Kingdom of Saudi Arabia
Al-Imam Mohammad Ibn Saud Islamic University
College of Science

المملكة العربية السعودية
جامعة الإمام محمد بن سعود الإسلامية
كلية العلوم

Department of Mathematics and Statistics

قسم الرياضيات والإحصاء

Measures of Noncompactness and Applications

Research Project by:

Almonther ALARFAJ

Supertvisor:

Prof. Ismail DJEBALI   Department of Mathematics and Statistics - IMSIU

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Dedication

To my MOTHER peace be upon her: I wish you were alive to share my happiness of my achievement.

To my beloved wife NADA: I really appreciate your patience, great sacrifice, and understanding for being too busy with my studies rather than spending time with you.

To my lovely children: I hope I have done something that makes you proud of your FATHER.

To my supervisor Prof. Smail DJEBALI*, for his patience and efforts and who has been always generous during all phases of this research project that has resulted in this valuable work.*

I owe a deep debt of gratitude to our department for giving me the opportunity to complete this work.

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Abstract

The aim of the project is first to present the concept of measure of noncompactness (MNC for short). More precisely, Kuratowski and Hausdorff MNCs will be investigated together with their main properties. Related nonlinear mappings, generalizing classic Lipschitz type functions are then introduced. Some fixed point theorems extending Schauder's fixed point theorem are also presented together with some proofs. The theory is finally used to investigate the solvability of some nonlinear integral equations of Hammerstein type. Existence results are established in appropriate classical functional spaces.

Introduction

The measure of noncompactness (MNC for short) measures the degree of noncompactness of a set in some metric space for it is zero for a relatively compact set. The first MNC was introduced by the polish mathematician K. Kuratowski in 1930[1], where he extended Cantor's intersection theorem by using his MNC, denoted throughout α.

In 1955, G. Darbo[2] used Kuratowski MNC to prove a general fixed point theorem which extends both Schauder's Theorem (for compact mappings) and Banach's Contraction Principle (1922) (for contractive mappings). Indeed Darbo introduced the notion of k-set contractive mappings extending the class of Lipschitz mappings also contractive mappings.

Later, in 1967, B.N. Sadovskii[3] generalized Darbo's fixed point theorem to a wider class of mappings, the so-called condensing mappings. Roughly speaking, a condensing mapping is a map such that the image of a set is, in a certain sense, "more compact" than the set itself.

The Hausdorff measure, denoted χ, was introduced by L.S. Goldestein *et al.* in 1957[4]. In 1972, the romanian mathematician V.I. Istratescu *et al.* [5]defined the β MNC.

Now, we can find in the literature several MNCs that are developed for special functional setting. A MNC can even be defined in an axiomatic approach (see [1] for details).

As mentioned above, classes of mappings involving MNCs may be alternatives to compact mappings and thus of great importance in fixed point theory. For example, it will be checked in this work that the sum of a contractive mapping and a compact one is a strict k-set contaction, which is the basic idea in Darbos' fixed point theorem. In addition, condensing mappings have nice properties similar to compact ones. This may explain their usefulness in several applications in Topology and Functional Analysis.

Detailed Plan of the Project:

1. Review of the completeness and compactness theories in topological spaces with focus on the main properties in metric spaces and especially in Banach spaces. This is the main of preliminary chapter 1.

2. The concept of measure of noncompactness (MNC) in metric spaces through Kuratowski and Hausdorff MNCs is introduced in Section 2. Their main properties are presented in detail, including some special case in the framework of normed spaces..

[1] Sur les espaces complets, Fund. Math., 15 (1930) 301-309 (French)

[2] Punti uniti in transformazioni a codominio non compatto, Rend. Sem. Mat. Univ. Padova, 24 (1955) 58-92 (Italian)

[3] On a fixed point principle, Funcktsional. Anal. i Prilozhen, 2 (1967) 74-76 (Russian)

[4] Investigation of some properties of bounded linear operators and of the connection with their g-norm, Uchen. Zap. Kishinev. Gos. Univ., 29 (1957) 29-36 (Russian)

[5] A generalization of collectively compact sets of operators. I, Rev. Roumaine de Math. Pures et Appli., 17 (1972) 33-37

3. Darbo and Sadovskii's fixed point theorems for k-set contractions and condensing mappings are discussed and proved in Section 3. They extend the classical Brouwer and Schauder fixed point theorems.

4. Final chapter 4 is devoted to applying the MNC to Hammerstien type nonlinear integral equation.

1 Completeness and Compactness in Metric Spaces

1.1 Completeness

Definition 1.1.1. *Let (X, d_X) be a metric space. A sequence $(x_n)_{n=1}^{\infty}$ in X is called a Cauchy sequence if*

$$\forall\, \varepsilon > 0\, \exists\, n_\varepsilon \in \mathbb{N} :\ \forall\, n, m \in \mathbb{N},\ n, m \geq n_\varepsilon \Rightarrow d_X(x_n, x_m) < \varepsilon.$$

Proposition 1.1.2. *Let (X, d_X) be a metric space and let $(x_n)_{n=1}^{\infty}$ be a convergent sequence in X. Then $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.*

Proof. Let $x := \lim_{n\to\infty} x_n$ and let $\varepsilon > 0$. Then

$$\exists\, n_\varepsilon \in \mathbb{N},\ \text{such that}\ \forall\, n \geq n_\varepsilon, d(x_n, x) < \frac{\varepsilon}{2}.$$

For all $n, m \in \mathbb{N}$ such that $n, m \geq n_\varepsilon$, we have

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. □

Remark 1.1.3. *In case of general metric spaces, the converse does not hold. For example,* $(\frac{1}{n})_{n=1}^{\infty}$ *is a Cauchy sequence in the metric space* $((0,1), |\cdot|)$. *However, this sequence has no limit in this metric space.*

Proposition 1.1.4. *Every Cauchy sequence in a metric space is bounded.*

Proof. Let $\varepsilon = 1$. Then there exists $N_1 \in \mathbb{N}$ such that for all $n \geq m \geq N_1, d(x_m, x_n) < 1$. Let $p \in X$ and let $k = \max_{i \leq m} d(p, x_i)$. Then $d(p, x_n) \leq d(p, x_m) + d(x_m, x_n) < k + 1$, which implies that $(x_n)_n$ is bounded. □

Definition 1.1.5. *A metric space* (X, d) *is said to be complete if every Cauchy sequence in* X *converges.*

Example 1.1.6.

(*a*) *In a discrete metric space, every Cauchy sequence is eventually constant. So it is convergent. Hence discrete metric spaces are complete.*

(*b*) *Let* $S \neq \phi$ *and* Y *be a metric space. A function* $f : S \to Y$ *is said to be bounded if* $\sup_{(x,y)\in S^2} d(f(x), f(y)) < \infty$.

Let $B(S,Y) = \{f : S \to Y \text{ bounded}\} \subset \mathscr{F}(S,Y)$. Then
Claim 1 : $B(S,Y)$ is a metric space with the distance $D(f,g) = \sup_{x\in S} d(f(x), g(x))$.
Firstly, since f and g are bounded, this makes a sense. We have

1. $D(f,g) = \sup_{x\in S} d(f(x), g(x)) \geq 0$ for (Y,d) is a metric and

$$\begin{aligned} D(f,g) = 0 &\Leftrightarrow \sup_{x\in S} d(f(x), g(x)) = 0 \\ &\Leftrightarrow d(f(x), g(x)) = 0 \ \forall\, x \in S \\ &\Leftrightarrow f(x) = g(x) \ \forall\, x \in S \\ &\Leftrightarrow f = g. \end{aligned}$$

2. $D(f,g) = \sup_{x\in S} d(f(x), g(x)) = \sup_{x\in S} d(g(x), f(x)) = D(g,f)$.

3.

$$\begin{aligned} d(f(x), g(x)) &\leq d(f(x), h(x)) + d(h(x), g(x)) \\ &\leq \sup_{x\in S} d(f(x), h(x)) + \sup_{x\in S} d(h(x), g(x)) \end{aligned}$$

$\Rightarrow \sup_{x\in S} d(f(x), g(x)) \leq \sup_{x\in S} d(f(x), h(x)) + \sup_{x\in S} d(h(x), g(x))$
$\Rightarrow D(f,g) \leq D(f,h) + D(h,g)$.

Claim 2: $B(S,Y)$ is a complete space if Y is complete. Let $(f_n)_n$ be a Cauchy sequence in $B(S,Y)$. We have

$$\begin{aligned} &\forall \varepsilon>0,\ \exists n_\varepsilon\in\mathbb{N}:\forall n,m\in\mathbb{N}:\ n>m\geq n_\varepsilon &&\Rightarrow\ D(f_n,f_m)<\varepsilon \\ \Leftrightarrow &\forall \varepsilon>0,\ \exists n_\varepsilon\in\mathbb{N}:\forall n,m\in\mathbb{N}:\ n>m\geq n_\varepsilon &&\Rightarrow\ \sup_{x\in S} d(f_n(x),f_m(x))<\varepsilon \\ \Rightarrow &\ \varepsilon>0,\ \exists n_\varepsilon\in\mathbb{N}:\forall n,m\in\mathbb{N}:\ n>m\geq n_\varepsilon &&\Rightarrow\ d(f_n(x),f_m(x))<\varepsilon\ \forall x\in S. \end{aligned}$$

Hence $(f_n(x))_n$ is a Cauchy sequence in Y, $\forall x\in S$. Since Y is complete, then $\lim_{n\to\infty} f_n(x)=f(x)$, $\forall x\in S$. We have to check that

1. $f\in B(S,Y)$
2. $\lim_{n\to\infty} D(f_n,f)=0$

1. Let $x,y\in S$, we have

$$\begin{aligned} d(f(x),f(y)) &\leq d(f(x),f_n(x))+d(f_n(x),f_n(y))+d(f_n(y),f(y)) \\ &\leq d(f(x),f_n(x))+\sup_{x\in S} d(f_n(x),f_n(y))+d(f_n(y),f(y)). \end{aligned}$$

   Since $(f_n)_n\subset B(S,Y)$. Then $\sup_{x,y\in S} d(f_n(x),f_n(y))<M$. So as $n\to\infty$

$$d(f(x),f(y))<M,\ \forall x,y\in S\Rightarrow \sup_{x,y\in S} d(f(x),f(y))<M\Rightarrow f\in B(S,Y).$$

2. Let us show that $\lim_{n\to\infty} D(f_n,f)=0$. Since $(f_n)_n$ is a Cauchy sequence in $B(S,Y)$, then

$$\forall\varepsilon\geq 0,\ \exists n_\varepsilon\in\mathbb{N}:\ \forall n,m\in\mathbb{N},\ m>n\geq n_\varepsilon\Rightarrow D(f_n,f_m)<\varepsilon.$$

   We have

$$d(f_n(x),f_m(x))\leq D(f_n,f_m)\ \forall x\in S,$$

   which implies

$$\Rightarrow d(f_n(x),f_m(x))<\varepsilon,\ \forall x\in S,\ \forall m>n\geq n_\varepsilon.$$

   Therefore

$$\forall x\in S,\ d(f_n(x),f(x))\leq d(f_n(x),f_m(x))+d(f_m(x),f(x))<\varepsilon+d(f_m(x),f(x)).$$

   Letting n fixed and $m\to\infty$, we get

$$\begin{aligned} & d(f_n(x),f(x))<\varepsilon,\ \forall n\geq n_\varepsilon \\ \Rightarrow\ & \sup_{x\in S} d(f_n(x),f(x))<\varepsilon\ \forall n\geq n_\varepsilon \\ \Rightarrow\ & f_n \underset{n\to\infty}{\longrightarrow} f\ in\ B(S,Y). \end{aligned}$$

Proposition 1.1.7. *Let (X,d) be a metric space and let $Y\subset X$. Then*

(i) *If X is complete and Y is closed in X, then Y is complete.*

(ii) *If Y is complete then Y is closed.*

Proof.

(i) Suppose that X is complete and Y is a closed subset of X. Let $(x_n)_n \subset Y$ be a Cauchy sequence. Then $(x_n)_n$ is a Cauchy sequence in X, hence converges in X for X is complete. However, Y being closed, the sequence $(x_n)_n$ converges in Y. Hence Y is complete.

(ii) Suppose that Y is complete and let $(x_n)_n \subset Y$ be a convergent sequence to some limit x. Since a convergent sequence is a Cauchy sequence and Y is complete, then $x \in Y$. Hence Y is closed. □

Example 1.1.8. *Let (X, d_X) and (Y, d_Y) be two metric space and let $C(X,Y) = \{f : X \to Y \text{ continuous}\}$. Then $C_b(X,Y) = C(X,Y)\bigcap B(X,Y)$ is closed in $B(X,Y)$ and therefore complete if (Y, d_Y) is.*

Clearly, $C_b(X,Y)$ is a subspace of $(B(X,Y), D)$. Let $(f_n)_n$ be a sequence in $C_b(X,Y)$ that converges to $f \in B(X,Y)$, that is $\lim\limits_{n\to\infty} D(f_n, f) = 0$. We have to prove that $f : X \to Y$ is continuous, that is for all sequence $(x_n)_n \subset X$ such that $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} f(x_n) = f(x)$. Let $(x_n)_n \subset X$ be a sequence such that $\lim\limits_{n\to\infty} x_n = x$. By assumption, we have that

$$\forall \varepsilon \geq 0 \, \exists n_\varepsilon \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_\varepsilon,\ \ D(f_n, f) < \varepsilon.$$

Hence

$$D(f_n, f) < \varepsilon \Rightarrow \sup_{x \in S} d(f_n(x), f(x)) < \varepsilon.$$

Also f_n being continuous $\forall\, n \in \mathbb{N}$, we have

$$\forall\, n \in \mathbb{N},\ \lim_{m\to\infty} f_n(x_m) = f_n(x)\ \forall\, (x_m)_m \subset X \text{ converges to } x.$$

Then

$$\begin{aligned} d_Y(f(x_m), f(x)) &\leq d_Y(f(x_m), f_n(x_m)) + d_Y(f_n(x_m), f_n(x)) + d_Y(f_n(x), f(x)), \\ &< \varepsilon + d_Y(f_n(x_m), f_n(x)) + \varepsilon. \end{aligned}$$

As $m \to \infty$, we find

$$\lim_{m\to\infty} d_Y(f(x_m), f(x)) \leq 2\varepsilon,\ \forall\, \varepsilon > 0 \Rightarrow \lim_{m\to\infty} d_Y(f(x_m), f(x)) = 0.$$

Definition 1.1.9. *Let (X, d) be a metric space and $A \subset X$ with $A \neq \phi$. The diameter of A is defined by* $\mathrm{diam(A)} = \sup\limits_{\mathrm{x,y \in A}} \mathrm{d(x,y)}$.

Clearly, if $A \subseteq B$ then $\mathrm{diam(A)} \leq \mathrm{diam(B)}$.

Proposition 1.1.10. *Let (X, d) be a metric space and $A \subset X$ with $A \neq \phi$. Then*

1. $\mathrm{diam(A)} = \mathrm{diam(\overline{A})}$.

2. *If* $A = B_r(x_0)$ *then* $\mathrm{diam(A)} \leq 2\mathrm{r}$.

3. *A is bounded* $\Leftrightarrow \mathrm{diam(A)} < \infty$.

Proof. 1. Since $A \subset \overline{A}$ then $\mathrm{diam(A)} \leq \mathrm{diam}(\overline{\mathrm{A}})$. Let $x, y \in \overline{A}$. Then there exist sequences $(x_n)_n, (y_n)_n \subset A$ such that $\underset{n\to\infty}{lim}\, x_n = x$ and $\underset{n\to\infty}{lim}\, y_n = y$. We have

$$\begin{aligned} d(x,y) &\leq d(x,x_n) + d(x_n,y_n) + d(y_n,y), \\ &\leq d(x,x_n) + \mathrm{diam(A)} + \mathrm{d(y_n,y)}. \end{aligned}$$

Letting $n \to \infty$, we find

$$\begin{aligned} d(x,y) &\leq \mathrm{diam(A)}, \quad \forall \mathrm{x,y} \in \overline{\mathrm{A}}. \\ \Rightarrow \sup_{x,y\in\overline{A}} d(x,y) &\leq \mathrm{diam(A)}. \\ \Rightarrow \mathrm{diam}(\overline{\mathrm{A}}) &\leq \mathrm{diam(A)}. \end{aligned}$$

Hence $\mathrm{diam}(\overline{\mathrm{A}}) = \mathrm{diam(A)}$.

