Kingdom of Saudi Arabia
Imam Mohammad Ibn Saud Islamic University
College of Science

Department of Mathematics and Statistics

Master of Science in Mathematics

Research Project Report (MAT699)

On Transfer Operators for Chaotic Maps

By

Hadeel Alhatlani

Supervisor:

Prof. Dr. Mohamed Hmissi

Academic Year: 1441-1442

Contents

Acknowledgement

In completing this project, many people have helped me. I would like to thank all those who are related to this project.
Primarily, I would thank God for being able to complete this project. Then I will thank

Prof. Dr. Mohamed Hmissi

who gave me this opportunity to work on this project. I got to learn a lot from this project.
Finally, I must also thank my parents for the immense support and help during this project. Without their help, completing this project would have been very difficult.

Introduction

The evolution of many real-life phenomena is modeled by a recurrent sequence of the form

$$x_{n+1} = \tau(x_n); \quad x_0 \in X$$

where X is Borel subset of $\mathbb{R}^d$ and $\tau : X \to X$ is a transformation of X.
A natural problem is to predict the future of such a system.
The transformation τ is said to be deterministic, if each associated recurrent sequence converges to some $l \in X$, which is always a fixed point of τ. In this case, the system starting from the state x_0, evolves over time to states $x_1, x_2 \cdots$ and so one. And for n big enough, the state x_n approaches the final state l.
If τ is not deterministic, it is called chaotic (in elementary sense). For example, in one dimensional case, piecewise monotonic and not monotonic maps are always chaotic. For chaotic maps, the states of some recurrent sequence never approach a final state. In fact, the states x_n evolve chaotically in the set X even for n big enough. Thus, one cannot predict the future for such system by considering recurrent sequences of points.
In order to solve this problem, we consider a large number of recurrent sequences at the same time. This means that we pick a large number of initial states:

$$I_0 = \{x_0^1, x_0^2, ..., x_0^N\} \subset X$$

The density associated to I_0 is the function $f_0 : X \to [0, \infty)$ which is integrable and such that

$$\int_A f_0(x)dx = \frac{1}{N} \sum_{j=1}^{N} 1_A(x_0^j)$$

for any Borel subset $A \subset X$. Now, let $I_1 = \{x_1^1, x_1^2, ..., x_1^N\}$ where

$$x_1^1 = \tau(x_0^1), x_1^2 = \tau(x_0^2), ..., x_1^N = \tau(x_0^N)$$

and let f_1 be the density defined by I_1. Then

$$\begin{aligned}\int_A f_1(x)dx &= \frac{1}{N}\sum_{j=1}^{N} 1_A(x_1^j) = \frac{1}{N}\sum_{j=1}^{N} 1_A(\tau(x_0^j)) \\ &= \frac{1}{N}\sum_{j=1}^{N} 1_{\tau^{-1}(A)}(x_0^j) = \int_{\tau^{-1}(A)} f_0(x)dx\end{aligned}$$

Hence by induction, we obtain a recurrent sequence of densities (f_n) on X such that

$$\int_A f_{n+1}(x)dx = \int_{\tau^{-1}(A)} f_n(x)dx$$

where f_n is the density associated the the states $I_n = \{x_n^1, x_n^2, ..., x_n^N\}$ with

$$x_n^1 = \tau^n(x_0^1), x_n^2 = \tau^n(x_0^2), ..., x_n^N = \tau^n(x_0^N)$$

Using the Radon-Nikodym Theorem, we introduce in this project the so called transfer operator of τ; namely $P_\tau : L^1(X) \to L^1(X)$ defined implicitly by the formula

$$\int_A P_\tau f(x)dx = \int_{\tau^{-1}(A)} f(x)dx$$

for any Borel subset A of X. Hence the recurrent relation of (f_n) becomes

$$f_{n+1} = P_\tau f_n$$

If a subsequence of (f_n) converges in $L^1(X)$ to some $f_\star$, then $f_\star$ is an invariant density of P_τ, that is $f_\star = P_\tau f_\star$ and the distribution f_n of I_n approaches the final distribution $f_\star$. In this case, the measure absolutely continuous invariant $\nu(dx) := f_\star(x)dx$ is τ-invariant.
In this project, we also prove the existence of such measures for expanding piecewise monotonic maps. Finally, we study similar problems for random maps.

