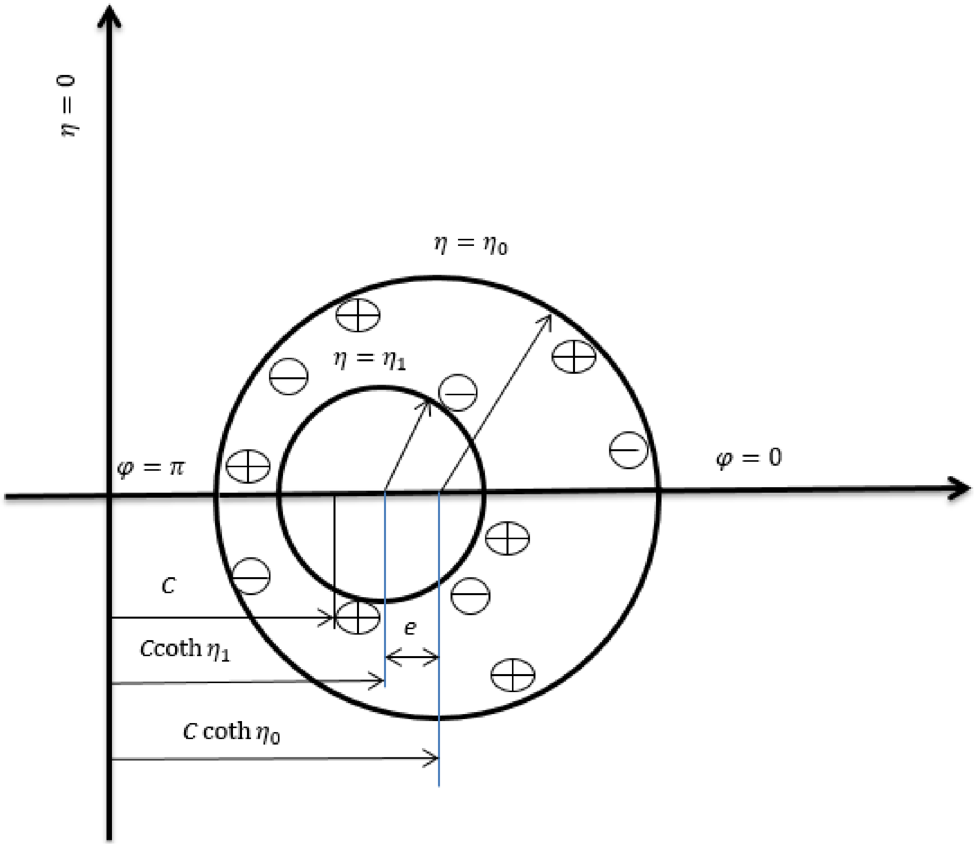
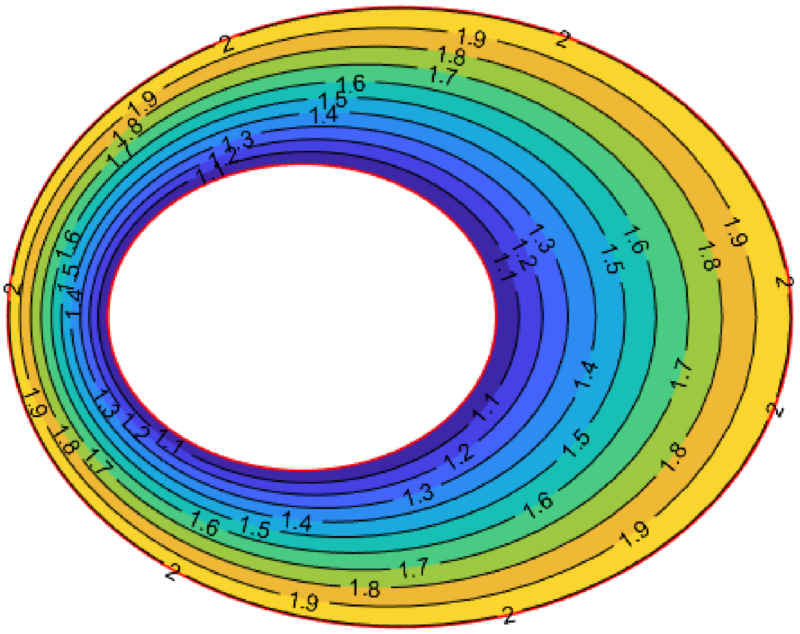
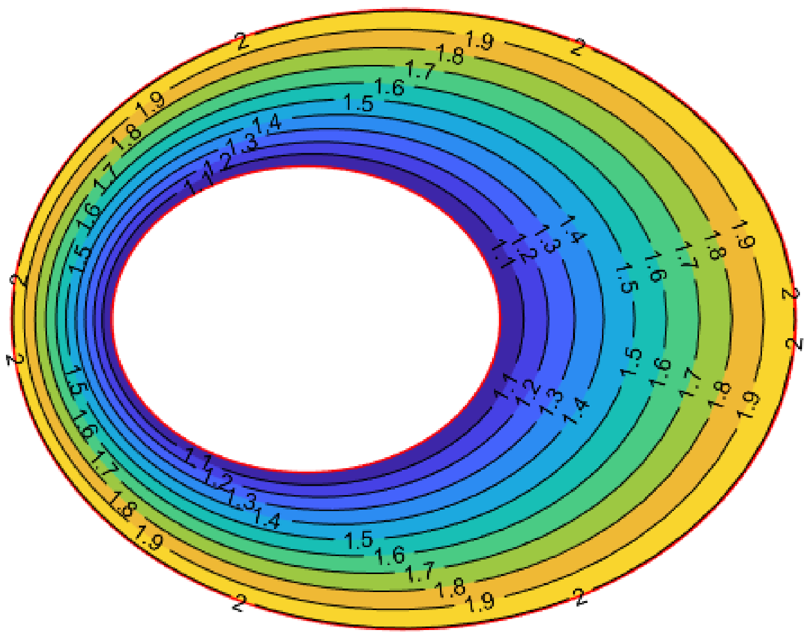