2. Let $x, y \in A = B_r(x_0)$. Then $d(x,x_0) < r$ and $d(y,x_0) < r$. So

$$d(x,y) \leq d(x,x_0) + d(x_0,y) < 2r \Rightarrow \sup_{x,y\in A} d(x,y) \leq 2r \Rightarrow \mathrm{diam(A)} \leq 2\mathrm{r}.$$

3. Suppose that A is bounded. Then there exists a ball $B_r(x_0)$ such that $A \subset B_r(x_0)$. Hence $\mathrm{diam(A)} \leq 2\mathrm{r} < \infty$. Conversely, $\mathrm{diam(A)} < \infty$, then $x_0 \in A^o$, which implies that $A \subset B_{r_0}(x_0)$, where $r_0 = \mathrm{diam(A)}$. Hence A is bounded.

□

Proposition 1.1.11. *Let* $(E, ||.||_E)$ *be a normed space and let* $A = B_r(x_0) \subset X$. *Then*

$$\mathrm{diam}(\overline{\mathrm{A}}) = \mathrm{diam}(\partial \mathrm{A}) = 2\mathrm{r}.$$

Proof. First let us show that $\mathrm{diam}(\partial\mathrm{A}) = 2\mathrm{r}$. Let $x, y \in S_r[x_0] = \partial A$ for $(E, ||.||_E)$ is normed space. Then

$$||x_0 - x|| = r \;\; and \;\; ||x_0 - y|| = r.$$

$$\begin{aligned} \Rightarrow \;\; ||x-y|| &\leq ||x-x_0|| + ||x_0-y||, \\ &= r + r = 2r. \\ \Rightarrow \;\; ||x-y|| &\leq 2r, \quad \forall\, x, y \in S_r[x_0]. \end{aligned}$$

Hence

$$\mathrm{diam}(\partial\mathrm{A}) = \sup_{\mathrm{x,y}\in\partial(\mathrm{A})} ||\mathrm{x}-\mathrm{y}|| \leq 2\mathrm{r}.$$

Conversely, let $x \in \partial A$ and choose $y = 2x_0 - x$. Then $y \in \partial A$ since

$$||x_0 - y|| = ||x_0 - 2x_0 + x|| = || - x_0 + x|| = ||x - x_0|| = r.$$

Therefor

$$||x - y|| = ||x - 2x_0 + x|| = 2||x - x_0|| = 2r.$$

Hence

$$\mathrm{diam}(\partial \mathrm{A}) = \sup_{x,y \in \partial A} ||x - y|| \geq 2r.$$

Finally

$$\mathrm{diam}(\partial \mathrm{A})) = 2r.$$

Secondly let us show that $\mathrm{diam}(\overline{A}) = \mathrm{diam}(\partial A)$. We have $\overline{A} = A \bigcup \partial A$ that means $\partial A \subset \overline{A}$. Then

$$2r = \mathrm{diam}(\partial \mathrm{A}) \leq \mathrm{diam}\overline{\mathrm{A}} \leq 2r.$$

Hence

$$\mathrm{diam}(\overline{\mathrm{A}}) = \mathrm{diam}(\mathrm{A}) = \mathrm{diam}(\partial \mathrm{A}) = 2r.$$

□

Definition 1.1.12. *Let (X, d) be a metric space, $\phi \neq A \subset X$, and $x \in X$. Then*

$$d(x, A) = inf\{d(x, a) : \ a \in A\}.$$

Proposition 1.1.13. *Let (X, d) be a metric space, $\phi \neq A \subset X$ and $x \in X$. Then*

$$x \in \overline{A} \ \Leftrightarrow \ d(x, A) = 0.$$

Proof. Let $x \in \overline{A}$. Then there exists $(x_n)_n \subset A$ such that $\lim_{n\to\infty} x_n = x$ (i.e, $\lim_{n\to\infty} d(x_n, x) = 0$. Hence

$$0 \leq d(x, A) = \inf_{a \in A} d(x, a) \leq d(x, x_n), \quad \forall \, n.$$

As $n \to \infty$, we get $d(x, A) = 0$. Conversely, let $d(x, A) = 0$. Then

$$\forall \, n \in \mathbb{N}, \, \exists \, x_n \in A \, : \, 0 \leq d(x, x_n) < \frac{1}{n}.$$

Hence there exists $(x_n)_n \subset A$ such that $\lim_{n\to\infty} d(x, x_n) = 0$. As a consequence $x \in \overline{A}$. □

Theorem 1.1.14. *(Cantor's Intersection Theorem) Let (X, d) be a complete metric space and $(F_n)_{n=1}^{\infty}$ a sequence of nonempty closed subsets of X such that $F_{n+1} \subset F_n \; \forall \, n$ and* $\lim_{n\to\infty} \mathrm{diam}(\mathrm{F_n}) = 0$. *Then $\bigcap_{n=1}^{\infty} F_n$ is nonempty and reduces to a singleton.*

Proof. Let $(x_n)_n \subset X$ be a sequence such that $x_n \in F_n\ \forall\, n$. Firstly, we show that $(x_n)_n$ is a Cauchy sequence. Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} \mathrm{diam(F_n)} = 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\mathrm{diam(F_n)} \le \varepsilon$, $\forall\, \mathrm{n} \ge \mathrm{n}_\varepsilon$. Let $m, n > n_\varepsilon$. Then $x_m, x_n \in F_{n_\varepsilon}$ for $F_{n+1} \subset F_n\ \forall\, n$. Hence $d(x_m, x_n) \le \mathrm{diam(F_{n_\varepsilon})} \le \varepsilon$. So $(x_n)_n$ is a Cauchy sequence. Secondly, we show that $\bigcap\limits_{n=1}^{\infty} F_n \neq \phi$. Since (X, d) is complete, there exists $x \in X$ such that $x = \lim\limits_{n\to\infty} x_n$. Moreover F_n closed $\forall\, n$, then $\bigcap\limits_{n=1}^{\infty} F_n$ closed. We show that $x \in \bigcap\limits_{n=1}^{\infty} F_n$. Let $n \in \mathbb{N}$ be arbitrary. We have

$$\forall\, m > n,\ x_m \in F_n \Rightarrow\ x = \lim_{m\to\infty} x_m \in \bar{F_n} = F_n.$$

Then

$$\forall\, n \in \mathbb{N},\ x \in F_n \Leftrightarrow x \in \bigcap_{n=1}^{\infty} F_n.$$

We show that $\bigcap\limits_{n=1}^{\infty} F_n = \{x\}$. We have

$$\bigcap_{n=1}^{\infty} F_n \subseteq F_n \Rightarrow 0 \le \mathrm{diam}\left(\bigcap_{\mathrm{n=1}}^{\infty} \mathrm{F_n}\right) \le \mathrm{diam(F_n)}.$$

Hence $\lim\limits_{n\to\infty} \mathrm{diam(F_n)} = 0 \Rightarrow \mathrm{diam}(\bigcap\limits_{\mathrm{n=1}}^{\infty} \mathrm{F_n}) = 0 \Rightarrow \bigcap\limits_{n=1}^{\infty} F_n = \{x\}$. □

1.2 Compactness

Definition 1.2.1. *Let (X, d) be a metric space and $S \subset X$.*

(a) An open cover for S is a collection of open subsets of X $(U_\lambda)_{\lambda\in\Lambda}$ such that $S \subset \bigcup\limits_{\lambda\in\Lambda} U_\lambda$.

(b) S is called compact if, every open cover $(U_\lambda)_{\lambda\in\Lambda}$ of S, there are $U_1, U_2, \cdots, U_n \in (U_\lambda)_{\lambda\in\Lambda}$ such that $S \subset \bigcup\limits_{i=1}^{n} U_i$.

(c) S is called precompact or totally bounded if

$$\forall\, \varepsilon > 0\ \exists \{x_1, x_2, \cdots, x_{N_\varepsilon}\} \subset X \text{ such that } S \subset \bigcup_{k=1}^{N_\varepsilon} B_\varepsilon(x_k).$$

(d) S is called sequentially compact if every sequence in S has a convergent subsequence in S.

Proposition 1.2.2. *Let (X, d) be a compact metric space. Then (X, d) is totally bounded.*

Proof. Let $\varepsilon > 0$. Then $X = \bigcup\limits_{x\in X} B(x)$ which is an open cover. Since X is compact, there exits $N \in \mathbb{N}$ such hat $X = \bigcup\limits_{k=1}^{N} B(x_k)$, that is X is totally bounded. □

Example 1.2.3.

(a) *Let* (X,d) *be a metric space and* $S \subset X$ *be finite:* $S = \{x_1, x_2, \cdots, x_n\}$. *Let* $(U_\lambda)_{\lambda\in\Lambda}$ *be an open cover of* S. *Then, for each* $i = 1,2,\cdots,n$ *there is* $U_i \in (U_\lambda)_{\lambda\in\Lambda}$ *such that* $x_i \in U_i$. *Hence* $S \subset \bigcup_{i=1}^{n} U_i$.

(b) *Let* $X = (0,1)$ *be equipped with the usual metric. Let* $U_n = (\frac{1}{n}, 1)$. *Then* $\bigcup_{i=1}^{\infty} (\frac{1}{n}, 1)$ *is an open cover for* $(0,1)$ *which has no finite subcover.*

Proposition 1.2.4. *Let* (X,d) *be a metric such that* (X,d) *is totally bounded. Then* (X,d) *is bounded.*

Proof. Let $\varepsilon > 0$. Then there exists $N_\varepsilon \in \mathbb{N}$ such that $x_1, x_2, \cdots, x_{N_\varepsilon}$ with $X = \bigcup_{k=1}^{N_\varepsilon} B_\varepsilon(x_k)$. Let $x_0 \in X$ and $R > \varepsilon + \max_{1\le k\le N_\varepsilon} \{d(x_k, x_0)\}$. Then $B_\varepsilon(x_k) \subset B_R(x_0)\ \forall\, 1 \le k \le N_\varepsilon$. For $d(x, x_k) < \varepsilon$, we have

$$d(x, x_0) \le d(x, x_k) + d(x_k, x_0) < \varepsilon + R - \varepsilon = R.$$

Then $X \subset B(x_0, R)$, i.e., X is bounded. □

Remark 1.2.5. *By Proposition 1.2.2, every compact metric space is bounded.*

Proposition 1.2.6. *Let* (X,d) *be a metric space, and let* Y *be a subspace of* X.

(i) *If* X *is compact and* Y *closed in* X, *then* Y *is compact.*

(ii) *If* Y *is compact, then it is closed in* X.

Proof.

(i) Let $(U_\lambda)_{\lambda\in\Lambda}$ be an open cover of Y; then $Y \subset \bigcup_{\lambda\in\Lambda} U_\lambda$. Since Y is closed then $X \setminus Y$ is open. Hence $X \subset (X\setminus Y) \bigcup (\bigcup_{\lambda\in\Lambda} U_\lambda)$. Since X is compact, we have $X \subset (X\setminus Y) \bigcup (\bigcup_{k=1}^{N} U_k)$. Hence $Y \subset \bigcup_{k=1}^{N} U_k$, which is a finite open subcover. Then Y is compact.

(ii) To show that Y is closed, we show that $X \setminus Y$ is open. Let $x \in X \setminus Y$ then

$$\forall\, y \in Y\ \exists \varepsilon_y > 0\ \exists \delta_y > 0 :\ B_{\varepsilon_y}(x) \cap B_{\delta_y}(y) = \phi,$$

Since x is a metric space, hence it is a Hausdorff space. We have $Y \subset \bigcup_{y\in Y} B_{\delta_y}(y)$ which is an open covering. Since Y is compact, then

$$Y \subset \bigcup_{k=1}^{N} B_{\delta_{y_k}}(y_k)$$

Now, let $\varepsilon = \min_{1\le k\le N}(\varepsilon_{y_k}) > 0$. Then

$$B_\varepsilon(x)\cap Y \subset B_\varepsilon(x)\bigcap\left(\bigcup_{k=1}^{N} B_{\delta_{y_k}}(y_k)\right) = \phi.$$

Hence $B_\varepsilon(x)\subset X\setminus Y$ and thus $X\setminus Y$ is open. □

Proposition 1.2.7. *Let $(X,\mathcal{T})$ and $(X',\mathcal{T})$ be two topological spaces and $f : X \to X'$ be a continuous function. Then, if X is compact then $f(X)$ is compact.*

Proof. Let $(U_\lambda)_{\lambda\in\Lambda}$ be an open cover of $f(X)$, i.e., $f(X)\subset \bigcup_{\lambda\in\Lambda} U_\lambda$. Since f is continuous, then $f^{-1}(\bigcup_{\lambda\in\Lambda} U_\lambda)$ is an open set. Moreover, $X\subset f^{-1}(\bigcup_{\lambda\in\Lambda} U_\lambda) = \bigcup_{\lambda\in\Lambda} f^{-1}(U_\lambda)$. Since X is compact, then $X\subset \bigcup_{i=1}^{N} f^{-1}(U_i)$. Hence

$$f(X)\subset f(\bigcup_{i=1}^{N} f^{-1}(U_i)) = f(f^{-1}(\bigcup_{i=1}^{N} U_i))\subset \bigcup_{i=1}^{N} U_i.$$

Then $f(X)$ is compact. □

Definition 1.2.8. *Let $(X,\mathcal{T})$ be a topological space, $A\subset X$, and $x\in X$. We say that*

1. *x is an adherent point of A, if*

$$\forall\, U\in N_x,\ U\cap A\neq\phi(i.e., x\in\bar{A}).$$

2. *x is a cluster point of A, if*

$$\forall\, U\in N_x,\ U\setminus\{x\}\cap A\neq\phi(i.e., x\in A').$$

3. *x is an accumulation point or limit point of A, if*

$$\forall\, U\in N_x,\ U\cap A \text{ contains infinitely many points } (i.e., x\in\mathscr{A}).$$

Remark 1.2.9. *From above, we have*

1. *$\mathscr{A}\subset A'\subset\bar{A}$.*

2. *$\bar{A} = A'\cup A$.*

3. *$L = \bar{A}\setminus A'$ is the set of isolated points.*

Definition 1.2.10. *Let $(x_n)_n$ be a sequence in a topological space $(X,\mathcal{T})$ and $x\in X$.*

1. *x is called an accumulation point of $(x_n)_n$, if*

$$\forall\, U \in N_x,\ U \text{ contains an infinite number of elements of } (x_n)_n.$$

2. *x is called a limit point of $(x_n)_n$, if*

$$\forall\, U \in N_x,\ \forall\, n \in \mathbb{N},\ \exists\, n_0 > n \text{ such that } x_{n_0} \in U.$$

Then, an accumulation point is a limit point in a metric space, and conversely.

Remark 1.2.11. *(a) The concepts adherent point and cluster point are still valid for sequences.*
(b)

1. *Let $(x_n)_n$ be a sequence in a metric space (X, d) and $x \in X$. Denote by $\mathscr{A}$ the set of limit points of $(x_n)_n$ and $\mathscr{A}_n = \{x_n, x_{n+1}, \cdots\}$. Then we have*

$$\begin{aligned}
x \in \mathscr{A} &\Leftrightarrow \forall\, \varepsilon > 0,\ \forall\, n \in \mathbb{N},\ \exists\, n_0 > n : x_{n_0} \in B_\varepsilon(x) \\
&\Leftrightarrow \forall\, \varepsilon > 0,\ \forall\, n \in \mathbb{N},\ \exists\, n_0 > n : d(x, x_{n_0}) < \varepsilon \\
&\Leftrightarrow \forall\, n \in \mathbb{N},\ \forall\, \varepsilon > 0,\ \exists\, n_0 > n : d(x, x_{n_0}) < \varepsilon \\
&\Leftrightarrow \forall\, n \in \mathbb{N},\ \forall\, \varepsilon > 0,\ B_\varepsilon(x) \cap \mathscr{A}_n \neq \phi \\
&\Leftrightarrow \forall\, n \in \mathbb{N},\ x \in \bar{\mathscr{A}}_n \\
&\Leftrightarrow x \in \bigcap_{n \in \mathbb{N}} \bar{\mathscr{A}}_n.
\end{aligned}$$

Hence $\mathscr{A} = \bigcap_{n \in \mathbb{N}} \bar{\mathscr{A}}_n$.