Chapter 1

Basics on Lebesgue Integration

In this chapter, we review some essential concepts from measure theory and the theory of Lebesgue integration. In particular, we present the Radon-Nikodym theorem which allow to introduce in the next chapter, the concept of transfer operator. For more details, we will refer to the monographs [1, 5, 8].

1.1 Measure spaces

Definition 1.1. Let X be a nonempty set. A collection $\mathcal{A}$ of subsets of X is called a *σ-algebra* of X if:

1. $X \in \mathcal{A}$;
2. $\mathcal{A}$ is stable by completion, that is, if $A \in \mathcal{A}$ then $X \setminus A \in \mathcal{A}$;
3. $\mathcal{A}$ is σ-stable, that is, for any finite or infinite sequence $\{A_k\}$ of $\mathcal{A}$, the union $\cup_k A_k \in \mathcal{A}$.

In this case, the pair $(X, \mathcal{A})$ is called *measurable space* and elements of $\mathcal{A}$ are called be *measurable sets*.

It follows immediately that

4. $\emptyset \in \mathcal{A}$;
5. For any finite or infinite sequence $\{B_k\}$ of $\mathcal{A}$, the intersection $\cap_k B_k \in \mathcal{A}$.

If X is a topological space then it is always endowed with its Borel σ-algebra $\mathcal{A} = \sigma(\mathcal{O})$ (generated by $\mathcal{O}$), the smallest σ-algebra of X which contains all open subsets $O \in \mathcal{O}$. $A \in \mathcal{A} = \sigma(O)$ is called a Borel set.
Notice that the Borel σ-algebra of $X = \mathbb{R}$ is also generated by all intervals of $\mathbb{R}$

Definition 1.2. A *measure* on a measurable space $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ satisfying

(i) $\mu(\emptyset) = 0$;

(ii) for any finite or in finite sequence $\{A_k\}$ of pairwise disjoint set of $\mathcal{A}$ (that is $A_i \cap A_j = \emptyset$ for $i \neq j$)

$$\mu(\bigcup_k A_k) = \sum_k \mu(A_k) \tag{1.1}$$

In this case $(X, \mathcal{A}, \mu)$ is called a *measure space.*

Notice that

$$0 \leq \mu(A) \leq \infty; \quad A \in \mathcal{A}$$

Moreover, $(X, \mathcal{A}, \mu)$ is called

- *finite* if $\mu(X) < \infty$;
- *probabilistic* (or *normalized*) if $\mu(X) = 1$;
- *σ-finite* if there is an infinite sequence $\{A_k\}$ of sets of A such that $X = \cup_k A_k$ and $\mu(A_k) < \infty$ for all k.

Examples 1.3. 1- Let $X = \mathbb{R}$ and let $\mathcal{A} = \sigma(I) = \sigma(O)$ the Borel σ-algebra of $\mathbb{R}$. The *Lebesgue measure* μ on $\mathbb{R}$ is the unique measure satisfying

$$\mu([a, b]) \;=\; b - a; \quad a, b \in \mathbb{R}, a < b$$

$(\mathbb{R}, \mathcal{A}, \mu)$ is a σ-finite measure space.
2- Let $(X, \mathcal{A}, \mu)$ be a measure space and let Y be a measurable space subset of X. Put $\mathcal{A}_Y = \{A \cap Y : A \in \mathcal{A}\}$ and $\mu_Y(A) = \mu(A)$ for any $A \in \mathcal{A}, A \subset Y$. Then $(Y, \mathcal{A}_Y, \mu_Y)$ is measure space called restriction of $(X, \mathcal{A}, \mu)$ to Y. It is also denoted by $(Y, \mathcal{A}, \mu)$ for simplicity.