Figure 3.2: Schematic of the physical problem along with the coordinate system

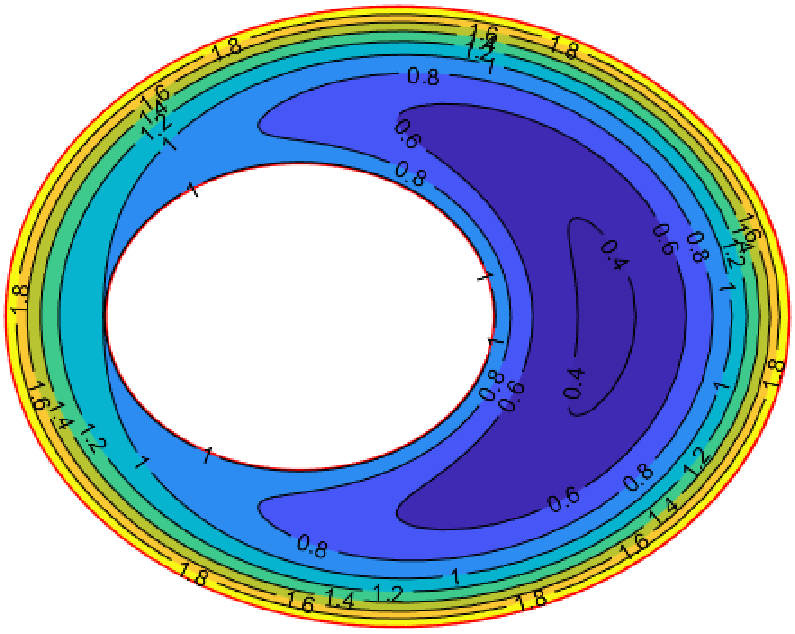


(a) For DHA

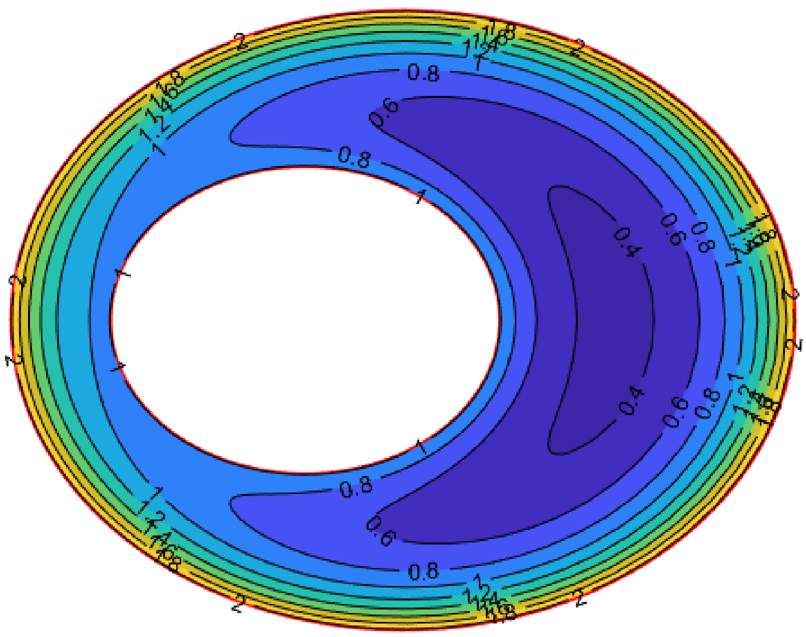


(b) For FPBE

Figure 3.3: The electric potential distribution $\lambda = 1$, e=1/2

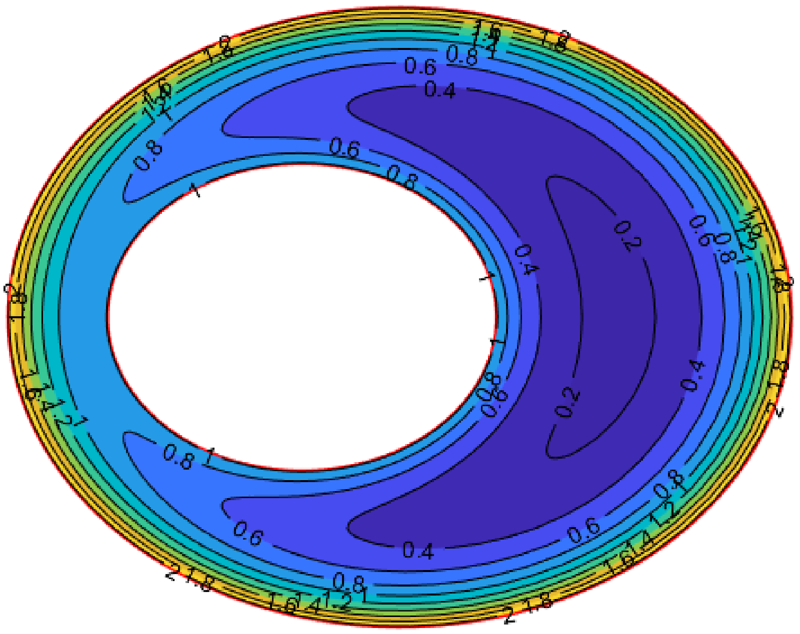


(a) For DHA