2. *In a metric space, every limit of a sequence is a limit point of the sequence.*

   Indeed, let $x := \lim_{n \to \infty} x_n$. Then $\forall\, \varepsilon > 0,\ \exists\, n_0 \in \mathbb{N},\ \forall\, n \geq n_0 \Rightarrow d(x_n, x) \leq \varepsilon$. Let $\varepsilon > 0$ and $n > 0$. Then there exists n_0 satisfying $\forall\, n \geq n_0,\ d(x_n, x) \leq \varepsilon$. For $n_1 > \max(n, n_0)$, we have $d(x_{n_1}, x) \leq \varepsilon$.

3. *Clearly, in any metric space, every limit point of a subsequence is a limit point of the sequence.*

Proposition 1.2.12. *Let $(x_n)_n$ be a sequence of a metric space (X, d) and $x \in X$. Then x is a limit point of $(x_n)_n \Leftrightarrow \exists\, (x_{n_k})_k \subset X$ a subsequence such that $\lim_{k \to \infty} x_{n_k} = x$.*

Proof.

($\Leftarrow$) Suppose $\exists (x_{n_k})_k \subset X$ subsequence of $(x_n)_n$ such that $\lim_{k\to\infty} x_{n_k} = x$. Then x is a limit point of $(x_{n_k})_k$ implies that x is a limit point of $(x_n)_n$.

($\Rightarrow$) Let x be a limit point of $(x_n)_n$. By definition, we have that for all $\varepsilon > 0$, $n \in \mathbb{N}$, there exists $n_0 > n : d(x_{n_0}, x) < \varepsilon$.

$$\begin{array}{lll} \text{For } \varepsilon = 1, & \forall\, n \in \mathbb{N}\, \exists\, n_1 > n & : d(x_{n_1}, x) < 1 \\ \text{For } \varepsilon = \frac{1}{2}, & \exists\, n_2 > n_1 & : d(x_{n_2}, x) < \frac{1}{2} \\ \vdots & \vdots & \vdots \\ \text{For } \varepsilon = \frac{1}{k}, & \exists\, n_k > n_{k-1} & : d(x_{n_k}, x) < \frac{1}{k} \end{array}$$

Then we construct $(x_{n_k})_k$ a subsequence of $(x_n)_n$ and $d(x_{n_k}, x) < \frac{1}{k}$, $\forall\, k = 1, 2, \ldots$. This implies that $\lim_{k\to\infty} x_{n_k} = x$. □

Corollary 1.2.13. *Let (X, d) be a metric space and $(x_n)_n \subset X$. Then*

$$\lim_{n\to\infty} x_n = x \Rightarrow \mathscr{A} = \{x\}.$$

Proof. Since $(x_n)_n$ converges to x, then any subsequence also converges to x. By Proposition 1.2.12 , x is the only limit point of this sequence, then $\mathscr{A} = \{x\}$. □

Proposition 1.2.14. *Let $(x_n)_n$ be a sequence of a metric space (X, d). Then*

$$(x_n)_n \text{ Cauchy sequence} \Leftrightarrow \lim_{\mathrm{n}\to\infty} \operatorname{diam}(\mathscr{A}_{\mathrm{n}}) = 0.$$

Proof. Let $(x_n)_n$ be a Cauchy sequence. Then

$$\forall\, \varepsilon > 0,\ \exists\, n_0 \in \mathbb{N} \text{ such that } \forall\, k, k' > n_0 : d(x_k, x_{k'}) < \varepsilon.$$

Note that since $k, k' > n_0 \Rightarrow x_k, x_{k'} \in \mathscr{A}_{n_0}$. Hence

$$\begin{aligned} & \forall\, \varepsilon > 0,\ \exists\, n_0 \in \mathbb{N} : \sup_{k,k' > n_0} d(x_k, x_{k'}) < \varepsilon \\ \Leftrightarrow\ & \forall\, \varepsilon > 0,\ \exists\, n_0 \in \mathbb{N} : \operatorname{diam}(\mathscr{A}_{\mathrm{n}_0}) < \varepsilon \\ \Leftrightarrow\ & \forall\, \varepsilon > 0,\ \exists\, n_0 \in \mathbb{N},\ \forall\, n \geq n_0 : \operatorname{diam}(\mathscr{A}_{\mathrm{n}}) \leq \operatorname{diam}(\mathscr{A}_{\mathrm{n}_0}) < \varepsilon \\ & (\text{for } \mathrm{n} > \mathrm{n}_0 \Rightarrow \mathscr{A}_{\mathrm{n}} \subseteq \mathscr{A}_{\mathrm{n}_0}) \\ \Leftrightarrow\ & \lim_{n\to\infty} \operatorname{diam}(\mathscr{A}_{\mathrm{n}}) = 0. \end{aligned}$$

□

Corollary 1.2.15. *If a Cauchy sequence in a metric space (X, d) has a limit point x, then x is the limit of the sequence.*

Proof. Let $(x_n)_n$ be a Cauchy sequence, then $\lim_{n\to\infty} \mathrm{diam}(\mathscr{A}_\mathrm{n}) = 0$, i.e., $\forall\, \varepsilon > 0\, \exists\, n_0 \in \mathbb{N}:$ $\mathrm{diam}(\mathscr{A}_\mathrm{n}) < \varepsilon$. But $\mathrm{diam}\,(\bar{\mathscr{A}_\mathrm{n}}) = \mathrm{diam}(\mathscr{A}_\mathrm{n})$. Then

$$\forall\, \varepsilon > 0\, \exists\, n_0 \in \mathbb{N},\ \forall\, n \geq n_0,\ (d(x, x_n) \leq \mathrm{diam}(\bar{\mathscr{A}_\mathrm{n}}) \leq \varepsilon),$$

for

$$(x \in \mathscr{A} \Rightarrow x \in \bigcap_{n\geq 1} \bar{\mathscr{A}_n}) \Rightarrow \lim_{n\to\infty} d(x, x_n) = 0 \Rightarrow \lim_{n\to\infty} x_n = x.$$

□

Proposition 1.2.16. *(Cantor's Intersection Theorem for Compact Spaces) Let $(X, \mathscr{T})$ be a compact topological space and $(F_n)_n$ a decreasing sequence of nonempty closed subsets. Then*

$$\bigcap_{n\geq 1} F_n \neq \phi.$$

Proof. By contradiction. Assume that $\bigcap_{n\geq 1} F_n = \phi$. Since X is compact, then there exits $p \in \mathbb{N}$ such that $\bigcap_{n=1}^{p} F_n = \phi$. But $(F_n)_n$ is decreasing, then $F_p = \phi$, a contradiction. □

Corollary 1.2.17. *If X is compact, then $\mathscr{A} \neq \phi$.*

Proof. Let $(x_n)_n$ be a sequence and $F_n = \bar{\mathscr{A}_n}$. Then $(F_n)_n$ is a decreasing sequence of closed nonempty sets. By Cantor's intersection theorem for compact spaces, $\bigcap_n F_n \neq \phi$. Then $\bigcap_n F_n = \bigcap_n \bar{A}_n = \mathscr{A} \neq \phi$. □

Remark 1.2.18. *By Corollary 1.2.17 and Proposition 1.2.12, if X is compact, then every sequence has a convergent subsequence. So if X is compact, then X is sequentially compact.*

Corollary 1.2.19. *Let (X, d) be a metric space. Then*

$$X \text{ compact} \implies X \text{ complete}.$$

Proof. Let $(x_n)_n$ be a Cauchy sequence. Then $\mathscr{A} \neq \phi$. By Corollary 1.2.17, $(x_n)_n$ has a limit point, then by Corollary 1.2.15 $\lim_{n\to\infty} x_n = x$. Hence X is complete. □

Corollary 1.2.20. *Let $(x_n)_n$ be a sequence in a compact metric space (X, d). Then*

$$\lim_{n\to\infty} x_n = x \Leftrightarrow \mathscr{A} = \{x\}.$$

Proof.

(⇒) $\lim_{n\to\infty} x_n = x \Rightarrow \mathscr{A} = \{x\}$. This is proved By Corollary 1.2.13.

(⇐) Let us show that $\mathscr{A} = \{x\} \Rightarrow \lim_{n\to\infty} x_n = x$. Let $U \in \mathscr{N}_x$ and $F_n = \bar{A}_n \cap C_x U$. Then F_n closed $\forall\, n$. Hence $\bigcap_n F_n = \mathscr{A} \cap C_x U = \phi$ for $\mathscr{A} = \{x\}$. We have

$$\begin{aligned}
X \text{ is compact} &\Rightarrow \exists\, n_0 \in \mathbb{N} : \bigcap_{n=1}^{n_0} F_n = \phi \\
&\Rightarrow F_{n_0} = \phi, \text{ for } (\mathrm{F_n})_\mathrm{n} \text{ is decreasing} \\
&\Rightarrow \bar{A}_{n_0} \subset U \\
&\Rightarrow A_n \subset A_{n_0} \subset \bar{A}_{n_0} \subset U,\ \forall\, n \geq n_0 \\
&\Rightarrow x_n \in U,\ \forall\, n \geq n_0 \\
&\Rightarrow \lim_{n\to\infty} x_n = x.
\end{aligned}$$

□

Example 1.2.21. *Let* $(X,d) = (\mathbb{R}, |\cdot|)$ *and the sequence*

$$x_n = \begin{cases} n, & n \text{ odd;} \\ \frac{1}{n}, & n \text{ even.} \end{cases}$$

Then 0 *is a limit point. Indeed* $x_{2n} = \frac{1}{2n} \to 0$, *then* 0 *is a limit point and the other subsequence diverges. So* 0 *is the unique limit point. However,* $\lim_{n\to\infty} x_n \neq 0$, *and the reason is that* $(\mathbb{R}, |\cdot|)$ *is not compact.*

Proposition 1.2.22. *[5, 8] Let* (X,d) *be a metric space. If* X *is totally bounded, then it separable.*

Proposition 1.2.23. *Let* (X,d) *be a metric space. If* X *is sequentially compact, then* X *is compact.*

Proof. Let U be an open cover of X. Since X is sequentially compact, then from Lebesgue Number lemma , there exists $\delta > 0$ such that for every $x \in X$, there is an open set $O \in U$ for which $B(x,\delta) \subset O$. Also we have X is totally bounded (for X is sequentially compact) , then there exist $x_1, x_2, ..., x_n \in X$ such that

$$X = \bigcup_{i=1}^{n} B(x_i, \delta).$$

We Know from above for $x_i \in X$ there is $O_i \in U$ such that

$$B(x_i, \delta) \subset O_i,\, i = 1, 2, ..., n.$$

Hence

$$X = \bigcup_{i=1}^{n} O_i, O_i \in U.$$

Therefore X is compact. □

Proposition 1.2.24. *Let (X,d) be a metric space. If X is sequentially compact, then X is totally bounded and complete.*

Proof. Suppose by contradiction that X is not totally bounded. Then there exists $\varepsilon > 0$ such that for every finite subset $S \subset X$, $X \neq \bigcup_{s\in S} B_\varepsilon(s)$. Let $x_0 \in X$, $x_1 \in X \setminus B_\varepsilon(x_0), \cdots, x_n \in X \setminus \bigcup_{i=0}^{n-1} B_\varepsilon(x_i)$. Then the sequence $(x_n)_n$ has no subsequence which is convergent, for $d(x_j, x_k) > \varepsilon$, $\forall j \neq k$. So it has no Cauchy subsequence, then it has no convergent subsequence which is contradiction, then X is totally bounded. □

Finally X sequentially compact $\Rightarrow X$ compact $\Rightarrow X$ complete. We can even prove that X totally bounded and complete implies that X is sequentially compact.

2 Measures of Noncompactness

2.1 Measures of Noncompactness in Metric Spaces

[1, 2, 3, 4]

Definition 2.1.1. *Let (X,d) be a metric space and $A \subset X$ a bounded subset. We introduce the following sets:*

1. $K(A) = \{D > 0 : \exists N \in \mathbb{N},\ \exists (A_i)_{i=1}^{N} \subset X \text{ such that, } A \subseteq \bigcup_{i=1}^{N} A_i \text{ with } \mathrm{diam}(A_i) \leq D,\ \forall\, 1 \leq i \leq N\}$.
2. $H(A) = \{r > 0 : \exists N \in \mathbb{N},\ \exists \{x_1, x_2, \cdots, x_N\} \subset X \text{ such that, } A \subseteq \bigcup_{i=1}^{N} B_r(x_i)\}$.

Remark 2.1.2. $2H(A) \subset K(A) \subset H(A)$. *Indeed*

(i) *let $D \in K(A)$. Then*

$$\exists N \in \mathbb{N},\ \exists (A_i)_{i=1}^{N} \subset X \text{ such that } A \subseteq \bigcup_{i=1}^{N} A_i \text{ with } \mathrm{diam}(A_i) \leq D,\ \forall\, 1 \leq i \leq N.$$

Since A_i is bounded for all $1 \leq i \leq N$, then

$$A_i \subset B(x_i, D), \text{ with } x_i \in A_i^{\circ},\ \forall\, 1 \leq i \leq N.$$

Hence

$$\bigcup_{i=1}^{N} A_i \subset \bigcup_{i=1}^{N} B(x_i, D) \Rightarrow A \subset \bigcup_{i=1}^{N} B(x_i, D).$$

As a consequence $D \in H(A)$ *and* $K(A) \subset H(A)$.

(ii) *Let* $r \in H(A)$. *Then there exit* $N \in \mathbb{N}$ *and* $\{x_1, x_2, \cdots, x_N\} \subset X$ *such that* $A \subseteq \bigcup_{i=1}^{N} B_r(x_i)$ *with* $\mathrm{diam}(B_r(x_i)) \leq 2r,\ \forall\, 1 \leq i \leq N$. *Then*

$$2r \in K(A) \Rightarrow 2H(A) \subset K(A).$$

Definition 2.1.3. *We say that* A *has an* ε*-net* $(\varepsilon > 0)$ *if there exists* $N \in \mathbb{N}$ *such that* $A \subset \bigcup_{i=1}^{N} B_\varepsilon(x_i)$ *and* $\{x_1, x_2, \cdots, x_N\} \subset X$.

Remark 2.1.4. *(a)* A *is totally bounded if and only if* A *has an* ε*-net for all* $\varepsilon > 0$.

(b) $H(A) = \{r > 0 : A$ *has an* r*-net*$\}$.

Definition 2.1.5. *1. The Kuratowski measure of noncompactness is defined by*

$$\alpha(A) := \inf(K(A)).$$

2. The Hausdorff measure of noncompactness is defined by

$$\chi(A) := \inf(H(A)).$$

Proposition 2.1.6. $\chi(A) \leq \alpha(A) \leq 2\chi(A)$.

Proof. From Remark 2.2.11, we have

$$2H(A) \subseteq K(A) \subseteq H(A).$$

Then

$$\inf(H(A)) \leq \inf(K(A)) \leq 2\inf(H(A))$$

and so

$$\chi(A) \leq \alpha(A) \leq 2\chi(A).$$

□

As a consequence $\alpha(A) = 0 \Leftrightarrow \chi(A) = 0$.

Proposition 2.1.7. $\chi(A) = 0 \Leftrightarrow$ *A is totally bounded.*

Proof.

$$\begin{aligned} \chi(A) = 0 \quad &\Leftrightarrow \quad \inf\{r > 0 : A \text{ has an } r-\text{net}\} = 0 \\ &\Leftrightarrow \quad A \text{ has an } r-\text{net},\ \forall\, r > 0 \\ &\Leftrightarrow \quad A \text{ totally bounded.} \end{aligned}$$

□

Definition 2.1.8. *A is relatively compact if $\bar{A}$ compact.*

Proposition 2.1.9. $0 \leq \chi(A) \leq \alpha(A) \leq \mathrm{diam(A)}$.

Proof. $A \subseteq A$ with $N = 1$ and $D = \mathrm{diam(A)}$. Then

$$\begin{aligned} \mathrm{diam(A)} \in \mathrm{K(A)} \quad &\Rightarrow \quad \inf(K(A)) \leq \mathrm{diam(A)}, \\ &\Rightarrow \quad \alpha(A) \leq \mathrm{diam(A)}, \\ &\Rightarrow \quad \chi(A) \leq \alpha(A) \leq \mathrm{diam(A)}. \end{aligned}$$

□

Proposition 2.1.10. *Let A and B be bounded subsets of a metric space X such that $A \subset B$. Then*

1. $\alpha(A) \leq \alpha(B)$.
2. $\chi(A) \leq \chi(B)$.