3- Let $X = \mathbb{R}$, let $\mathcal{A}$ be the Borel σ-algebra, and let μ the Lebesgue measure on $Y = [0,1]$ then $([0,1], \mathcal{A}, \mu)$ is a probability space.
4- Let $X = \mathbb{R}$ endowed with its Borel σ-algebra. Consider the real valued function ν on $\mathcal{A}$ defined by

$$\nu(A) \;=\; 1_A(0); \quad A \in \mathcal{A} \tag{1.2}$$

Then ν is a probability measure on $(\mathbb{R}, \mathcal{A})$, called Dirac measure at 0. It is always denoted by δ_0.

Remark 1.4. In this project, we consider a σ-finite measure space $(X, \mathcal{A}, \mu)$. Always X is a Borel subset of $\mathbb{R}^d; d \geq 1$, $\mathcal{A}$ is the Borel σ-algebra of X, and often μ is the Lebesgue measure on $(X, \mathcal{A})$.
For the most applications, X will be a compact interval.

1.2 Lebesgue Integration

Let $(X, \mathcal{A}, \mu)$ be a measure space.
If a property involving points of X is true except for a subset having measure zero, we say that this property is true *almost everywhere* (a.e).
Let $f, g : X \to \mathbb{R}$. Define

$$f^+(x) = max(0, f(x)) \quad and \quad f^-(x) = max(0, -f(x))$$

We have

$$f = f^+ - f^- \quad and \quad |f| = f^+ + f^- \tag{1.3}$$

Definition 1.5. *A real-valued function $f : X \to \mathbb{R}$ is said to be measurable if $f^{-1}(I) \in \mathcal{A}$ for any interval $I \subset \mathbb{R}$.*

Examples 1.6. 1- If f is measurable, then f^+, f^- and $|f|$ are measurable.
2- If X is a topological space endowed with its Borel σ-algebra, then each a.e. continuous function is measurable.

Definition 1.7. Let $A_1, A_2, ..., A_n$ be a finite sequence of pairwise disjoint subset of X and let $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{R}$. The function

$$g(x) = \sum_{i=1}^{n} \lambda_i 1_{A_i}(x); \quad x \in X$$

is called *simple function.*

Notice that $A_1, A_2, ..., A_n$ are measurable if and only if the associated simple function $g = \sum_{i=1}^{n} \lambda_i 1_{A_i}$ is measurable in the sense of Definition 1.7.

Definition 1.8. Let $g = \sum_{i=1}^{n} \lambda_i 1_{A_i}$ be a measurable simple function. The *Lebesgue integral* of g is defined by:

$$\int_x g(x)\mu(dx) \;=\; \sum_{i=1}^{n} \lambda_i \mu(A_i) \tag{1.4}$$

Lemma 1.9. *Let $f : X \to \mathbb{R}$ be a nonnegative bounded measurable function. Then there exist a sequence of simple function g_n on X converging uniformly to f; that is*

$$\sup_{x\in I} |f(x) - g_n(x)| \to 0 \quad \textit{as } n \to \infty$$

Proof. There exit a constant $M > 0$ such that $0 \le f(x) \le M$ for all $x \in X$. Consider the partition of the interval $[0, M]$ defined by

$$a_i = \frac{iM}{n}; \quad 0 \le i \le n$$

Define for any $1 \le i \le n$

$$A_i = \{x \in I : f(x) \in [a_{i-1}, a_i)\}$$

$A_i = f^{-1}([a_{i-1}, a_i)) \in \mathcal{A}$ since f is measurable. For any $n \ge 1$, define the function g_n by

$$g_n = \sum_{i=1}^{n} a_{i-1} 1_{A_i}$$

Then each g_n is a measurable simple function and

$$|f(x) - g_n(x)| \le \frac{M}{n}; \quad x \in X$$

We conclude that $\{g_n\}$ converges uniformly to f. □

Definition 1.10. Let $f : X \to \mathbb{R}$ be a nonnegative bounded measurable function and let $\{g_n\}$ be a sequence of simple functions which converges uniformly to f. The *Lebesgue integral* of f is defined by

$$\int_X f(x)\mu(dx) \;=\; \lim_{n\to\infty} \int_X g_n(x)\mu(dx) \tag{1.5}$$

Remarks 1.11. 1- By Lemma 1.9, it is proved that there exist sequences of simple functions which converge uniformly to f.