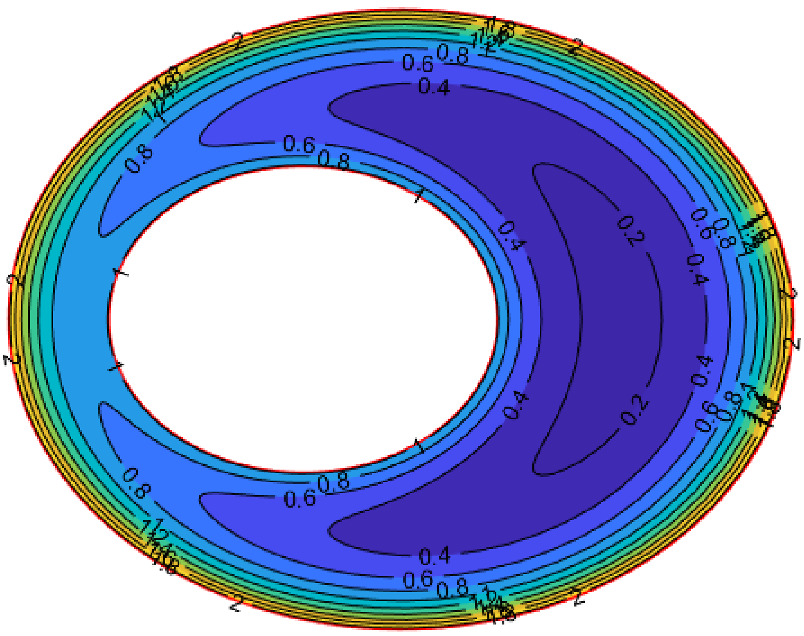


(b) For FPBE

Figure 3.4: The electric potential distribution $\lambda = \sqrt{50}$,e=1/2

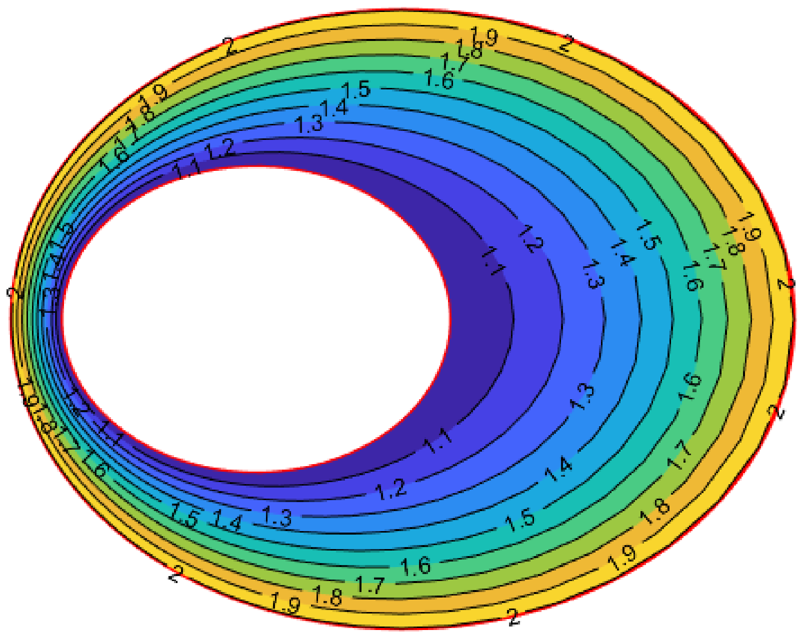


(a) For DHA

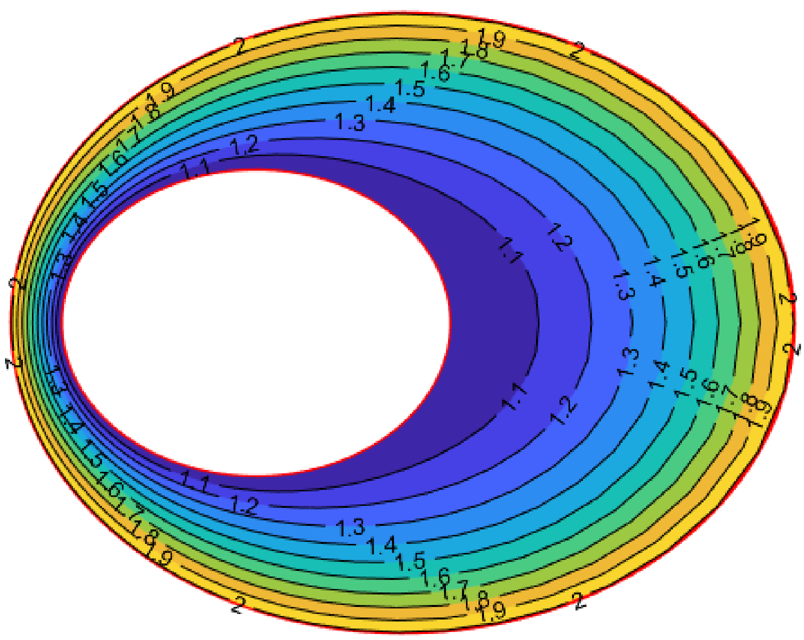


(b) For FPBE

Figure 3.5: The electric potential distribution $\lambda = 10$,e=1/2

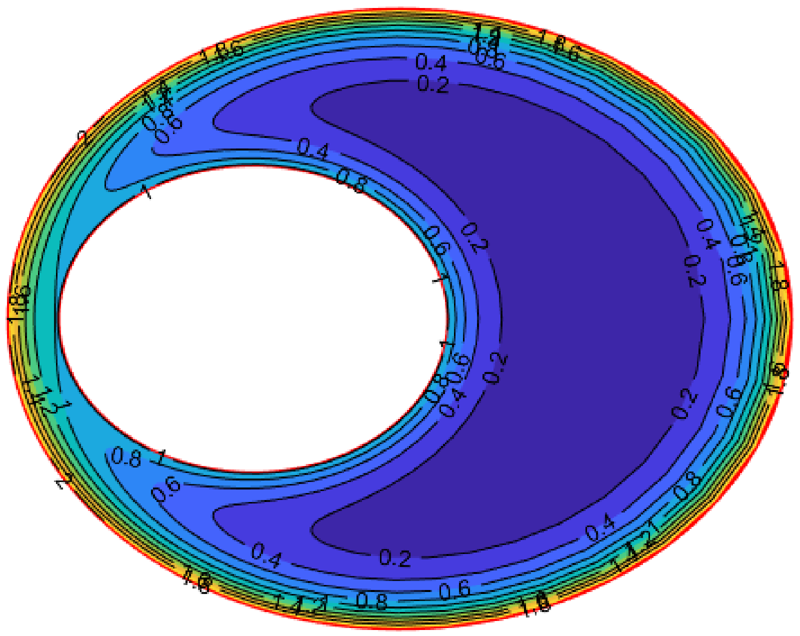