Proof. 1. Let $D \in K(B)$. Then

$$\exists N \in \mathbb{N},\ \exists (A_i)_{i=1}^{N} \subset X \text{ such that } \mathrm{B} \subseteq \bigcup_{\mathrm{i=1}}^{\mathrm{N}} \mathrm{A_i}$$
$$\text{with } \mathrm{diam(A_i)} \leq \mathrm{D},\ \forall\, 1 \leq \mathrm{i} \leq \mathrm{N}.$$

Since $A \subset B \subset \bigcup_{i=1}^{N} A_i$. Then

$$\begin{aligned} D \in K(A) \quad &\Rightarrow \quad K(B) \subset K(A), \\ &\Rightarrow \quad \inf K(A) \leq \inf K(B), \\ &\Rightarrow \quad \alpha(A) \leq \alpha(B). \end{aligned}$$

2. We show that $\chi(A) \leq \chi(B)$, by the same way as in part 1.

□

Proposition 2.1.11. $\alpha(A) = \alpha(\bar{A})$ *and* $\chi(A) = \chi(\bar{A})$.

Proof. Since $A \subseteq \bar{A}$, then by Proposition 2.1.10, $\alpha(A) \leq \alpha(\bar{A})$. Let $D \in K(A)$. Then there exist $N \in \mathbb{N}$, $(A_i)_{i=1}^{N} \subset X$ such that $A \subseteq \bigcup_{i=1}^{N} A_i$ with $\mathrm{diam(A_i)} \leq \mathrm{D},\ \forall\, 1 \leq \mathrm{i} \leq \mathrm{N}$. Hence

$$\bar{A} \subset \overline{\bigcup_{i=1}^{N} A_i} = \bigcup_{i=1}^{N} \bar{A}_i$$

with $\mathrm{diam}(\bar{A}_i) = \mathrm{diam}(A_i) \le D$, $\forall\, 1 \le i \le N$. As a consequence $D \in K(\bar{A})$ and so $K(A) \subseteq K(\bar{A}), \alpha(\bar{A}) \le \alpha(A)$. We conclude that $\alpha(A) = \alpha(\bar{A})$. Let us show that $\chi(A) = \chi(\bar{A})$. We have

$$A \subseteq \bar{A} \Rightarrow \chi(A) \le \chi(\bar{A}). \tag{2.1}$$

Let $r \in H(A)$. Then $A \subseteq \bigcup_{i=1}^{N} B_r(x_i)$, where $\{x_1, x_2, \cdots, x_N\} \subset X$. Then

$$\bar{A} \subset \overline{\bigcup_{i=1}^{N} B_r(x_i)} = \bigcup_{i=1}^{N} \overline{B_r(x_i)} = \bigcup_{i=1}^{N} B_r[x_i] \subset \bigcup_{i=1}^{N} B_{r+\varepsilon}(x_i),\ , \forall \varepsilon > 0.$$

Hence

$$r + \varepsilon \in H(\bar{A}),\ \forall \varepsilon > 0 \Rightarrow \chi(\bar{A}) \le r + \varepsilon,\ , \forall \varepsilon > 0,$$

which implies that $\chi(\bar{A}) \le r$. As a consequence $\chi(\bar{A}) \le \chi(A)$. We conclude that $\chi(A) = \chi(\bar{A})$. □

Corollary 2.1.12. *Let (X, d) be a complete metric space and A be a bounded subset of X. Then*

$$\alpha(A) = 0 \Leftrightarrow \chi(A) = 0 \Leftrightarrow A \text{ is relatively compact.}$$

Proof.

($\Rightarrow$) We have $\alpha(A) = \alpha(\bar{A}) = 0$, then by Proposition 2.1.7 $\bar{A}$ is totally bounded and since $\bar{A}$ is closed in a complete metric space, then $\bar{A}$ is compact.

($\Leftarrow$) Since A is relatively compact then $\bar{A}$ is compact. Hence $\bar{A}$ is totally compact and so

$$\chi(\bar{A}) = 0 \Rightarrow \chi(A) = 0.$$

□

Remark 2.1.13. *Let A and B be bounded subsets of a metric space X such that $A \subset B$. Then*

1. *B is relatively compact $\Rightarrow$ A is relatively compact.*
2. *Let A be a relatively compact subset of X. Then*

$$0 = \alpha(A) \le \alpha(B).$$

For that, α and χ are called measures of noncompactness.

Proposition 2.1.14. *Let A and B be bounded subsets of a metric space (X, d). Then*

1. $\alpha(A\cup B)=\max(\alpha(A),\alpha(B))$.

2. $\chi(A\cup B)=\max(\chi(A),\chi(B))$.

3. $\alpha(A\cap B)\le\min(\alpha(A),\alpha(B))$.

4. $\chi(A\cap B)\le\min(\chi(A),\chi(B))$.

Proof. 1. We have $A\subset A\cup B$ and $B\subset A\cup B$. Then

$\alpha(A)\le\alpha(A\cup B)$ and $\alpha(B)\le\alpha(A\cup B)$. Hence

$$\max(\alpha(A),\alpha(B))\le\alpha(A\cup B). \tag{2.2}$$

By the characteristic property of the infimum, we have

$$\forall\,\varepsilon>0,\ \exists D_\varepsilon,\ \exists N\in\mathbb{N},\ \exists(A_i)_{i=1}^N \text{ such that } A\subset\bigcup_{i=1}^N A_i$$

with

$$\mathrm{diam(A_i)\le D_\varepsilon,\ \forall\, i\in[1,N]\ \text{ and } D_\varepsilon<\alpha(A)+\varepsilon\le\max(\alpha(A),\alpha(B))+\varepsilon.}$$

Also there exist D'_ε, $M\in\mathbb{N}$, $(B_j)_{j=1}^M$ such that $B\subset\bigcup_{j=1}^M B_j$ with $\mathrm{diam(B_j)\le D'_\varepsilon,\ \forall\, j\in[1,M]}$ and $D'_\varepsilon<\alpha(B)+\varepsilon\le\max(\alpha(A),\alpha(B))+\varepsilon$. Then

$$A\cup B\subset\left(\bigcup_{i=1}^N A_i\right)\cup\left(\bigcup_{j=1}^M B_j\right)=\bigcup_{K=1}^{N+M} C_k,$$

where

$$C_k=\begin{cases} A_i, & \forall\, k\in[1,N];\\ B_j, & \forall\, k\in[N+1,M]\end{cases}$$

and $\mathrm{diam(C_k)<\max(\alpha(A),\alpha(B))+\varepsilon,\ \forall\,\varepsilon>0}$. So $\max(\alpha(A),\alpha(B))+\varepsilon\in K(A\cup B)$, $\forall\,\varepsilon>0$. Hence

$$\alpha(A\cup B)\le\max(\alpha(A),\alpha(B))+\varepsilon,\ \forall\,\varepsilon>0.$$

Then

$$\alpha(A\cup B)\le\max(\alpha(A),\alpha(B)). \tag{2.3}$$

From 2.2 and 2.3, we get $\alpha(A\cup B)=\max(\alpha(A),\alpha(B))$.

2. Clearly

$$\max(\chi(A),\chi(B))\le\chi(A\cup B). \tag{2.4}$$

Let $d = \max(\chi(A), \chi(B))$. Then by the characteristic property of the infimum, we have

$$\forall \varepsilon > 0,\ \exists r_\varepsilon > 0,\ \exists \{x_1, x_2, \cdots, x_N\} \subset X \text{ such that } A \subset \bigcup_{i=1}^{N} B_{r_\varepsilon}(x_i),$$

$$\exists r'_\varepsilon,\ \exists \{x_1, x_2, \cdots, x_M\} \subset X \text{ such that } B \subset \bigcup_{j=1}^{M} B_{r'_\varepsilon}(x_j)$$

with

$$r_\varepsilon < \chi(A) + \varepsilon \le d + \varepsilon \text{ and } r'_\varepsilon < \chi(B) + \varepsilon \le d + \varepsilon.$$

So $A \subset \bigcup_{i=1}^{N} B_{d+\varepsilon}(x_i)$ and $B \subset \bigcup_{j=1}^{M} B_{d+\varepsilon}(x_j)$. Hence

$$A \cup B \subset \bigcup_{k=1}^{N+M} B_{d+\varepsilon}(z_k), \text{ where } z_k = \begin{cases} x_i, & \forall k \in [1, N]; \\ x_j, & \forall k \in [N+1, M]. \end{cases}$$

Then $\chi(A \cup B) \le d + \varepsilon$, $\forall \varepsilon > 0$, and

$$\chi(A \cup B) \le d = \max(\chi(A), \chi(B)). \tag{2.5}$$

From 2.4 and 2.5 , we get $\chi(A \cup B) = \max(\chi(A), \chi(B))$.

3. We have $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Then

$$\alpha(A \cap B) \le \alpha(A) \text{ and } \alpha(A \cap B) \le \alpha(B).$$

$$\alpha(A \cap B) \le \min(\alpha(A), \alpha(B)).$$

4. We have $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Then

$$\chi(A \cap B) \le \chi(A) \text{ and } \chi(A \cap B) \le \chi(B) \Rightarrow \chi(A \cap B) \le \min(\chi(A), \chi(B)).$$

□

Lemma 2.1.15. *Let* $\mathcal{N}_r(A) = \{x \in X : d(x, A) < r\}$. *Then*

$$\mathrm{diam}(\mathcal{N}_r(A)) \le \mathrm{diam}(A) + 2r.$$

Proof. Let x and $y \in \mathcal{N}_r(A)$. Then $d(x, A) < r$ and $d(y, A) < r$. By the characteristic property of the infimum, we have

$$\forall \varepsilon > 0,\ \exists x_\circ \in A,\ \exists y_\circ \in A : d(x, x_\circ) < r + \varepsilon,\ d(y, y_\circ) < r + \varepsilon.$$

So for all $x, y \in \mathcal{N}_r(A)$, we have

$$\begin{aligned} d(x,y) &\leq d(x,x_\circ)+d(x_\circ,y_\circ)+d(y_\circ,y), \\ &< r+\varepsilon+\mathrm{diam(A)}+\mathrm{r}+\varepsilon \\ &= \mathrm{diam(A)}+2\mathrm{r}+2\varepsilon. \end{aligned}$$

Hence

$$\mathrm{diam}(\mathcal{N}_\mathrm{r}(\mathrm{A})) = \sup_{\mathrm{x,y}\in\mathcal{N}_\mathrm{r}(\mathrm{A})} \mathrm{d(x,y)} \leq \mathrm{diam(A)}+2\mathrm{r}+2\varepsilon,\ \forall\, \varepsilon>0.$$

Then $\mathrm{diam}(\mathcal{N}_\mathrm{r}(\mathrm{A})) \leq \mathrm{diam(A)}+2\mathrm{r}$. □

Lemma 2.1.16. *Let* $A = B_{r_\circ}(x_\circ)$. *Then*

$$\mathcal{N}_r(A) \subset B_{r_\circ+r}(x_\circ).$$

Proof. Let $x \in \mathcal{N}_r(A)$. Then

$$d(x,A) < r \Leftrightarrow \inf_{y\in A} d(x,y) < r.$$

So for all $y \in A$, we have since $y \in A$

$$d(x,x_\circ) \leq d(x,y)+d(y,x_\circ) \leq d(x,y)+r_\circ.$$

Hence

$$\begin{aligned} & d(x,x_\circ) \leq \inf_{y\in A} d(x,y)+r_\circ, \\ \Rightarrow\ & d(x,x_\circ) \leq d(x,A)+r_\circ, \\ \Rightarrow\ & d(x,x_\circ) < r+r_\circ, \\ \Rightarrow\ & x \in B_{r_\circ+r}(x_\circ). \end{aligned}$$

□

Lemma 2.1.17. *Let* $A \subseteq \bigcup_{i\in I} A_i \Rightarrow \mathcal{N}_r(A) \subseteq \bigcup_{i\in I} \mathcal{N}_r(A_i)$.

Proof. Let $x \in \mathcal{N}_r(A)$. Then $d(x,A) < r$ and thus $A \subseteq B \Rightarrow d(x,B) \leq d(x,A)$. Hence

$$\begin{aligned} & d := d(x, \bigcup_{i\in I} A_i) < r, \\ \Rightarrow\ & \exists\, i_\circ \in I : d(x, A_{i_\circ}) < r, \\ \Rightarrow\ & x \in \mathcal{N}_r(A_{i_\circ}) \Rightarrow x \in \bigcup_{i\in I} \mathcal{N}_r(A_i). \end{aligned}$$

Otherwise, assume by contradiction that $\forall\, i \in I$, $d(x, A_i) \geq r$. By the characteristic property of the infimum we have

$$\forall\, \varepsilon > 0,\ \exists\, y_\varepsilon \in \bigcup_{i\in I} A_i : d \leq d(x, y_\varepsilon) < d+\varepsilon.$$

Then

$$\forall \varepsilon > 0, \exists i_\circ \in I : y_\varepsilon \in A_{i_\circ},$$

with $r \leq d(x, A_{i_\circ}) \leq d(x, y_\varepsilon) < d + \varepsilon$. Hence $r < d + \varepsilon$, $\forall \varepsilon > 0 \Rightarrow r < d$, which is a contradiction. □

Proposition 2.1.18. *1.* $\chi(\mathcal{N}_r(A)) \leq \chi(A) + r$.

2. $\alpha(\mathcal{N}_r(A)) \leq \alpha(A) + 2r$.

Proof. 1. Let $\varepsilon > 0$. Then

$$\exists r_\circ > 0 : A \subset \bigcup_{i=1}^{N} B_{r_\circ}(x_i) \text{ with } \{x_1, x_2, \cdots, x_N\} \subset X.$$

By the characteristic property of the infimum,

$$\chi(A) \leq r_\circ < \chi(A) + \varepsilon.$$

By Lemma 2.1.17, we have

$$\mathcal{N}_r(A) \subseteq \bigcup_{i=1}^{N} \mathcal{N}_r(B_{r_\circ}(x_i)),$$

And by Lemma 2.1.16, we have

$$\mathcal{N}_r(B_{r_\circ}(x_\circ)) \subset B_{r_\circ + r}(x_\circ), \ \forall i \in [1, N].$$

Hence $\mathcal{N}_r(A) \subset \bigcup_{i=1}^{N} B_{r_\circ + r}(x_\circ)$, and so

$$r_\circ + r \in H(\mathcal{N}_r(A)) \Rightarrow \chi(\mathcal{N}_r(A)) \leq r_\circ + r < \chi(A) + \varepsilon + r, \ \forall \varepsilon > 0.$$

Then

$$\chi(\mathcal{N}_r(A)) \leq \chi(A) + r.$$

2. Let $\varepsilon > 0$. Then there exist $D_\varepsilon > 0$, $N \in \mathbb{N}$, $(A_i)_{i=1}^{N} \subset X$ such that $A \subset \bigcup_{i=1}^{N} A_i$ with $\mathrm{diam(A_i) \leq D_\varepsilon, \ \forall i \in [1, N]}$. Hence

$$\alpha(A) \leq D_\varepsilon < \alpha(A) + \varepsilon.$$

By Lemma 2.1.17, we have

$$\mathcal{N}_r(A) \subset \bigcup_{i=1}^{N} \mathcal{N}_r(A_i),$$

And by Lemma 2.1.15, we have

$$\mathrm{diam}(\mathcal{N}_r(A_i)) \le \mathrm{diam}(A_i) + 2r \le D_\varepsilon + 2r < \alpha(A) + \varepsilon + 2r,\ \forall\, \varepsilon > 0.$$

We conclude that

$$\forall\, \varepsilon > 0,\ \alpha(A) + 2r + \varepsilon \in K(\mathcal{N}_r(A)).$$

Then

$$\alpha(\mathcal{N}_r(A)) \le \alpha(A) + 2r + \varepsilon,\ \forall\, \varepsilon > 0.$$

As a consequence

$$\alpha(\mathcal{N}_r(A)) \le \alpha(A) + 2r.$$

□

Proposition 2.1.19. *(Cantor's Generalized Intersection Theorem) Let (X, d) be a complete metric space and $(F_n)_n$ a decreasing sequence of closed nonempty subsets of X such that $\lim\limits_{n\to\infty} \alpha(F_n) = 0$. Then $F_\infty := \bigcap\limits_n F_n$ is nonempty and compact.*

Proof. Let us show that $F_\infty \neq \phi$. Let $(x_n)_n \subset X$ be a sequence such that $x_n \in F_n$, $\forall\, n$ and $\mathcal{A}_n = \{x_n, x_{n+1}, \cdots\}$. Since $\mathcal{A}_n \subset F_n$, $\forall\, n$, then $\mathcal{A}_1 \subset \{x_1, \dots, x_n\} \bigcup F_n$. Hence

$$\alpha(\mathcal{A}_1) \le \alpha(F_n),\ \forall\, n.$$

Passing to the limit as $n \to \infty$, we get $\alpha(\mathcal{A}_1) = 0$, which implies that $\mathcal{A}_1$ is relatively compact. Hence the sequence $(x_n)_n$ has a limit point $x \in \bigcap\limits_n \bar{A}_n$. Since $\bigcap\limits_n \bar{A}_n \subset \bigcap\limits_n \bar{F}_n$, then $x \in \bigcap\limits_n F_n = F_\infty$.