The limit in the preceding definition exist in the large sense and is independent of the choice of the sequence of simple functions as long as they converge uniformly to f.
2- If $\mu(x) < \infty$ then $\int_X f(x)\mu(dx) < \infty$ since f is bounded.
In general, since μ is σ- finite (only) may be that $\int_X f(x)\mu(dx) = \infty$. In this case, we consider $f_k = f1_{E_k}$ where $\mu(E_k) < \infty$ and $\cup_k E_k = X$.

Let $f : X \to [0, \infty]$ be measurable and unbounded. Define for any constant $M > o$:

$$f_M(x) = \begin{cases} f(x) & if\ 0 \leq f(x) \leq M \\ M & if\ M < f(x) \end{cases}$$

It is clear that $\{f_M\}$ increases to f as M increases to $+\infty$. Moreover, f_M is bounded for each $M > 0$.

Definition 1.12. The *Lebesgue integral* of the (unbounded) function f is defined by

$$\int_X f(x)\mu(dx) = \lim_{M\to\infty} \int_X f_M(x)\mu(dx) \tag{1.6}$$

It is clear that $\int_X f_M(x)\mu(dx)$ increase as $M \to \infty$ and the limit may be $+\infty$.

Definition 1.13. Let $f : X \to \mathbb{R}$ be measurable. The *Lebesgue integral* of f is defined by

$$\int_X f(x)\mu(dx) = \int_X f^+(x)\mu(dx) - \int_X f^-(x)\mu(dx) \tag{1.7}$$

if at least one of terms $\int_X f^+(x)\mu(dx)$ or $\int_X f^-(x)\mu(dx)$ is finite.

Definition 1.14. A measurable function $f : X \to \mathbb{R}$ is said to be *Lebesgue integrable* if both $\int_X f^+(x)$ and $\int_X f^-(x)\mu(dx)$ are finite.
In this case the integral of f defined by (1.7) is finite.

Since $|f| = f^+ + f^-$, then a function f is integrable if and only if f is measurable and $\int_X |f|(x)\mu(dx) < \infty$.

For any measurable subset E of X and any measurable function f defined on X

$$\int_E f(x)\mu(dx) \;=\; \int_X f(x)1_E(x)\mu(dx) \tag{1.8}$$

Example 1.15. Suppose that μ is the Lebesgue measure on $X = \mathbb{R}$ endowed with its Borel σ-algebra $\mathcal{A}$. Recall that μ is the unique measure in $(\mathbb{R}, \mathcal{A})$ such that $\mu[c, d] = d - c$ for all $c < d \in \mathbb{R}$.
If $f : [a, b] \to \mathbb{R}$ is Riemann integrable then f is Lebesgue integrable on the measure space $([a, b], \mathcal{A}, \mu)$ and

$$\int_{[a,b]} f(x)\mu(dx) = \int_a^b f(x)\mu(dx) \tag{1.9}$$

For this reason, we use always the notation $\mu(dx) = dx$ if μ is the Lebesgue measure on $\mathbb{R}$.
Notice that the converse is not true. For example consider the Dirichlet's function $f = 1_{\mathbb{Q}\cap[0,1]}$ is not Riemann integrable but f is Lebesgue integrable on $[0, 1]$ and $\int_{[0,1]} f(x)\mu(dx) = 0$.

Theorem 1.16. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space.*

1. *Let $f, g : X \to \mathbb{R}$ measurable.*

   - *If g is nonnegative and integrable and if $|f| \le g$; a.e then f is integrable and*

     $$|\int_X f(x)\mu(dx)| \le \int_X g(x)\mu(dx)$$

   - *$\int_X |f|(x)\mu(dx) = 0$ if and only if $f = 0$; a.e*
   - *If f and g are integrable, then $(af + bg)$ is integrable for all $a, b \in \mathbb{R}$ and*

     $$\int_X (af + bg)(x)\mu(dx) = a\int_X f(x)\mu(dx) + b\int_X g(x)\mu(dx) \tag{1.10}$$