(a) For DHA

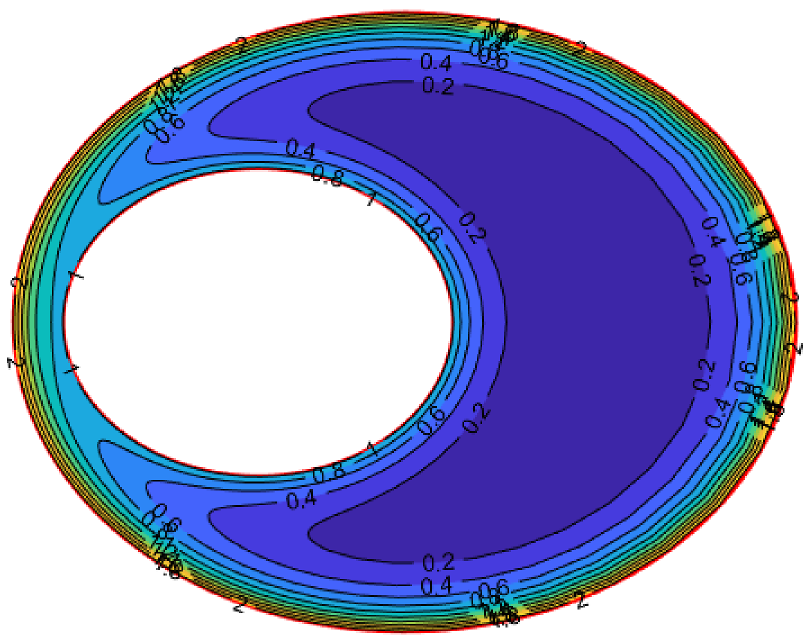


(b) For FPBE

Figure 3.6: The electric potential distribution $\lambda = 1$,e=3/4

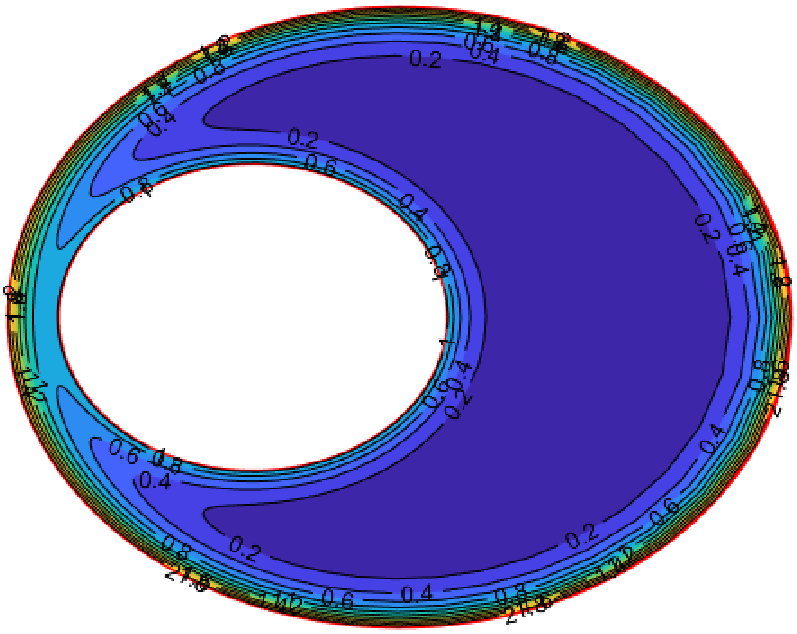


(a) For DHA

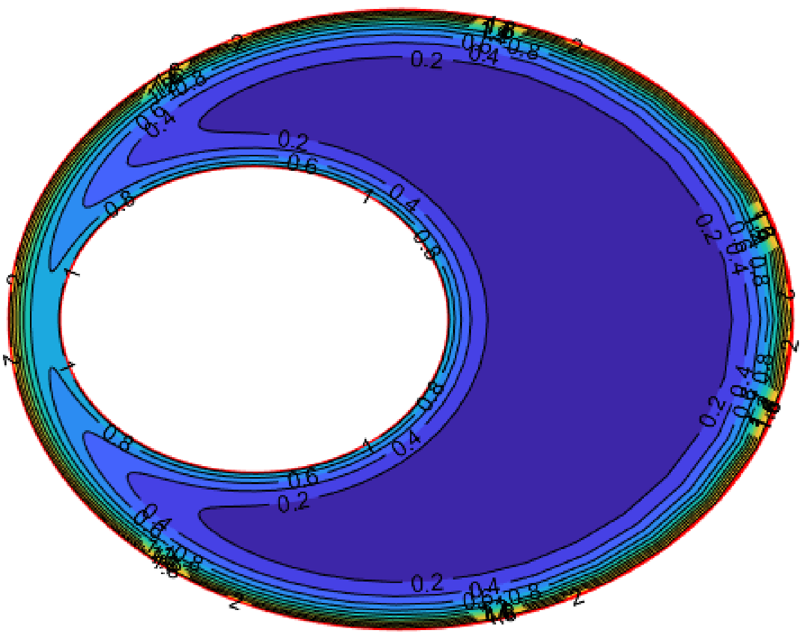


(b) For FPBE

Figure 3.7: The electric potential distribution $\lambda = \sqrt{50}$,e=3/4