Let us show that F_∞ is compact. Since $F_\infty \subset F_n\ \forall\, n$, then

$$0 \le \alpha(F_\infty) \le \alpha(F_n),\ \forall\, n.$$

Hence $\alpha(F_\infty) = 0,\ as\ n \to \infty$, that is F_∞ is relatively compact, hence compact. □

2.2 Measures of Noncompactness in Normed Spaces

Lemma 2.2.1. *Let $(X, \|\cdot\|)$ be a normed space, A, B two bounded subsets of X, and $\lambda \in \mathbb{R}$. Then,*

(a) $\mathrm{diam}(A + B) \le \mathrm{diam}(A) + \mathrm{diam}(B)$.

(b) $\mathrm{diam}(\lambda A) = |\lambda| \mathrm{diam}(A)$.

Proof.

(*a*) Let $x, y \in A+B$. Then there exist $a_1, a_2 \in A$, $b_1, b_2 \in B$ such that $x = a_1 + b_1$ and $y = a_2 + b_2$. Thus

$$\begin{aligned} \|x-y\| &= \|a_1+b_1-a_2-b_2\| \\ &\leq \|a_1-a_2\|+\|b_1-b_2\| \\ &\leq \text{diam(A)}+\text{diam(B)}. \end{aligned}$$

Then

$$\text{diam(A+B)} \leq \text{diam(A)}+\text{diam(B)}.$$

(*b*) Let $x, y \in \lambda A$ with $\lambda \neq 0$. Then

$$\exists\, a_1, a_2 \in A : x = \lambda a_1 \text{ and } y = \lambda a_2.$$

So

$$\|x-y\| = \|\lambda a_1 - \lambda a_2\| = |\lambda|\,\|a_1-a_2\| \leq |\lambda|\,\text{diam(A)}.$$

Then

$$\text{diam}(\lambda\text{A}) \leq |\lambda|\text{diam(A)}.$$

Conversely, let $a_1, a_2 \in A$. Then $\lambda a_1, \lambda a_2 \in \lambda A$. So

$$\|a_1-a_2\| = \|\frac{\lambda}{\lambda} a_1 - \frac{\lambda}{\lambda} a_2\| = \frac{1}{|\lambda|}\|\lambda a_1 - \lambda a_2\| \leq \frac{1}{|\lambda|}\text{diam}(\lambda\text{A}).$$

Then

$$\text{diam(A)} \leq \frac{1}{|\lambda|}\text{diam}(\lambda\text{A})$$

Hence $\text{diam}(\lambda\text{A}) = |\lambda|\text{diam(A)}$. □

Proposition 2.2.2. *Let γ be α or χ a measure of noncompactness in a normed space $(X, \|\cdot\|)$. Then for all $A, C \subset X$ and bounded and for all $\lambda \in \mathbb{R}^*$, we have*

(*a*) $\gamma(A+C) \leq \gamma(A)+\gamma(C)$ *(subaddivity)*

(*b*) $\gamma(A+\{x\}) = \gamma(A)$ *(invariance under shift)*

(*c*) $\gamma(\lambda A) = |\lambda|\gamma(A)$ *(homogeneity)*

Proof.

(a) Let us show that $\alpha(A+C) \le \alpha(A) + \alpha(C)$. Let $(A_i)_{i=1}^N$ and $(C_j)_{j=1}^M$ cover A and C respectively. Then $(A_i)_{i=1}^N + (C_j)_{j=1}^M$ cover $A+C$. Let $\varepsilon > 0$. Then there exist

$$D_1, D_2 > 0,\ (A_i)_{i=1}^N,\ (C_j)_{j=1}^M : A \subset \bigcup_{i=1}^{N} A_i,\ C \subset \bigcup_{j=1}^{M} C_j,$$

with $\mathrm{diam(A_i)} \le \mathrm{D_1},\ \forall \mathrm{i} \in [1,\mathrm{N}]$, $\mathrm{diam(C_j)} \le \mathrm{D_2},\ \forall \mathrm{j} \in [1,\mathrm{N}]$, and

$$\alpha(A) \le D_1 < \alpha(A) + \frac{\varepsilon}{2} \text{ and } \alpha(C) \le D_2 < \alpha(C) + \frac{\varepsilon}{2}.$$

By Lemma 2.2.1, we have

$$\begin{aligned}
\mathrm{diam(A_i + C_j)} &\le \mathrm{diam(A_i) + diam(C_j)},\ \forall\, \mathrm{i} \in [1,\mathrm{N}],\ \mathrm{j} \in [1,\mathrm{M}] \\
&\le D_1 + D_2, \\
&< \alpha(A) + \frac{\varepsilon}{2} + \alpha(C) + \frac{\varepsilon}{2},\ \forall\, \varepsilon > 0 \\
&= \alpha(A) + \alpha(C) + \varepsilon,\ \forall\, \varepsilon > 0.
\end{aligned}$$

Hence $\alpha(A+C) < \alpha(A)+\alpha(C)+\varepsilon,\ \forall\, \varepsilon > 0$, that is $\alpha(A+C) \le \alpha(A)+\alpha(C)$. Let us show that $\chi(A+C) \le \chi(A)+\chi(C)$. Let $\varepsilon > 0$. Then there exist $r_1, r_2 > 0$, $\{a_1, a_2 \cdots, a_N\} \subset X\, \{c_1, c_2 \cdots, c_j\} \subset X$ such that

$$A \subset \bigcup_{i=1}^{N} B_{r_1}(a_i) \text{ and } C \subset \bigcup_{j=1}^{M} B_{r_2}(c_j),$$

such that

$$\chi(A) \le r_1 < \chi(A) + \frac{\varepsilon}{2} \text{ and } \chi(C) \le r_1 < \chi(C) + \frac{\varepsilon}{2}.$$

Since $(B_{r_1}(a_i))_{i=1}^N$ cover A and $(B_{r_2}(c_j))_{j=1}^M$ cover C, then

$$A + C \subset \bigcup_{i=1}^{N} B_{r_1}(a_i) + \bigcup_{j=1}^{M} B_{r_2}(c_j).$$

Let us show that

$$\bigcup_{i=1}^{N} B_{r_1}(a_i) + \bigcup_{j=1}^{M} B_{r_2}(c_j) \subseteq \bigcup_{k=1}^{N+M} B_{r_1+r_2}(a_k + c_k),$$

where $a_k = 0,\ \forall\, k > N$ and $c_j = 0,\ \forall\, k \le N$. Let $x_\circ \in \bigcup_{i=1}^{N} B_{r_1}(a_i) + \bigcup_{j=1}^{M} B_{r_2}(c_j)$. Then there exist

$$i_\circ \in [1, N],\ j_\circ \in [1, M] : x_0 \in B_{r_1}(a_{i_\circ}) + B_{r_2}(c_{j_\circ}).$$

Hence

$$\exists\, t_1 \in B_{r_1}(a_{i_\circ}) \text{ and } \exists\, t_2 \in B_{r_2}(c_{j_\circ}) : x_\circ = t_1 + t_2,$$

which implies

$$\|t_1 - a_{i_\circ}\| < r_1 \text{ and } \|t_2 - c_{j_\circ}\| < r_2.$$

Then

$$\begin{aligned} \|x_\circ - (a_{i_\circ} + c_{j_\circ})\| &= \|t_1 + t_2 - a_{i_\circ} - c_{j_\circ}\|, \\ &\leq \|t_1 - a_{i_\circ}\| + \|t_2 - c_{j_\circ}\|, \\ &< r_1 + r_2. \end{aligned}$$

So

$$x_\circ \in B_{r_1+r_2}(a_{i_\circ} + c_{j_\circ}) \Rightarrow x_\circ \in \bigcup_{k=1}^{N+M} B_{r_1+r_2}(a_k + c_k).$$

Hence $A + C \subset \bigcup_{k=1}^{N+M} B_{r_1+r_2}(a_k + c_k)$ and then

$$\chi(A+C) \leq r_1 + r_2 < \chi(A) + \frac{\varepsilon}{2} + \chi(C) + \frac{\varepsilon}{2}, \ \forall \varepsilon > 0,$$

which implies $\chi(A+C) \leq \chi(A) + \chi(C)$.

(b) From (a) we have $\gamma(A + \{x\}) \leq \gamma(A) + \gamma(\{x\}) = \gamma(A)$. Note that $\gamma(\{x\}) \leq \mathrm{diam}(\{x\}) = 0 \Rightarrow \gamma(\{x\}) = 0$. Then

$$\begin{aligned} A = A + \{x\} + \{-x\} \Rightarrow \gamma(A) &= \gamma(A + \{x\} + \{-x\}) \\ &\leq \gamma(A + \{x\}) + \gamma(\{-x\}) \\ &\Rightarrow \gamma(A) \leq \gamma(A + \{x\}). \end{aligned}$$

Hence $\gamma(A + \{x\}) = \gamma(A)$.

(c) Let $(A_i)_{i=1}^N$ cover A. Then $(\lambda A_i)_{i=1}^N$ cover λA. Indeed

$$\lambda A \subset \lambda \bigcup_{i=1}^N A_i = \bigcup_{i=1}^N \lambda A_i.$$

Let $\varepsilon > 0$, then $\exists D > 0$, $\exists (A_i)_{i=1}^N$, such that $A \subset \bigcup_{i=1}^N A_i$ with $\mathrm{diam}(A_i) \leq D$, $\forall i \in [1, N]$, such that $\alpha(A) \leq D < \alpha(A) + \varepsilon$. Then, for all $\varepsilon > 0$

$$\mathrm{diam}(\lambda A_i) = |\lambda| \, \mathrm{diam}(A_i) \leq |\lambda| D < |\lambda| (\alpha(A) + \varepsilon) = |\lambda| \, \alpha(A) + |\lambda| \varepsilon.$$

Then, for all $\varepsilon > 0$

$$\alpha(\lambda A) \leq |\lambda| \, \alpha(A) + |\lambda| \varepsilon.$$

Hence

$$\alpha(\lambda A) \leq |\lambda|\, \alpha(A). \tag{2.6}$$

Let $\varepsilon > 0$, then there exists $D' > 0$, and $(\lambda A_i)_{i=1}^{M}$ such that $\lambda A \subset \bigcup_{i=1}^{M} \lambda A_i$ with $\mathrm{diam}(\lambda \mathrm{A_i}) \leq$ D', $\forall\, i \in [1, M]$ and

$$\alpha(\lambda A) \leq D' < \alpha(\lambda A) + \varepsilon.$$

Then

$$\mathrm{diam}(\mathrm{A_i}) = \frac{\mathrm{diam}(\lambda \mathrm{A_i})}{|\lambda|} \leq \frac{\mathrm{D}'}{|\lambda|} < \frac{\alpha(\lambda \mathrm{A})}{|\lambda|} + \frac{\varepsilon}{|\lambda|},\ \forall\, \varepsilon > 0.$$

So

$$\alpha(A) \leq \frac{\alpha(\lambda A)}{|\lambda|} + \frac{\varepsilon}{|\lambda|} \Rightarrow \alpha(A) \leq \frac{\alpha(\lambda A)}{|\lambda|},\ \forall\, \varepsilon > 0.$$

Hence

$$|\lambda|\, \alpha(A) \leq \alpha(\lambda A). \tag{2.7}$$

From (2.6) and (2.7), we deduce that

$$\alpha(\lambda A) = |\lambda|\, \alpha(A),\ \forall\, \lambda \in \mathbb{R}.$$

Let us show that $\chi(\lambda A) = |\lambda|\, \chi(A)$. Firstly, we will show that

(a) $\bigcup_{i=1}^{N} \lambda B_r(a_i) = \bigcup_{i=1}^{N} B_{|\lambda| r}(\lambda a_i)$, where $(a_i)_{i=1}^{N} \subset X$. Let $x \in \bigcup_{i=1}^{N} \lambda B_r(a_i)$. Then there exists $i_\circ \in [1, N] : x \in \lambda B_r(a_i)$. Hence there exists $y \in B_r(a_i) : x = \lambda y$ and $\|y - a_{i_\circ}\| < r$. As a consequence

$$\|x - \lambda a_{i_\circ}\| = \|\lambda y - \lambda a_{i_\circ}\| = |\lambda|\, \|y - a_{i_\circ}\| < |\lambda|\, r.$$

Hence $x \in B_{|\lambda| r}(\lambda a_{i_\circ}) \Rightarrow x \in \bigcup_{i=1}^{N} B_{|\lambda| r}(\lambda a_i)$. Conversely, let $x \in \bigcup_{i=1}^{N} B_{|\lambda| r}(\lambda a_i)$. Then there exists $i_\circ \in [1, N] : x \in B_{|\lambda| r}(\lambda a_{i_\circ})$. Hence

$$\begin{aligned}
& \|x - \lambda a_{i_\circ}\| < r|\lambda| \Rightarrow |\lambda|\, \|\frac{x}{\lambda} - a_{i_\circ}\| < r\,|\lambda| \\
\Rightarrow\ & \|\frac{x}{\lambda} - a_{i_\circ}\| < r \\
\Rightarrow\ & \frac{x}{\lambda} \in B_r(a_{i_\circ}) \Rightarrow x \in \lambda B_r(a_{i_\circ}).
\end{aligned}$$

(b) $r \in H(A) \Rightarrow |\lambda|\, r \in H(\lambda A)$.

$$\begin{aligned}
r \in H(A) \quad & \Rightarrow \quad \exists N \in \mathbb{N},\ \exists \{a_1, a_2, \cdots, a_N\} \subset X : A \subset \bigcup_{i=1}^{N} B_r(a_i) \\
& \Rightarrow \quad \lambda A \subset \bigcup_{i=1}^{N} \lambda B_r(a_i) \subset \bigcup_{i=1}^{N} B_{|\lambda| r}(\lambda a_i),\ (by\ (a)) \\
& \Rightarrow \quad r\,|\lambda| \in H(\lambda A).
\end{aligned}$$

(c) If $r \in H(\lambda A)$, then

$$\begin{aligned} & \exists N \in \mathbb{N},\ \exists\{\lambda a_1, \lambda a_2, \cdots, \lambda a_N\} \subset X : \lambda A \subset \bigcup_{i=1}^{N} B_r(\lambda a_i) \\ \Rightarrow\ & A \subset \bigcup_{i=1}^{N} \frac{1}{\lambda} B_r(\lambda a_i) \subset \bigcup_{i=1}^{N} B_{\frac{r}{|\lambda|}}(a_i),\ (by\ (a)) \\ \Rightarrow\ & \frac{r}{|\lambda|} \in H(A). \end{aligned}$$

Secondly, let $\varepsilon > 0$. Then there exist $r_1, > 0$ and $\{a_1, a_2, \cdots, a_N\} \subset X$ such that $A \subset \bigcup_{i=1}^{N} B_{r_1}(a_i)$ and

$$\chi(A) \le r_1 < \chi(A) + \varepsilon.$$

By using (b), we get

$$\begin{aligned} r_1 \in H(A) &\Rightarrow |\lambda| r_1 \in H(\lambda A) \\ &\Rightarrow \chi(\lambda A) \le |\lambda| r_1 < |\lambda| \chi(A) + |\lambda|\varepsilon,\ \forall \varepsilon > 0. \\ &\Rightarrow \chi(\lambda A) \le |\lambda| \chi(A). \end{aligned}$$

Conversely, let $\varepsilon > 0$. Then there exist $r_2 > 0$, $\{\lambda a_1, \lambda a_2, \cdots, \lambda a_N\} \subset X$ such that $\lambda A \subset \bigcup_{i=1}^{N} B_{r_1}(\lambda a_i)$ and

$$\chi(\lambda A) \le r_2 < \chi(\lambda A) + \varepsilon.$$

By using (c), we get

$$\begin{aligned} r_2 \in H(\lambda A) &\Rightarrow \frac{r_2}{|\lambda|} \in H(A) \\ &\Rightarrow \chi(A) \le \frac{r_2}{|\lambda|} < \frac{\chi(\lambda A) + \varepsilon}{|\lambda|} < \frac{\chi(\lambda A)}{|\lambda|} + \frac{\varepsilon}{|\lambda|},\ \forall \varepsilon > 0. \\ &\Rightarrow |\lambda| \chi(A) < \chi(\lambda A). \end{aligned}$$

Hence $\chi(\lambda A) = |\lambda| \chi(A)$. □

Definition 2.2.3. *A set A is convex if $\lambda x + (1-\lambda) y \in A$, $\forall x, y \in A$ and $\lambda \in [0,1]$.*

Definition 2.2.4. *The convex hull of A, denoted $conv(A)$, is the smallest convex set that contains A) (i.e. $conv\ A$ is the intersection of all convex sets containing A.*

Proposition 2.2.5. *[1, 2, 3] (Invariance under the convex hull). Let A be a subset of a normed space and $\gamma = \alpha$ or χ. Then*

$$\gamma(A) = \gamma(conv\ A).$$

Proof. $A \subset conv\ A \Rightarrow \gamma(A) \leq \gamma(conv\ A)$. The converse is based on the following fact

1. $\text{conv } A = \{\sum_{i=1}^{n} \lambda_i a_i,\ a_i \in A,\ \lambda_i > 0,\ \sum_{i=1}^{n} \lambda_i = 1\}$.
2. diam(A) = diam(conv A).