2. **The Lebesgue monotone convergence theorem(MCT):**
Let $f, f_n : X \to \mathbb{R}$ be measurable functions for all $n \in \mathbb{N}$ such that $0 \leq f_1 \leq f_2 \leq ... \leq f_n$ a.e and $\{f_n\}$ converge to f a.e, then

$$\lim_{n\to\infty} \int_X f_n(x)\mu(dx) \;=\; \int_X f(x)\mu(dx) \tag{1.11}$$

3. **The Lebesgue dominated convergence theorem(DCT):**
Let $f, g, f_n : X \to \mathbb{R}$ be measurable functions for all $n \in \mathbb{N}$ such that $|f_n| \leq g$; a.e and $\{f_n\}$ converge to f a.e. If g is integrable then f_n and f are integrable and (1.11) holds.

4. *Let $f : X \to \mathbb{R}$ be integrable and let $\{A_n\}$ be a sequence of pairwise disjoint sets in $\mathcal{A}$. Then*

$$\int_{\cup_n A_n} f(x)\mu(dx) \;=\; \sum_n \int_{A_n} f(x)\mu(dx) \tag{1.12}$$

Remark 1.17. The Lebesgue integral is stated in four distinct steps:

- Step 1: f is a simple function.
- Step 2: f is nonnegative and bounded function.
- Step 3: f is nonnegative and unbounded function.
- Step 4: f is any measurable function.

If $f : X \to \mathbb{R}$ is integrable, then from this 4 steps construction, there exist a sequence of simple functions $f_n = \sum_i \lambda_{i,n} 1_{A_{i,n}}$ such that

$$\lim_n f_n = f; \; \textit{a.e and } |f_n| \leq |f|$$

Thus (1.11) holds by the DCT.
This observation will be used later in simplifying proofs by considering two steps:

- First: Verify some formula for simple functions.
- Second: Pass to the limit using the DCT.

The Banach space $L^1(X)$

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. Recall that, the relation

$$f \sim g \Leftrightarrow f = g \ a.e$$

is an equivalent relation. So we do not distinguish, two functions which are equal a.e.
For any measurable function $f : X \to \mathbb{R}$, the *norm* of f is defined by

$$\|f\| = \int_X |f(x)|\mu(dx) \tag{1.13}$$

The $\mathbb{R}$-linear space $L^1(X)$ is the set of (class) of functions f which are Lebesgue integrable , that is

$$L^1(X) = \{f : X \to \mathbb{R} : \|f\| < \infty\} \tag{1.14}$$

Then

- $(L^1(x), \|.\|)$ is a Banach space over $\mathbb{R}$.
- If $\{f_n\} \to f$ in $L^1(X)$, then there exist a subsequence of $\{f_n\}$ which converges a.e to f.
- If $X = [a, b]$ is a compact interval of $\mathbb{R}$ then the space

  $$C^1([a, b]) = \{f : [a, b] \to \mathbb{R} : f \ and \ f' \ are \ continuous \ on \ [a, b]\}$$

  of C^1 functions is dense in $L^1([a, b])$

1.3 The Radon-Nikodyn Theorem

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space.

Proposition 1.18. *Let $f : X \to \mathbb{R}$ be an integrable function. Then $f = 0$, a.e. if and only if for any $\int_A f(x)\mu(dx) = 0$ for any $A \in \mathcal{A}$.*

PROOF. Suppose that $f = 0$, a.e. and let $A \in \mathcal{A}$. Then

$$\int_A f(x)\mu(dx) = \int_{A\cap\{f=0\}} f(x)\mu(dx) + \int_{A\cap\{f\neq0\}} f(x)\mu(dx) \tag{1.15}$$