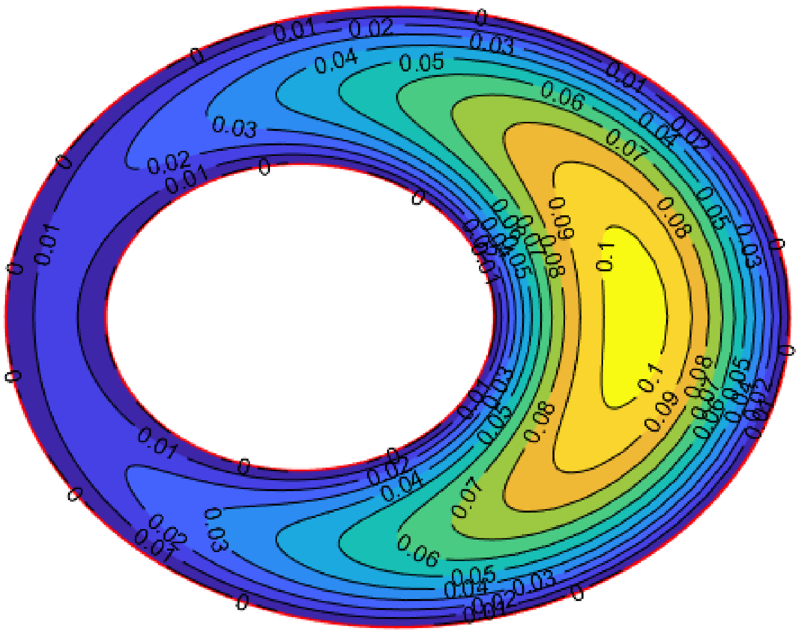


(a) For DHA

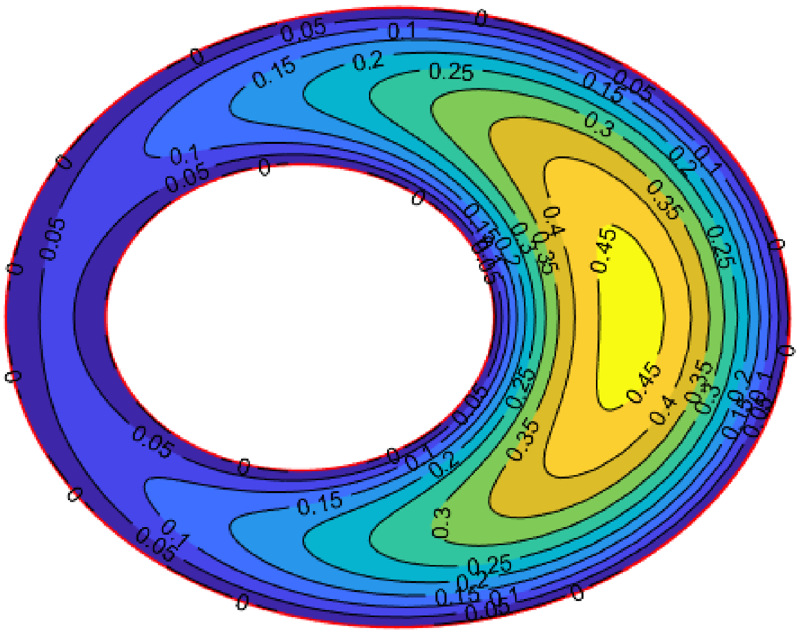


(b) For FPBE

Figure 3.8: The electric potential distribution $\lambda = 10$,e=3/4

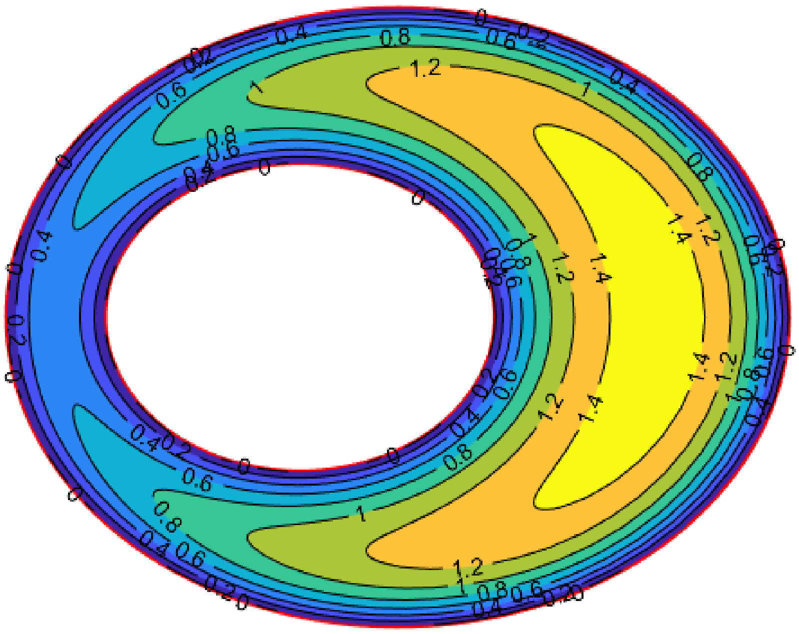


(a) For $\dfrac{dp}{DZ} = 1$

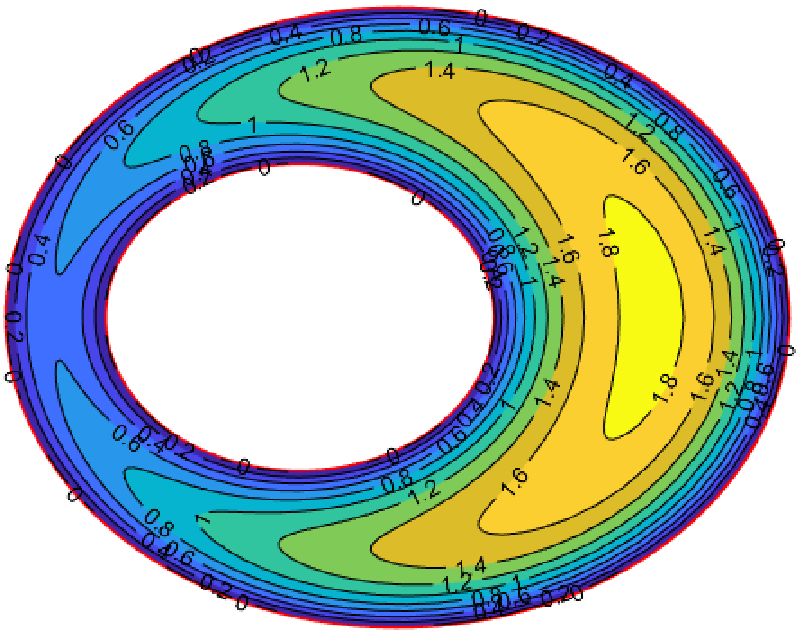


(b) For $\dfrac{dp}{DZ} = 10$