□

Lemma 2.2.6. *Let* $(X, \|\cdot\|)$ *be a normed space and* $x_\circ \in X$. *Then*

$$B(x_\circ, R) = \{x_\circ\} + RB(0,1).$$

Proof. $y \in B(x_\circ, R) \Leftrightarrow \|y - x_\circ\| < R$. Let $u = \frac{y - x_\circ}{R}$. Then $u \in B(0,1)$ and $y = x_\circ + Ru \in \{x_\circ\} + RB(0,1)$. □

Lemma 2.2.7. *(Riesz Theorem) [7] A normed linear space is finite-dimensional if and only if the closed unit ball is compact.*

Proposition 2.2.8. *Let* $B = B(0,1)$ *be the unit ball in a normed space* $(X, \|\cdot\|)$. *Then*

$$\chi(B) = \begin{cases} 0, & if\ dim(X) < \infty; \\ 1, & if\ dim(X) = \infty. \end{cases}$$

Proof. By Riesz Lemma, we have

$$\begin{aligned} dim(X) < \infty &\Leftrightarrow B(0,1) \text{ relatively compact}, \\ &\Leftrightarrow \chi(B) = 0 \text{ (for } X \text{ is complete)}. \end{aligned}$$

Assume that diam(X) = ∞. Then

$$B(0,1) \subset B(0,1) \Rightarrow 1 \in H(B) \Rightarrow \chi(B) \leq 1.$$

We need to prove that $\chi(B) = 1$. By contradiction, assume that $\chi(B) < 1$ and let $0 < \varepsilon < 1 - \chi(B)$. Then there exits $r > 0$, $N \in \mathbb{N}$, $\{x_1, x_2, \cdots, x_N\} \subset X$ such that $B \subset \bigcup_{i=1}^{N} B_r(x_i)$ and

$$\chi(B) \leq r < \chi(B) + \varepsilon < 1.$$

Since $B \subset \bigcup_{i=1}^{N} B_r(x_i)$, then

$$\begin{aligned} \chi(B) &\leq \max_{1 \leq i \leq N} \chi(B_r(x_i)) \\ &= \max_{1 \leq i \leq N} \chi(\{x_\circ\} + rB(0,1)) \text{ (by Lemma 2.2.6)} \\ &= \chi(rB(0,1)) = r\chi(B(0,1)). \end{aligned}$$

By Riesz Theorem, $\chi(B) \neq 0$, which is a contradiction with $1 > r$, and so $\chi(B) = 1$. □

Corollary 2.2.9. *Let $(X, \|\cdot\|)$ be a normed space and $B = B(x_\circ, r) \subset X$. Then*

$$\chi(B) = \begin{cases} 0, & if\ dim(X) < \infty; \\ r, & if\ dim(X) = \infty. \end{cases}$$

Proof. By Lemma 2.2.6, we know that $B(x_\circ, r) = \{x_\circ\} + rB(0,1)$. Then

$$\begin{aligned} \chi(B(x_\circ, r)) &= \chi(\{x_\circ\} + rB(0,1)), \\ &= \chi(rB(0,1)), \\ &= r\chi(B(0,1)), \\ &= \begin{cases} 0, & if\ dim(X) < \infty, \\ r, & if\ dim(X) = \infty. \end{cases} \end{aligned}$$

□

Lemma 2.2.10. *Let $(X, \|\cdot\|)$ be a normed space and S be the sphere of the unit ball $B(0,1)$. Then*

$$conv\ (S) = \bar{B}(0,1)$$

Proof. Clearly $S \subset B_1[0] = B(0,1)$. So $B_1[0]$ convex implies that, by definition, $conv\ (S) \subseteq B_1[0] = \bar{B}(0,1)$. Indeed, let $x, y \in B_1[0]$ and $\lambda \in [0,1]$. Then

$$\begin{aligned} \|\lambda y + (1-\lambda)x\| &\leq \|\lambda y\| + \|(1-\lambda)x\| \\ &= |\lambda|\,\|y\| + |1-\lambda|\,\|x\| \\ &\leq \lambda + 1 - \lambda = 1. \end{aligned}$$

So $\lambda y + (1-\lambda)x \in B_1[0] \Rightarrow B_1[0]$ is convex.

Let us show $B_1[0] \subset conv\ (S)$. Let $x \in B_1[0]$ and $\lambda = \frac{1+\|x\|}{2}$. Then $\lambda \in (0,1]$ and $x = \lambda \frac{-x}{\|x\|} + (1-\lambda)\frac{x}{\|x\|}$, where $\frac{\pm x}{\|x\|} \in S$. Hence $x \in conv\ (S) \Rightarrow B_1[0] = \bar{B}(0,1) \subseteq conv\ (S)$. We conclude that $conv\ (S) = \bar{B}(0,1)$. □

Remark 2.2.11. *By Lemma 2.2.10 and Proposition 2.2.5, we conclude that $\gamma(S) = \gamma(conv\ (S)) = \gamma(\bar{B}(0,1)) = \gamma(B(0,1))$.*

Lemma 2.2.12. *(Ljusternik-Schrinelman-Borsuk Theorem)[9] Let S be the sphere in a normed space X with $dim(X) = n$. Then, for every covering $(A_i)_{i=1}^n$ by closed sets, there exists at least one set $A_{i_\circ}$ that contains two antipodal points of the sphere S.*

Proposition 2.2.13. *Let* $(X, \|\cdot\|)$ *be a normed space and* $B = B(0,1)$ *be the unit ball in* X*. Then*

$$\alpha(B) = \begin{cases} 0, & if\ dim(X) < \infty; \\ 2, & if\ dim(X) = \infty. \end{cases}$$

Proof. By Riesz Lemma, we have

$$\begin{aligned} dim(X) \leq \infty \quad &\Rightarrow \quad B(0,1) \text{ is relatively compact,} \\ &\Rightarrow \quad \alpha(B) = 0. \end{aligned}$$

Assume that $dim(X) = \infty$. Then by Proposition 2.2.8

$$\chi(B) \leq \alpha(B) \leq 2\chi(B) \Rightarrow \alpha(B) \leq 2.$$

Suppose by contradiction, that $\alpha(S) = \alpha(B) < 2$.(by Remark 2.2.11) Then $\forall \varepsilon \in (0, 2 - \alpha(S))$, $\exists D > 0$, $\exists (A_i)_{i=1}^{N}$ (chosen closed): $S \subset \bigcup_{i=1}^{N} A_i$ with $\mathrm{diam}(\mathrm{A_i}) < \alpha(\mathrm{S}) + \varepsilon < 2$, $\forall \mathrm{i} \in [1, \mathrm{N}]$. Let $L = \{x_1, x_2, \cdots, x_N\}$ be a linearly independent subset of X and $E = [L]$. Then $\mathrm{diam(E)} = \mathrm{N}$. Let $S_N = \{x \in E : \|x\| = 1\}$. Then $S \cap S_N = S_N \subset \bigcup_{i=1}^{N} (S_N \cap A_i)$ with $\mathrm{diam}(\mathrm{S_N} \cap \mathrm{A_i}) \leq \mathrm{diam}(\mathrm{A_i}) < 2$, $\forall \mathrm{i} \in [1, \mathrm{N}]$. This is a contradiction with Lemma 2.2.12. So $\alpha(B) = 2$. □

Related Mappings and Fixed Point Theorems

3.1 Related Mappings

Definition 3.1.1. *Let* (X,d), (X',d') *be two metric spaces and* $f:(X,d)\to(X',d')$ *a mapping.*

(a) *We say that* f *is Lipschitz if*

$$\exists\, k\geq 0 : d'(f(x),f(y))\leq kd(x,y),\ \forall\, x,y\in X.$$

(b) *We say that* f *is contraction if* f *is Lipschitzian with* $0\leq k<1$.

Hereafter $\mathscr{P}_B(X)$ will denote the family of all bounded subsets of X.

Remark 3.1.2.

(a) *Every Lipschitzian function is uniformly continuous. Recall that* f *is uniformly continuous, if*

$$\forall\,\varepsilon>0,\ \exists\,\delta_\varepsilon>0,\ \forall\, x,y\in X : d(x,y)<\delta\Rightarrow d'(f(x),f(y))<\varepsilon.$$

By taking $\delta=\frac{\varepsilon}{k}$, *where* k *is the constant of Lipschitzian, we find that* f *is uniformly continuous.*

(b) $f\ k-Lipschitz \Leftrightarrow \forall A \in \mathscr{P}_B(X)$, $\mathrm{diam(f(A))} \leq \mathrm{k\,diam(A)}$. *Indeed*

$$\begin{aligned} f\ k-Lipschitz &\Rightarrow \exists k \geq 0 : d'(f(x), f(y)) \leq k\,d(x,y),\ \forall x,y \in A. \\ &\Rightarrow d'(f(x), f(y)) \leq k \sup_{x,y\in A} d(x,y) = k\,\mathrm{diam(A)}. \\ &\Rightarrow \mathrm{diam(f(A))} \leq \mathrm{k\,diam(A)}. \end{aligned}$$

Conversely, let $A = \{x, y\} \in \mathscr{P}_B(X)$. *Then*

$$\begin{aligned} \mathrm{diam(f(A))} \leq \mathrm{k\,diam(A)} &\Rightarrow d'(f(x), f(y)) \leq k\,d(x,y) \\ &\Rightarrow f\ is\ k-Lipschitz. \end{aligned}$$

Definition 3.1.3. *Let $A, B \subset X$ be bounded subsets. The Hausdorff distance between A and B is defined by*

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\},$$

where $d(a,B) = \inf_{b\in B} d(a,b)$ *and* $d(b,A) = \inf_{a\in A} d(b,a)$.

Proposition 3.1.4. $H_d(A,B) = \inf\{r > 0 : A \subset \mathcal{N}_r(B)$ *and* $B \subset \mathcal{N}_r(A)\}$.

Proof. Let $D = H_d(A,B)$ and $F = \{r > 0 : A \subset \mathcal{N}_r(B)$ and $A \subset \mathcal{N}_r(A)\}$. By definition of the Hausdorff distance, for all $a \in A$, we have

$$d(a,B) \leq \sup_{a\in A} d(a,B) \leq D < D + \varepsilon,\ \forall \varepsilon > 0.$$

Then

$$a \in \mathcal{N}_{D+\varepsilon}(B),\ \forall a \in A \Rightarrow A \subset \mathcal{N}_{D+\varepsilon}(B).$$

Likewise $B \subset \mathcal{N}_{D+\varepsilon}(A)$. Hence $D + \varepsilon \in F$, $\forall \varepsilon > 0$. Then, $\inf(F) \leq D + \varepsilon,\ \forall \varepsilon > 0$ and so

$$\inf(F) \leq D. \tag{3.1}$$

Let $r \in F$. Then

$$\begin{aligned} \forall a \in A &\Rightarrow a \in \mathcal{N}_r(B), \\ &\Rightarrow d(a,B) < r,\ \forall a \in A. \\ &\Rightarrow \sup_{a\in A} d(a,B) < r,\ \forall r \in F. \\ &\Rightarrow \sup_{a\in A} d(a,B) \leq \inf(F). \end{aligned} \tag{3.2}$$

Likewise

$$\sup_{b\in B} d(b,A) \le \inf(F). \tag{3.3}$$

From (3.2) and (3.3) we get

$$D = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\} \le \inf(F). \tag{3.4}$$

Then by (3.1) and (3.4) we conclude that $D = \inf(F)$. □

Proposition 3.1.5. *H_d is a distance over $\mathscr{P}_{cl}(X)$ the family of all closed subsets of X.*

Proof. 1. (i) $H_d(A,B) \ge 0,\ \forall\, A,B \in \mathscr{P}_{cl}(X)$.

(ii) $A = B \Rightarrow H_d(A,B) = 0$

$$\begin{aligned} H_d(A,B) = 0 \quad &\Rightarrow \quad d(a,B) = 0,\ \forall\, a \in A \text{ and } d(b,A) = 0,\ \forall\, b \in B.\\ &\Rightarrow \quad a \in \bar{B}\ \forall\, a \in A \text{ and } b \in \bar{A},\ \forall b \in B.\\ &\Rightarrow \quad A \subset \bar{B} \text{ and } B \subset \bar{A}\\ &\Rightarrow \quad \bar{A} = \bar{B}\\ &\Rightarrow \quad A = B, \text{ if } A \text{ and } B \text{ are closed.} \end{aligned}$$

2. $H_d(A,B) = H_d(B,A)$.

3. Let $a \in A,\ b \in B$, and $c \in C$. Then

$$\begin{aligned} & d(a,B) \le d(a,b) \le d(a,c) + d(c,b),\ \forall\, b \in B.\\ \Rightarrow\ & d(a,B) \le d(a,c) + d(c,B),\ \forall\, c \in C,\\ & \text{(by passing to the infimum over } b \in B).\\ \Rightarrow\ & d(a,B) \le d(a,c) + \sup_{c\in C} d(c,B),\ \forall\, c \in C.\\ \Rightarrow\ & d(a,B) \le d(a,C) + \sup_{c\in C} d(c,B),\ \forall\, a \in A.\\ \Rightarrow\ & d(a,B) \le \sup_{a\in A} d(a,C) + \sup_{c\in C} d(c,B).\\ \Rightarrow\ & \sup_{a\in A} d(a,B) \le \sup_{a\in A} d(a,C) + \sup_{c\in C} d(c,B).\\ \le\ & H_d(A,C) + H_d(C,B). \end{aligned}$$

Inverting the roles of A and B, we get

$$\sup_{b\in B} d(b,A) \le H_d(B,C) + H_d(C,A).$$

Finally

$$H_d(A,B) \le H_d(A,C) + H_d(C,B), \ \forall\, A,B,C \in \mathscr{P}_{cl}(X).$$

Hence H_d is a distance over $\mathscr{P}_{cl}(X)$.

□

Proposition 3.1.6. *Let $A, B \subset X$ be such that A and B are bounded. Then*

(a) $|\alpha(A) - \alpha(B)| \le 2H_d(A,B)$.

(b) $|\chi(A) - \chi(B)| \le H_d(A,B)$.

Proof.

(a) By Proposition 3.1.4, we have

$H_d(A,B) = \inf\{r > 0 : A \subset \mathscr{N}_r(B) \text{ and } B \subset \mathscr{N}_r(A)\}$.