But

$$\int_{A\cap\{f=0\}} f(x)\mu(dx) = \int_{A\cap\{f=0\}} 0\mu(dx) = 0 \tag{1.16}$$

and

$$\int_{A\cap\{f\neq 0\}} f(x)\mu(dx) = 0 \tag{1.17}$$

since

$$0 \leq \mu(A \cap \{f \neq 0\}) \leq \mu(\{f \neq 0\})$$

by definition of $f = 0$; a.e. We conclude from (1.15), (1.16) and (1.17) that $\int_A f(x)\mu(dx) = 0$.
Conversely, suppose that $\int_A f(x)\mu(dx) = 0$ for any $A \in \mathcal{A}$.
Define A_1 and A_2 by:

$$A_1 = \{x \in X : \ f(x) > 0\} \text{ and } A_2 = \{x \in X : \ f(x) \leq 0\}$$

Notice that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = X$.
By hypothesis, we have

$$\int_{A_1} f(x)\mu(dx) = \int_{A_2} f(x)\mu(dx) = 0$$

Hence

$$0 = \int_{A_1} f(x)\mu(dx) - \int_{A_2} f(x)\mu(dx) = \int_{A_1} f(x)\mu(dx) + \int_{A_2} (-f)(x)\mu(dx) \tag{1.18}$$

Since

$$\mid f \mid (x) = \begin{cases} f(x) \ if \ x \in A_1 \\ -f(x) \ if \ x \in A_2 \end{cases}$$

therefore (1.18) becomes

$$\begin{aligned} 0 &= \int_{A_1} \mid f \mid (x)\mu(dx) + \int_{A_2} \mid f \mid (x)\mu(dx) \\ &= \int_{A_1 \cup A_2} \mid f \mid (x)\mu(dx) = \int_X \mid f \mid (x)\mu(dx) \end{aligned}$$

We conclude that $\int_X \mid f \mid (x)\mu(dx) = 0$ and therefore $f = 0$; a.e in view of 1.16. □

Corollary 1.19. *Let $f_1, f_2 : X \to \mathbb{R}$ be integrable functions. Then $f_2 = f_2$; a.e. if and only if $\int_A f_1(x)\mu(dx) = \int_A f_2(x)\mu(dx)$ for any $A \in \mathcal{A}$.*

PROOF. It suffices to apply Proposition 1.18 to $g = f_1 - f_2$. □

Proposition 1.20. *Let $f : X \to \mathbb{R}$ be an integrable function. Then $f \geq 0$, a.e. if and only if for any $\int_A f(x)\mu(dx) \geq 0$ for any $A \in \mathcal{A}$.*

PROOF. If $f \geq 0$, a.e. then obviously $\int_A f(x)\mu(dx) \geq 0$ for any $A \in \mathcal{A}$. Conversely, let $A = \{f < 0\}$. If $\mu(A) > 0$, then $\int_A f(x)\mu(dx) < 0$ which gives a contradiction. □

Proposition 1.21. *Let $f : X \to \mathbb{R}$ be a nonnegative integrable function, then the real valued function $(f.\mu)$ defined by*

$$(f.\mu)(A) \;=\; \int_A f(x)\mu(dx); \quad A \in \mathcal{A} \tag{1.19}$$

is a finite measure on $(X, \mathcal{A})$.

PROOF. Since $f \geq 0$ and integrable, then for each $A \in \mathcal{A}$, we have

$$0 \leq (f.\mu)(A) = \int_A f(x)\mu(dx) \leq \int_X f(x)\mu(dx) = (f.\mu)(X) < \infty$$

In particular $(f.\mu)$ is nonnegative and finite.

- $(f.\mu)(\emptyset) = \int_\emptyset f(x)\mu(dx) = 0$
- Let $\{A_k\}$ be a finite or infinite sequence of pairwise disjoint set of A. Then

$$\begin{aligned}(f.\mu)(\cup_k A_k) &= \int_{\cup_k A_k} f(x)\mu(dx) \\ &= \sum_k \int_{A_k} f(x)\mu(dx) \text{ by (1.12)} \\ &= \sum_k (f.\mu)(A_k)\end{aligned}$$

Hence $(f.\mu)$ is σ-additive and therefore $(f.\mu)$ is a finite measure on $(X, \mathcal{A})$. □

Definition 1.22. The measure $(f.\mu)$ defined in Proposition 1.21 is called measure with *density* f with respect to μ.