Figure 3.9: Velocity distribution $\lambda = 1$, e=1/2

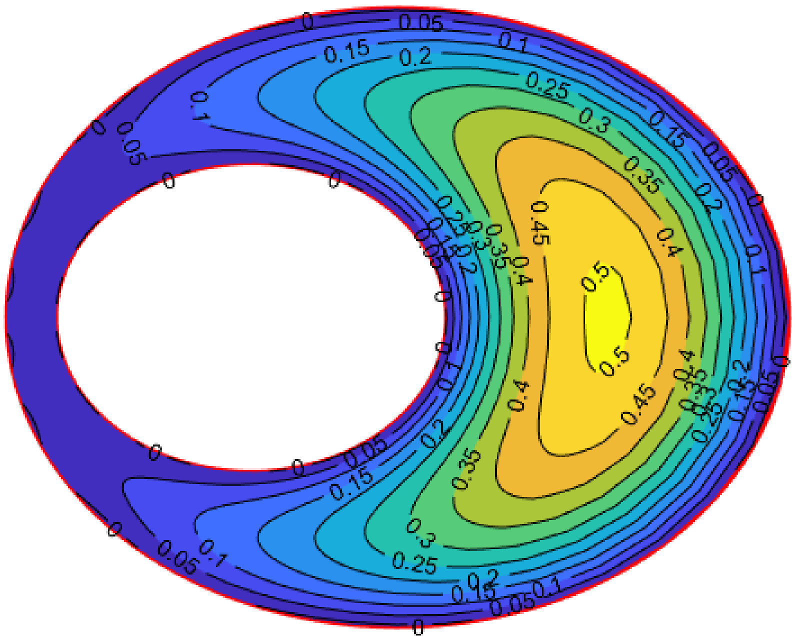


(a) For $\frac{dp}{DZ} = 1$

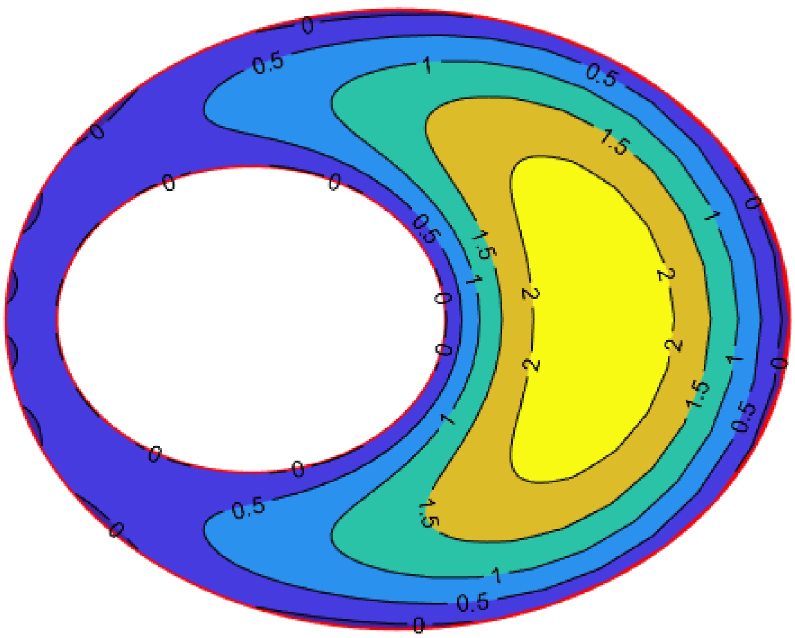


(b) For $\frac{dp}{DZ} = 10$

Figure 3.10: Velocity distribution $\lambda = 10$, e=1/2



(a) For $\frac{dp}{DZ} = 1$



(b) For $\frac{dp}{DZ} = 10$