Let $r \in F := \{r > 0 : A \subset \mathscr{N}_r(B) \text{ and } B \subset \mathscr{N}_r(A)\}$. Then $A \subset \mathscr{N}_r(B)$ and $B \subset \mathscr{N}_r(A)$, which implies that

$$\alpha(A) \le \alpha(\mathscr{N}_r(B)) \text{ and } \alpha(B) \le \alpha(\mathscr{N}_r(A)).$$

Using Proposition 2.1.18, we get

$$\alpha(A) \le \alpha(B) + 2r \ \text{ and } \ \alpha(B) \le \alpha(A) + 2r.$$

Then

$$|\alpha(A) - \alpha(B)| \le 2r, \forall\, r \in F.$$

Hence

$$|\alpha(A) - \alpha(B)| \le 2\inf(F) = 2H_d(A,B).$$

(b) From the proof of (a), we get $\chi(A) \le \chi(\mathscr{N}_r(B))$ and $\chi(B) \le \chi(\mathscr{N}_r(A))$. Hence

$$\Rightarrow \chi(A) \le \chi(B) + r \; and \; \chi(B) \le \chi(A) + r,$$

that is

$$|\chi(A) - \chi(B)| \le r, \forall\, r \in F \Rightarrow |\chi(A) - \chi(B)| \le \inf(F) = H_d(A,B).$$

□

Remark 3.1.7.

(a) $\alpha, \chi : \mathscr{P}_{cl}(X) \to \mathbb{R}^+$ *are Lipschitz functions with constants* 2 *and* 1 *respectively.*

(b) *If* $(X,d)=(X,\|\cdot\|)$ *and* $\gamma=\alpha$ *or* χ, *then*

$$|\gamma(A)-\gamma(B)|\le\gamma(B(0,1)).H_d(A,B).$$

Definition 3.1.8. *A function* $f:(X,d)\to(X',d')$ *is said to be compact if* $\overline{f(A)}$ *is compact,* $\forall\, A\in\mathscr{P}_B(X)$. *If* f *compact and continuous, then it is called completely continuous.*

Lemma 3.1.9. *Let* $f:(E,\|\cdot\|_E)\to(F,\|\cdot\|_F)$ *be a linear mapping. Then*

f *continuous* $\Leftrightarrow$ f *bounded* $\Leftrightarrow$ f *bounded over the unit ball.*

Proposition 3.1.10. *Let* $f:(E,\|\cdot\|_E)\to(F,\|\cdot\|_F)$ *be a linear mapping. Then*

(a) f *compact* $\Rightarrow$ f *continuous.*

(b) f *continuous with* $dim(E)<\infty\Rightarrow f$ *compact*

Proof.

(a) Let B be the unit ball. Then

$$\begin{aligned} f\text{ compact} &\Rightarrow \overline{f(B)}\text{ compact},\\ &\Rightarrow \overline{f(B)}\text{ bounded},\\ &\Rightarrow f(B)\text{ bounded}.\end{aligned}$$

By Lemma 3.1.9, we conclude that f is continuous.

(b) Let $A\in\mathscr{P}_B(X)$. Then

$$\begin{aligned} f\text{ continuous} &\Rightarrow f(A)\text{ bounded (by Lemma 3.1.9)}\\ &\Rightarrow \overline{f(A)}\text{ closed and bounded}\\ &\Rightarrow \overline{f(A)}\text{ compact } (dim(E)\le\infty).\\ &\Rightarrow f(A)\text{ compact}.\end{aligned}$$

□

Definition 3.1.11. *Let* $f:(X,d)\to(X',d')$ *be a bounded mapping and* $\gamma\,(\alpha$ *or* $\chi)$ *be a measure of noncompactness. Then*

(a) f *is called a* k*-set contraction, if there exists* $k\ge 0$, *such that*

$$\gamma(f(A))\le k\gamma(A),\ \forall\, A\in\mathscr{P}_B(X).$$

(b) *f is called a 1-set contraction, if $k = 1$.*

(c) *f is called a strict k-set contraction if $0 \leq k < 1$.*

(d) *f is called a condensing, if $\forall\, A \in \mathscr{P}_B(X)$ with $\gamma(A) > 0$, we have $\gamma(f(A)) < \gamma(A)$.*

Remark 3.1.12.

(a) *If f is a strict k-set contraction, then f is condensing, then f is a 1-set contraction where f is continuous and X is complete. Indeed let $A \in \mathscr{P}_B(X)$ with $\gamma(A) > 0$. Then, since f is a strict k-set contraction, there exists $0 \leq k < 1$ such that $\gamma(f(A)) \leq k\gamma(A) < \gamma(A)$, that is f is condensing.*

(b) *Suppose that f is condensing, continuous, and X is complete. Then*

- *if $\gamma(A) > 0$, then $\gamma(f(A)) \leq \gamma(A) \Rightarrow f$ 1-set contraction,*
- *if $\gamma(A) = 0$, then $\bar{A}$ is compact for X is complete. Hence $f(\bar{A})$ is compact for f is continuous. As a consequence $\gamma(f(A)) = 0 \leq \gamma(A)$ for $\overline{f(A)} \subset f(\bar{A})$ and $\gamma(f(\bar{A})) = 0)$.*

(c) *f compact $\Leftrightarrow$ f 0-set contraction, whenever (X', d') is complete. Indeed*

$$\begin{aligned} f \text{ compact} &\Rightarrow \overline{f(A)} \text{ compact, } \forall\, A \in \mathscr{P}_B(X), \\ &\Rightarrow \gamma(f(A)) = \gamma(\overline{f(A)}) = 0, \\ &\Rightarrow f\ 0-\text{set contraction.} \end{aligned}$$

Conversely

$$\begin{aligned} f\ 0-\text{set contraction} &\Rightarrow \gamma(\overline{f(A)}) = \gamma(f(A)) = 0,\ \forall\, A \in \mathscr{P}_B(X), \\ &\Rightarrow \overline{f(A)})\text{ compact, (since } X' \text{ is complete)}, \\ &\Rightarrow f \text{ compact.} \end{aligned}$$

(c) *Let $f : (E, \|\cdot\|_E) \to (F, \|\cdot\|_F)$ be a k-set contraction, and $g : (E, \|\cdot\|_E) \to (F, \|\cdot\|_F)$ be a compact function. Then $f + g$ is a k-set contraction. Indeed let $A \in \mathscr{P}_B(E)$. We have*

$$\begin{aligned} \gamma((f+g)(A)) &= \gamma(f(A) + g(A)), \\ &\leq \gamma(f(A)) + \gamma(g(A)) \\ &= \gamma(f(A)) + 0, \\ &\leq k\gamma(A). \end{aligned}$$

Hence $f + g$ is a k-set contraction.

Proposition 3.1.13. *f k-Lipschitz $\Rightarrow$ f k-set contraction (with respect to the kuratowski MNC).*

Proof. Let $A \in \mathscr{P}_B(X)$. Then

$$\forall \varepsilon > 0, \exists D_\varepsilon > 0, \exists N \in \mathbb{N}, \exists \{A_1, A_2, \cdots, A_N\} \subset X : A \subset \bigcup_{i=1}^{N} A_i,$$

with $\mathrm{diam}(A_i) \leq D_\varepsilon$, $\forall i \in [1, N]$ such that $\alpha(A) \leq D_\varepsilon < \alpha(A) + \varepsilon$. We have

$$f(A) \subset f\left(\bigcup_{i=1}^{N} A_i\right) \subset \bigcup_{i=1}^{N} f(A_i).$$

Then

$$\alpha(f(A)) \leq \alpha\left(\bigcup_{i=1}^{N} f(A_i)\right) \leq \max_{1\leq i\leq N} \alpha(f(A_i)) \leq \max_{1\leq i\leq N} \mathrm{diam}(f(A_i)).$$

By Remark 3.1.2, (b) we have

$$\begin{aligned} \alpha(f(A)) \leq \max_{1\leq i\leq N} \mathrm{diam}(f(A_i)) &\leq \max_{1\leq i\leq N} k\,\mathrm{diam}(A_i), \text{ (for } f \text{ is Lipschitz).} \\ &\leq kD_\varepsilon < k(\alpha(A) + \varepsilon), \ \forall \varepsilon > 0 \end{aligned}$$

Hence $\alpha(f(A)) \leq k\,\alpha(A)$. □

Remark 3.1.14. *In case of the Hausdorff MNC, we can find*

$$f\ k - \textit{Lipschitz} \Rightarrow f\ 2k - \textit{set contraction.}$$

Proposition 3.1.15.

$$\begin{aligned} \textit{Let } f : (X_1, d_1) &\to (X_2, d_2) \textit{ be a } k_1\textit{-set contraction and} \\ g : (X_2, d_2) &\to (X_3, d_3) \textit{ be a } k_2\textit{-set contraction. Then} \\ g \circ f : (X_1, d_1) &\to (X_3, d_3) \textit{ is a } k_1.k_2\textit{-set contraction.} \end{aligned}$$

Proof. Let $A \in \mathscr{P}_B(X_1)$. Then

$$\begin{aligned} \gamma(g(f(A))) &\leq k_2\,\gamma(f(A)) \text{ (for } g \text{ is } k_2\text{-set contraction).} \\ &\leq k_2.k_1\,\gamma(A) \text{ (for } f \text{ is } k_1\text{-set contraction).} \end{aligned}$$

□

Proposition 3.1.16.

$$\begin{aligned} \textit{Let } f : (X_1, \|\cdot\|_{X_1}) &\to (X_2, \|\cdot\|_{X_2}) \textit{ be a } k_1\textit{-set contraction and} \\ g : (X_1, \|\cdot\|_{X_1}) &\to (X_2, \|\cdot\|_{X_2}) \textit{ be a } k_2\textit{-set contraction. Then} \\ f + g : (X_1, \|\cdot\|_{X_1}) &\to (X_2, \|\cdot\|_{X_2}) \textit{ is a } (k_1 + k_2)\textit{-set contraction.} \end{aligned}$$

Proof. Let $A \in \mathscr{P}_B(X_1)$. Then

$$\begin{aligned} \gamma(f(A)+g(A)) &\leq \gamma(f(A))+\gamma(g(A)) \\ &\leq k_1\gamma(A)+k_2\gamma(A) \\ &= (k_1+k_2)\gamma(A). \end{aligned}$$

□

3.2 Fixed Point Theorems

Definition 3.2.1. *A fixed point of a function is an element of the function domain that is mapped to itself by the function, i.e., $x \in D$ and $x = f(x)$.*

Theorem 3.2.2. *(Brouwer's Fixed Point Theorem)[6, 9] Let $C \subset \mathbb{R}^n$ be a nonempty compact convex subset and $f : C \to C$ a continuous function. Then f has a fixed point.*

Example 3.2.3. *Let $f : [a,b] \to [a,b]$ be a continuous function where $[a,b] \subset \mathbb{R}$. Then f has a fixed point. Indeed, $[a,b]$ is compact convex subset of $\mathbb{R}$ and f is continuous, Then by Brouwer's Fixed point Theorem, we conclude that f has a fixed point. Recall that the Intermediat Value Theorem states that if $f : [a,b] \to \mathbb{R}$ continuous and $f(a)f(b) < 0$, then there exists $c \in (a,b) : f(c) = 0$. Let $g(x) = f(x) - x : [a,b] \to \mathbb{R}$. Then g is continuous and, if $g(a) = 0$ or $g(b) = 0$, then we are done. Otherwise $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$. By the Intermediate Value Theorem, there exists $c \in (a,b) : g(c) = 0 \Leftrightarrow f(c) = c$. Hence f has a fixed point.*

Remark 3.2.4. *Brouwer's Fixed Point Theorem is valid in any finite dimensional normed space.*

Theorem 3.2.5. *(Schauder's Fixed Point Theorem)[6, 9] Let X be a Banach space, $C \subset X$ a nonempty bounded closed convex subset, $f : C \to C$ be continuous and $f(C)$ is compact. Then f has a fixed point in C.*

Corollary 3.2.6. *(Schauder's Theorem Second Version) Let X be a Banach space and $C \subset X$ a nonempty compact convex subset. Then every continuous $f : C \to C$ has at least one fixed point.*

Proof. Since C is compact, then C is closed and bounded. In addition f continuous implies that $f(C)$ compact. By Schauder's Theorem, f has a fixed point. □

Remark 3.2.7. *Let X be a Banach space. If $C \neq \phi$ is a bounded closed convex subset of X and $f : C \to C$ is continuous, then f has not necessary a fixed point as shows the following counter-example. Let $X = \{x = (x_n)_n$ real sequence : $\lim\limits_{n\to\infty} x_n = 0\}$ with the norm $\|x\| = \sup\limits_{n\geq 1} |x_n|$, and $C = \bar{B}(0,1)$ (C is bounded, closed and convex). X being complete, define the mapping $f : C \to C$ by $f(x) = \left(\frac{1+\|x\|}{2}, x_1, \cdots, x_n, \cdots\right)$.*

Claim 1 : *$f(C) \subset C$. We have*

$$\|f(x)\| = \max\{\frac{1+\|x\|}{2}, \|x\|\} = \frac{1+\|x\|}{2} \leq 1, \ \forall\, x \in C.$$

Hence $f(x) \in C$, $\forall\, x \in C$.

Claim 2 : *f is continuous. Let $x, y \in C$ and $\|f(x) - f(y)\| = \max\{\frac{|\,\|x\|-\|y\|\,|}{2}, \|x-y\|\}$. We have $\|x - y\| \geq \left|\,\|x\| - \|y\|\,\right| \geq \left|\frac{\|x\|-\|y\|}{2}\right|$. So $\|f(x) - f(y)\| = \|x - y\|$. By taking $\delta = \varepsilon$, we can see that f is uniformly continuous.*

Claim 3 : *f is fixed point free. Assume by contradiction that $f(x) = x$. Then,*

$$\begin{aligned} x_1 &= \frac{1+\|x\|}{2} \\ x_2 &= x_1 \\ &\vdots \\ x_n &= x_{n-1} \\ &\vdots \end{aligned}$$

Hence $x_1 = x_2 = \cdots = x_n = \frac{1+\|x\|}{2}$. Since $\lim\limits_{n\to\infty} x_n = \frac{1+\|x\|}{2} \neq 0$, then $x \notin X$.

Theorem 3.2.8. *(Banach's Fixed Point Theorem)[9] Let (X, d) be a complete metric space and $f : X \to X$ (or $f : C \to C$ with $\phi \neq C \subset X$ closed) a contractive mapping. Then f has a unique fixed point.*

Proof. To show the uniqueness, suppose that f has two fixed points x and y such that $x \neq y$. Then

$$\begin{aligned} d(x,y) = d(f(x), f(y)) &\leq k\,d(x,y), \text{ with } 0 < k < 1, \\ &< d(x,y), \text{ for } d(x,y) \neq 0, \end{aligned}$$

which is a contradiction. □

Theorem 3.2.9. *(Darbo's Fixed Point Theorem) Let X be a Banach space and $C \subset X$ be nonempty closed bounded convex and $f : C \to C$ be a continuous and strict k-set contraction. Then f has at least one fixed point.*

Remark 3.2.10. *This theorem encompasses Schauder and Banach fixed point theorems. Also we can consider the sum of a compact and contractive mapping.*

Proof. Define recurrently a sequence of sets by

$$\begin{cases} C_\circ = C, \\ C_{n+1} = \overline{conv}(f(C_n)). \end{cases}$$

We have

- C_n is closed convex, $\forall\, n \geq 0$.
- $(C_n)_n$ is decreasing. Indeed

  $$C_1 = \overline{conv}(f(C_\circ)) = \overline{conv}(f(C)) \subset \overline{conv}(C) = C_\circ$$

  for C is closed convex and $f(C) \subset C$. By induction, if $C_n \subset C_{n-1}$, then:

  $$\overline{conv}(f(C_n)) \subset \overline{conv}(f(C_{n-1})) \Rightarrow C_{n+1} \subset C_n.$$

- $\lim\limits_{n\to\infty} \alpha(C_n) = 0$. Indeed, there exists $0 \leq k < 1$ such that

  $$\begin{aligned} \alpha(C_{n+1}) &= \alpha(\overline{conv}(f(C_n))) = \alpha(conv(f(C_n))) \\ &= \alpha(f(C_n)), \\ &\leq k\,\alpha(C_n), \\ &\leq k^2\,\alpha(C_{n-1}), \\ &\vdots \\ &\leq k^{n+1}\,\alpha(C_\circ). \end{aligned}$$

  Since $0 \leq k < 1$, then $\lim\limits_{n\to\infty} \alpha(C_{n+1}) = 0$.