Definition 1.23. Let μ and ν be two σ-finite measure defined on the same measurable space $(X, \mathcal{A})$. ν is said to be *absolutely continuous* (a.c) with respect to μ if

$$\mu(A) = 0 \ \Rightarrow \ \nu(A) = 0; \quad A \in \mathcal{A}$$

In this case, we write $\nu \ll \mu$.

Proposition 1.24. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and let $f : X \to \mathbb{R}$ be a nonnegative integrable function. Then the finite measure $(f.\mu)$ is a.c. with respect to μ.*

PROOF. If $\mu(A) = 0$ then $\mu_f(A) = \int_A f(x)\mu(dx) = 0$ in view of (1.8). □

The converse of the Proposition 1.24 is always true. This is given by the so-called Radon-Nikodyn theorem (RNT).

Theorem 1.25. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and let ν be a finite measure on $(X, \mathcal{A})$ which is a.c. with respect to μ. Then there exists a unique nonnegative integrable function f on X such that $\nu = (f.\mu)$.*

Remarks 1.26. 1- For the proof of Theorem 1.25, we refer to chapter 18 of [8].
2- According to Proposition 1.24 and Theorem 1.25, we have identified a.c. measures and measures with densities with respect to μ.
3- The RNT will be used in the next chapter in order to construct the transfer operator.
4- For a given σ-finite measure space $(X, \mathcal{A}, \mu)$, there exist measures ν which are not necessary a.c. with respect to μ. For example, The Dirac measure δ_0 defined by (1.2) is not a.c. with respect to the Lebesgue measure μ, since (for example) $\mu(\mathbb{N}) = 0$ but $\nu(\mathbb{N}) = 1$.
5- In Lebesgue integration, we may always replace real valued functions $f : X \to \mathbb{R}$ by functions $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ which are finite a.e, that is $\mu\{f = \pm\infty\} = 0$.

Chapter 2

Transfer Operators

In this chapter, we introduce first the concept of nonsingular transformation τ defined on a σ-finite measure space $(X, \mathcal{A}, \mu)$. Then we associate with τ, the transfer operator, namely the Frobenius-Perron operator P_τ. Finally, we characterize the τ-invariant measure which are absolutely continuous to μ. As reference, we will refer to [1, 2, 5].

2.1 Nonsingular transformations

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space.

Definition 2.1. Any application $\tau : X \to X$ is called *transformation* of X.

Recall that $\tau^{-1}(A) = \{x \in X : \tau(x) \in A\}$ for any $A \in X$. Notice that if $\{A_k\}$ is a sequence of subsets of X

1. $\tau^{-1}(\cap_k A_k) = \cap_k \tau^{-1}(A_k)$;
2. $\tau^{-1}(\cup_k A_k) = \cup_k \tau^{-1}(A_k)$;
3. If $\{A_k\}$ are pairwise disjoint then $\tau^{-1}(A_k)$ are pairwise disjoint;
4. For any $A \subset X$, we have

$$1_{\tau-1(A)} = 1_A \circ \tau \tag{2.1}$$

Definition 2.2. A transformation $\tau : X \to X$ is said to be *measurable* if $\tau^{-1}(A) \in \mathcal{A}$ for any $A \in \mathcal{A}$.

Always, X is a Borel subset of $\mathbb{R}^d$ and $\mathcal{A}$ be its Borel σ-algebra. In this case, if $\tau : X \to X$ is a.e. continuous, then τ is measurable.

Lemma 2.3. *Let $\tau : X \to X$ be a measurable transformation. Define*

$$(\mu\tau^{-1})(A) = \mu(\tau^{-1}(A)), \quad A \in \mathcal{A} \tag{2.2}$$

Then $(\mu\tau^{-1})$ is a measure on $(X, \mathcal{A})$.