Figure 3.11: Velocity distribution $\lambda = 1$, e=3/4

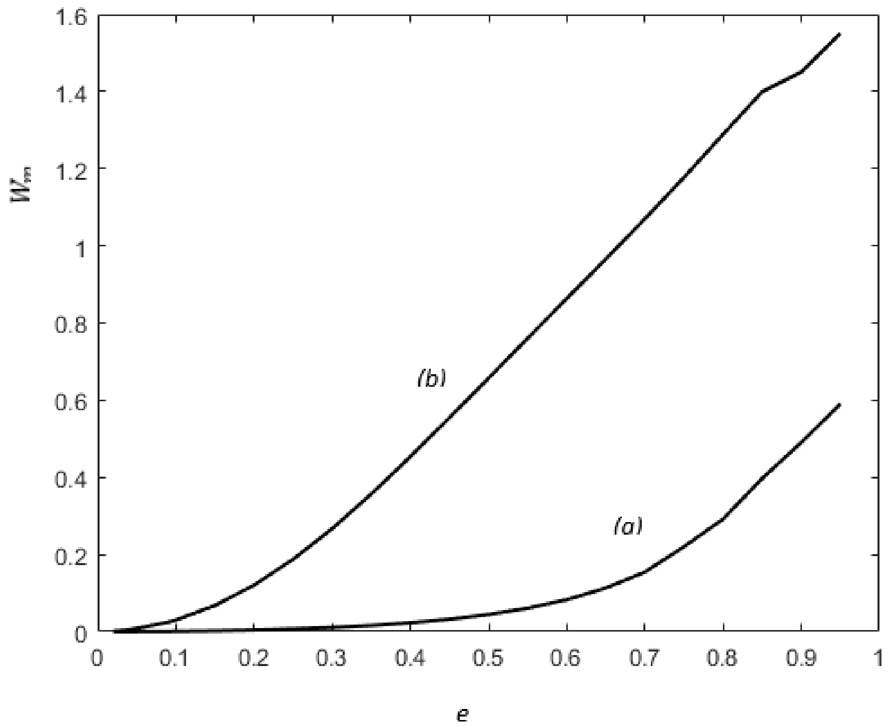


Figure 3.12: Average dimensionless velocity versus eccentricity (a) $\frac{dp}{DZ} = 1$,$\lambda = 1$, (b) $\frac{dp}{DZ} = 1$,$\lambda = \sqrt{50}$

Bibliography

[1] J. Shen,T. Tang, L. Wang; Spectral Methods Algorithm,Analysis, and Applications.