- Conclusion: by the generalized Cantor's intersection theorem, $C_\infty = \bigcap\limits_n C_n$ is compact and nonempty. Also we have C_∞ is convex as intersection of convex sets. Let us show that $f(C_\infty) \subset C_\infty$. For all $x \in C_\infty$, we have

  $$\begin{aligned} x \in C_n,\ \forall\, n \quad &\Rightarrow \quad f(x) \in f(C_n),\ \forall\, n, \\ &\Rightarrow \quad f(x) \in C_{n+1},\ \forall\, n, \\ &\Rightarrow \quad f(x) \in C_\infty. \end{aligned}$$

  Hence $f(C_\infty) \subset C_\infty$. Since $f : C_\infty \to C_\infty$ is continuous, then, by Schauder's Theorem (second version), there exists $x \in C_\infty : f(x) = x$.

□

Theorem 3.2.11. *(Sadovskii's Fixed Point Theorem) Darbo's Fixed point Theorem still holds for condensing mapping.*

Proof. Let $x_\circ \in C$ and $\mathscr{C}$ denotes the class of all bounded closed convex subsets F of C such that $x_\circ \in F$ and $f(F) \subset F$. Let $A = \bigcap_{F\in\mathscr{C}} F$ and $B = \overline{conv}(f(A) \cup \{x_\circ\})$. Then

- $x_\circ \in C \Rightarrow C \in \mathscr{C} \Rightarrow \mathscr{C} \neq \phi$.
- $x_\circ \in A \Rightarrow A \neq \phi$.
- $A \subset C \Rightarrow A$ bounded.

Moreover

$$f(A) = f\left(\bigcap_{F\in\mathscr{C}} F\right) \subset \bigcap_{F\in\mathscr{C}} f(F) \subset \bigcap_{F\in\mathscr{C}} F = A.$$

Also we have

- $x_\circ \in A$ and $f(A) \subset A$, then $(f(A) \cup \{x_\circ\}) \subset A$.
- A is convex and closed.

$$\text{Hence } B \subseteq A. \tag{3.5}$$

Then $f(B) \subset f(A) \subset \overline{conv}(f(A) \cup \{x_\circ\}) = B$. Also B is closed convex subset of C and $x_\circ \in B$.

$$\text{Hence } A \subseteq B. \tag{3.6}$$

From (3.5) and (3.6), we get $A = B$. Since f is condensing, then

$$\alpha(A) = \alpha(B) = \alpha(f(A)) < \alpha(A), \text{ if } \alpha(A) > 0,$$

a contradiction. Then, $\alpha(A) = 0 \Rightarrow A$ compact. A is nonempty closed bounded convex subset of X, and $f : A \to A$ continuous and $f(A)$ is compact. By Schauder's Theorem Second Version, f has a fixed point. □

4 Applications to Nonlinear Integral Equations

4.1 Existence of Local Solutions

Let $I \subset \mathbb{R}$ be an interval, $t_\circ \in I^\circ$, and

$$f, g : I \times \mathbb{R}^n \to \mathbb{R}^n$$

be two continuous functions such that g is contractive with respect to the second argument:

$$\exists\, k \in [0,1) : \|g(t,x) - g(t,y)\| \le k\,\|x-y\|,\ \forall\, t \in I,\ \forall\, x, y \in \mathbb{R}^n.$$

Theorem 4.1.1. *Then the nonlinear integral equation*

$$x(t) = g(t, x(t)) + \int_{t_\circ}^{t} f(s, x(s))\, ds, \quad t_\circ, t \in I \tag{4.1}$$

has at least one local solution, i.e., there exists $\delta > 0$ and a solution x defined on $I_\delta = [t_\circ - \delta, t_\circ + \delta] \subset I$.

Remark 4.1.2. *If the interval I is left bounded and $t_\circ$ is the left end point, then take $I_\delta = [t_\circ, t_\circ + \delta]$.*

Corollary 4.1.3. *(Peano's Theorem) Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and $(t_\circ, x_\circ) \in I \times \mathbb{R}^n$. Then, the initial value problem (Cauchy problem)*

$$\begin{cases} x'(t) = f(t, x(t)), \quad t, t_0 \in I, \\ x(t_0) = x_0 \end{cases} \tag{4.2}$$

has at least one local solution.

Remark 4.1.4. *If, in Corollary 4.1.3, f is locally Lipschitz, then (4.2) has a unique local solution. This is Cauchy-Lipschits local existence theorem. The proof of uniqueness is checked as follows. Given two possible solutions x and y, we have*

$$\begin{aligned} \|x(t) - y(t)\| &= \left\| \int_{t_0}^{t} \left[f(s, x(s)) - f(s, y(s)) \right] ds \right\|, \\ &\leq k \left| \int_{t_0}^{t} \|x(s) - y(s)\| \, ds \right| := k Z(t). \end{aligned}$$

Then $\forall\, t \geq t_0$, $Z'(t) = \|x(t) - y(t)\|$ and thus

$$Z'(t) \leq k Z(t) \Leftrightarrow (Z' - kZ)(t) \leq 0 \Leftrightarrow (Z(t) e^{-kt})' \leq 0.$$

Hence

$$0 \leq Z(t) e^{-kt} \leq Z(t_0) e^{-kt_0} = 0 \Rightarrow x(t) = y(t), \ \forall\, t \geq t_0.$$

A similar argument leads to $x(t) = y(t)$, $\forall\, t \leq t_0$.

Proof. We will prove Theorem 4.1.1 in two steps.

Step 1. Functional setting. Let $a > 0$, $\bar{g} = \sup\limits_{|t - t_0| \leq a} \|g(t, 0)\|$, and $b > \frac{\bar{g}}{1-k}$. Let $C = [t_0 - a, t_0 + a] \times B[0, b]$ be a cylinder. Since f is continuous, there exists $M > 0$ such that $\|f(t, x)\| < M$, $\forall\, (t, x) \in C$ Let $0 < \delta \leq \min(a, \frac{(1-k)b - \bar{g}}{M})$ and $J = [t_0 - \delta, t_0 + \delta]$. Consider the space $X = \mathscr{C}(J, \mathbb{R}^n)$ equipped with the norm $\|x\|_X = \sup\limits_{t \in J} \|x(t)\|$ and $D = \mathscr{C}(J, B[0, b])$. Then X is a Banach space. Since $\mathscr{C}(J, \mathbb{R}^n) = \mathscr{C}_b(J, \mathbb{R}^n)$ and from Example 1.1.8, we conclude that X is a Banach space. Moreover, $D = B_X[0, b]$. Indeed, $B_X[0, b] \subset D$ and let $x \in D$. Then x is a continuous function and $\forall\, t \in J$, $x(t) \in B[0, b]$, we have

$$\sup_{t \in J} \|x(t)\| \leq b \Rightarrow \|x\|_X \leq b \Rightarrow x \in B_X[0, b] \Rightarrow D \subset B_X[0, b].$$

Hence $D = B_X[0, b]$. So D is closed bounded and convex subset of X. Define the nonlinear mappings $F, G : D \to X$ by

$$Fx(t) = \int_{t_0}^{t} f(s, x(s)) \, ds \text{ and } Gx(t) = g(s, x(t)).$$

F is called a Hammerstein operator and G the Nemytskii operator associated with f and g respectively. Then Fx and Gx are continuous functions and we have

$$\begin{aligned} x \text{ is solution of equation (4.1)} &\Leftrightarrow x(t) = Fx(t) + Gx(t),\ \forall\, t \in I. \\ &\Leftrightarrow x = Fx + Gx, \\ &\Leftrightarrow x \text{ fixed point of the sum } (F+G). \end{aligned}$$

Step 2 : The mapping $F+G$ satisfies Darbo's Fixed Point Theorem.

(a) $(F+G)(D) \subset D$. For all $x \in D$, we have

$$\begin{aligned} \|Fx+Gx\|_X &= \sup_{t\in J} \|(Fx+Gx)(t)\|, \\ &= \sup_{t\in J} \left\| g(t,x(t)) + \int_{t_0}^{t} f(s,x(s))\,ds \right\|, \\ &\leq \sup_{t\in J} \Big[\|g(t,x(t)) - g(t,0)\| + \|g(t,0)\| \\ &\quad + \left| \int_{t_0}^{t} \|f(s,x(s))\|\,ds \right| \Big] \\ &\leq \sup_{t\in J} \left[k\,\|x(t)\| + \left\|g(t,0)\right\| + M|t-t_0| \right] \\ &\leq k\,\|x\|_X + M\delta + \bar{g} \\ &\leq kb + M\delta + \bar{g} \leq b. \end{aligned}$$

(b) G is a k-contraction because g is. Let $x, y \in D$. We have

$$\begin{aligned} \|Gx - Gy\|_X &= \sup_{t\in J} \|g(t,x(t)) - g(t,y(t))\| \\ &\leq k \sup_{t\in J} \|x(t) - y(t)\| \\ &= k\,\|x-y\|_X. \end{aligned}$$

Hence G is continuous.

(c) F is continuous. We check that F is sequentially continuous. Let $(x_n)_n \subset X$ be a sequence such that $\lim_{n\to\infty} x_n = x$. (i.e. $\lim_{n\to\infty} \sup_{t\in J} \|x_n(t) - x(t)\| = 0$). We have

$$\begin{aligned} \|Fx_n(t) - Fx(t)\| &= \left\| \int_{t_0}^{t} [f(s,x_n(s)) - f(s,x(s))]\,ds \right\| \\ &\leq \left| \int_{t_0}^{t} \|f(s,x_n(s)) - f(s,x(s))\|\,ds \right|. \end{aligned}$$

Since $(x_n)_n \overset{n\to\infty}{\longrightarrow} x$ in X for the sup-norm (i.e. uniformly convergent on J) and f is continuous, then $(f(\cdot,x_n(\cdot)))_n$ converges uniformly to $f(\cdot,x(\cdot))$. So

$$\lim_{n\to\infty} \|f(s,x_n(s)) - f(s,x(s))\| = 0,\ \forall\, s\in J.$$

Hence

$$\lim_{n\to\infty} \|Fx_n(t) - Fx(t)\|_X = 0,\ \forall\, t\in J.$$

We shall make use of an important compactness criterion.

Lemma 4.1.5 (**Ascoli-Arzela Lemma**). *[7] Let E,F be two metric spaces such that E is compact and F is complete, and $H\subset \mathscr{C}(E,F)$ be bounded. We have*

$$H \text{ relatively compact } \Leftrightarrow \begin{cases} H \text{ equicontinuous.} \\ \forall\, t\in E,\ H(t) \text{ is relatively compact in } F. \end{cases}$$

Definition 4.1.6. *H is equicontinuous if* $\forall\,\varepsilon>0,\ \exists\,\alpha=\alpha(\varepsilon)>0,\ \forall\, t,s\in E: d_E(s,t)<\alpha \Rightarrow d_F(f(s),f(t))<\varepsilon,\ \forall\, f\in H$

Corollary 4.1.7. *If F is a Banach space with* diam(F) $<\infty$, *then for every bounded subset H, we have*

$$H\subset\mathscr{C}(E,F) \text{ relatively compact} \Leftrightarrow H \text{ equicontinuous.}$$

We will apply this Corollary with $E=J$ and $F=\mathbb{R}^n$. Hence

$$H = F(D)\subset \mathscr{C}(J,\mathbb{R}^n).$$

For all $y\in F(D)$, there exists $x\in D: y=Fx$. For $\varepsilon>0$ and $t,s\in J$, we have

$$\begin{aligned} \|y(t)-y(s)\| &= \|Fx(t)-Fx(s)\| \\ &\le \left| \int_t^s \|f(u,x(u))\|\,du \right| \\ &\le M|t-s| < M\,\alpha < \varepsilon. \end{aligned}$$

If $|t-s|<\alpha$ and $0<\alpha<\frac{\varepsilon}{M}$. Hence $F(D)$ is equicontinuous.

(e) Conclusion. Since G is contraction and $F(D)$ relatively compact, then $(F+G)$ is strict k-set contraction. Since $F+G: D\to D$ and D nonempty, closed, bounded, and convex subset, then we conclude, by Darbo's Fixed Point Theorem, that $F+G$ has at least one fixed point $x\in D$ a solution of Equation (4.1) defined on J and continuous. □

4.2 Existence of Global Solutions

Let $I = [a, b]$ be a compact interval of the real line and $t_0 \in I$. Consider two continuous functions $f, g : I \times \mathbb{R}^n \to \mathbb{R}^n$, such that
(H_g) g is a contraction in the second variable.
(H_f) There exist $l \in L^1(I)$ and $\sigma > 0$ such that

$$\|f(t,x)\| \le l(t)(1+\|x\|^\sigma),\ \forall\,(t,x) \in I \times \mathbb{R}^n,$$

with either $(0 < \sigma < 1)$ or $(\sigma = 1$ and $k + \|l\|_1 < 1)$.

We have

Theorem 4.2.1. *Under Assumptions (H_g) and (H_f), the nonlinear integral equation:*

$$x(t) = g(t, x(t)) + \int_{t_0}^{t} f(s, x(s))\,ds, \quad t, t_0 \in I \tag{4.3}$$

has at least one global solution defined on I.

Corollary 4.2.2. *(Cauchy-Lipschitz Theorem) Suppose that $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz function with respect to the second variable. Then the initial value problem*

$$\begin{cases} x'(t) = f(t,x),\ t, t_0 \in I,\ x_0 \in \mathbb{R}^n, \\ x(t_0) = x_0. \end{cases} \tag{4.4}$$

has a unique solution defined on I provided that $k + \|f(\cdot, 0)\| < 1$.

Proof. We know that

$$(4.4) \Leftrightarrow x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds.$$

We let $g(t,x) = x_0$. Then

$$\begin{aligned} \|f(t,x)\| &\le \|f(t,x) - f(t,0)\| + \|f(t,0)\|, \\ &\le k\|x\| + \|f(t,0)\|, \\ &\le (k + \|f(t,0)\|)\|x\| + \|f(t,0)\| + k, \\ &= (k + \|f(t,0)\|)(1 + \|x\|). \end{aligned}$$

Since I is bounded, the function $l(t) = k + \|f(t,0)\| \in L^1(I)$. The uniqueness follows as in Corollary 4.1.4. □

Remark 4.2.3.

(a) *We can take* $l(t) = \max\{k, \|f(t,0)\|\}$.

(b) *We can also take* $x_\circ$ *instead of* 0, *with* $l(t) = \max\{k, 2k\|x_0\|, 2\|f(t,x_0)\|\}$.

Proof. We prove Theorem 4.2.1 in two steps.

Step 1 : Functional setting. Let $X = \mathscr{C}(I, \mathbb{R}^n)$ be the Banach space endowed with the sup-norm

$$\|x\|_X = \sup_{t\in I} \|x(t)\|.$$

Define the mappings $F, G : X \to X$, by

$$Fx(t) = \int_{t_0}^{t} f(s, x(s))\, ds,\; Gx(t) = g(t, x(t)),\; t \in I.$$

As in Theorem 4.1.1, we can prove that F and G are continuous and G is a contraction. Let $D = B_X[0,R] \subset X$ be a closed ball with radius $R > 0$ (to be determined). As Theorem 4.1.1, $F(D)$ is relatively compact by Ascoli-Arzela Lemma. As a consequence, $F+G$ is a strict k-set contraction. Thus, if we can find some $R > 0$ such that $(F+G)(D) \subset D$, Darbo's Fixed Point Theorem applies and provides a fixed point $x \in D : x = Fx + Gx \Leftrightarrow \forall\, t \in I,\; x(t) = Fx(t) + Gx(t)$, which implies that x solution of (4.3).

Step 2 : $(F+G)(D) \subset D$. Let $x \in D = B_X[0,R]$. Then

$$\begin{aligned}
\|Fx + Gx\|_X &= \sup_{t\in I} \|Fx(t) + Gx(t)\|,\\
&\le \sup_{t\in I}\left[\left|\int_{t_0}^{t} \|f(s,x(s))\|\, ds\right| + \|g(t,x(t))\|\right],\\
&\le \sup_{t\in I}\left|\int_{t_0}^{t} l(s)(1+\|x(s)\|^{\sigma})\, ds\right|\\
&\quad + \sup_{t\in I} k\|x(t)\| + \sup_{t\in I}\|g(t,0)\|,\\
&\le \|l\|_1(1+\|x(s)\|^{\sigma}) + kR + \bar{g},\\
&\le \|l\|_1(1+R^{\sigma}) + kR + \bar{g} \le R,
\end{aligned}$$

whenever $\sigma = 1$ and $\frac{\|l\|_1+\bar{g}}{1-(\|l\|_1+k)} \le R$, or, $0 < \sigma < 1$ in which case, $(\|l\|_1 + \bar{g})R^{-\sigma} + \|l\|_1 < (1-k)R^{1-\sigma}$, which is valid for R large enough. □

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