[2] L. C. Evans; Partial Differential Equations; 2nd Edition, American Mathematical Society, 2010.

[3] E. Kreyszig; Introductory Functional Analysis with Applications; Wiley.

[4] Kendall E. Atkinson, An Introduction to Numerical Analysis, second edition, Wiley, 1989.

[5] Reuss, F. F., 1809,"Charge-Induced Flow," Imperial Society of Naturalists of Moscow, Vol. 3, pp. 327344

[6] Helmholtz, H., 1879, "Studien uber electrische Grenzschichten," Ann. Phys.Chem., 243(7), pp. 337382.

[7] Smoluchowski, M., 1903, "`Contribution_a` la `th_eorie l0endosmose_electrique` et de quelques `ph_enome'nes corr_elatifs`," Bull. Acad. Sci. Cracovie, 8, pp. 182199.

[8] Debye, P., and Hückel, E., 1923, "Zur Theorie der Elektrolyte. I. Gefrierpunktserniedrigung und verwandte Erscheinungen," Phys. Z., 24, pp. 185206.

[9] Burgreen, D., and Nakache, F. R., 1963, "Electrokinetic Flow in Capillary Elements," Technical Report No. ASD-TDR-63-243.

[10] Burgreen, D., and Nakache, F. R., 1964, "Electrokinetic Flow in Ultrafine Capillary Slits," J. Phys. Chem., 68(5), pp. 10841091.

[11] Rice, C. L., and Whitehead, R., 1965, "Electrokinetic Flow in a Narrow Cylindrical Capillary," J. Phys. Chem., 69(11), pp. 40174024.

[12] Levine, S., Marriott, J. R., Neale, G., and Epstein, N., 1975, "Theory of Electrokinetic Flow in Fine Cylindrical Capillaries at High Zeta-Potentials," J. Colloid Interface Sci., 52(1), pp. 136149.

[13] Tsao, H.-K., 2000, "Electroosmotic Flow Through an Annulus," J. Colloid Interface Sci., 225(1), pp. 247250.

[14] Kang, Y., Yang, C., and Huang, X., 2002, "Electroosmotic Flow in a Capillary AnnulusWith High Zeta Potentials," J. Colloid Interface Sci., 253(2), pp. 285294.

[15] Sadeghi, M., Sadeghi, A., and Saidi, M. H., 2016, "Electroosmotic flow in hydrophobic microchannels of general cross section," ASME J. Fluids Eng., 138(3), p. 031104.

[16] Wang, C. Y., Kung, C. F & Chang, C. C. Approach to Analytic Solutions for Electroosmotic Flow in MicroDucts by Eigenfunctions of the Helmholtz Equation, Microfluid Nanofluid 20,111 (2016)

[17] W. T. Snyder and G. A. Goldstein, "An analysis of fully developed laminar flow in an eccentric annulus," AIChE J. 11, 462 1965.

[18] P. Kandaswamy , K. Murugesan and L. Debnath, On the Hydromagnetic Flow between Two Rotating Eccentric Cylinders, Zeitschrift fr Angewandte Mathematik und Mechanik(ZAMM), Volume74, Issue1, 1994.

[19] R. S. Alassar, Slip Flow in Eccentric Annuli, Journal of Fluids Engineering, Vol.139, APRIL 2017.

[20] A. AlSahlani and R. Mukherjee, DYNAMICS SIMULATION OF A CIRCULAR MEMBRANE WITH AN ECCENTRIC CIRCULAR AREAL CONSTRAINT, Proceedings of the ASME 2011 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference DETC/CIE 201August 28-31, 2011, Washington, DC, USA.

[21] G. S. Singh and L. S. Kothari, On the solution of the two dimensional Helmholtz equation, Journal of Mathematical Physics 25, 810 (1984).

[22] Morteza Sadeghi, Mohammad Hassan Saidi, Ali Moosavi, and Arman Sadeghi, Geometry effect on electrokinetic flow and ionic conductance in pH-regulated Nanochannels, Physics of Fluids 29, 122006 (2017).

[23] N. Gass, Solution of the Three Dimensional Scalar Helmholtz Equation by a Finite Element Formulation, Zeitschrift fr Angewandte Mathematik und Mechanik(ZAMM), Volume57, Issue2, 1977.

[24] Ylmazer, A., Kocar, C., A Novel Analytical Method for Heat Conduction in Convectively Cooled Eccentric Cylindrical Annuli, International Journal of Thermal Sciences, 63, Sept., pp. 1-15 (2014).

[25] A. Anderson, J.C. Tannehill, R.H. Pletcher, Computational Fluid Mechanics and Heat Transfer, Hemisphere, Washington, 1984.