PROOF. Since τ is measurable, then for any $A \in \mathcal{A}$, $\tau^{-1}(A) \in \mathcal{A}$ and therefore $\mu(\tau^{-1}(A)) = (\mu\tau^{-1})(A)$ is well defined. Moreover, since μ is a measure on $(X, \mathcal{A})$, then

$$(\mu\tau^{-1})(\emptyset) = \mu(\tau^{-1}(\emptyset)) = \mu(\emptyset) = 0$$

and, in view of (1.1), we get

$$(\mu\tau^{-1})(\cup_k A_k) = \mu(\tau^{-1}(\cup_k A_k)) = \mu(\cup_k \tau^{-1}(A_k)) = \sum_k (\mu\tau^{-1})(A_k)$$

for any finite or infinite sequence $\{A_k\} \subset \mathcal{A}$ of pairwise disjoint sets. □

Remark 2.4. Since $(\mu\tau^{-1})(X) = \mu(\tau^{-1}(X)) = \mu(X)$, then

- If μ is finite, then $(\mu\tau^{-1})$ is finite.
- If μ is a probability measure, then $(\mu\tau^{-1})$ is probability measure.

Definition 2.5. A measurable transformation $\tau : X \to X$ is said to be *nonsingular* (with respect to μ) if the measure $(\mu\tau^{-1})$ is absolutely continuous (a.c.) with respect to μ, that is $(\mu\tau^{-1}) \ll \mu$.
A non nonsingular transformation is said to be *singular*.

Examples 2.6. Let $X = [0, 1]$ endowed with the Borel σ-algebra $\mathcal{A}$ and the Lebesgue measure $\mu(dx) = dx$.
1- Let $\tau(x) = \frac{1}{2}; x \in [0, 1]$ and let $A = \{\frac{1}{2}\}$, then $\tau^{-1}(A) = [0, 1] = X$. Since $\mu(A) = 0$ and $(\mu\tau^{-1})(A) = 1$ we deduce that τ is singular.
2- Let $\tau(x) = x^2; x \in [0, 1]$. Let $A \in \mathcal{A}$, by the variable change $y = x^2$ and (2.1), we get

$$\begin{aligned}(\mu\tau^{-1})(A) &= \mu(\tau^{-1}(A)) = \int_0^1 1_{\tau^{-1}(A)}(x)dx = \int_0^1 1_A(\tau(x))dx \\ &= \int_0^1 1_A(x^2)dx = \int_0^1 1_A(y)\frac{dy}{2\sqrt{y}} = \int_A \frac{dy}{2\sqrt{y}}\end{aligned}$$

Hence $(\mu\tau^{-1})$ is a.c. with respect to μ, with density $\frac{1}{2\sqrt{x}}$, and therefore τ is nonsingular.

Remark 2.7. In general, if $(\mu\tau^{-1}) \ll \mu$ and $(\mu\tau^{-1})$ is finite, then there exist by the RNT (Theorem 1.25) a unique nonnegative integrable function f_τ such that $(\mu\tau^{-1}) = (f_\tau \cdot \mu)$, that is

$$\int_A f(x)(\mu\tau^{-1})(dx) \;=\; \int_{\tau^{-1}(A)} f(x)\mu(dx) \;=\; \int_A f(x)f_\tau(x)\mu(dx)$$

Moreover, if X is a Borel subset of $\mathbb{R}^d$, τ is invertible and τ^{-1} is differentiable then $f_\tau = J^{-1}$ the jacobian of τ^{-1}.

2.2 Frobenius-Perron operators

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and $\tau : X \to X$ be a nonsingular transformation of X.

Theorem 2.8. *For any integrable function $f \in L^1(X)$, there exist a unique integrable function denoted by $(P_\tau f) \in L^1(X)$ such that*

$$\int_A (P_\tau f)(x)\mu(dx) \;=\; \int_{\tau^{-1}(A)} f(x)\mu(dx); \quad A \in \mathcal{A} \tag{2.3}$$

PROOF. Let $f \in L^1(X)$.
First step: Suppose that f is nonnegative. Define $\nu : \mathcal{A} \to [0, +\infty]$ by

$$\nu(A) \;=\; \int_{\tau^{-1}(A)} f(x)\mu(dx); \quad A \in \mathcal{A} \tag{2.4}$$

Then ν is a measure on $(X, \mathcal{A})$ as image of the measure $(f \cdot \mu)$ by the transformation τ. Since

$$\nu(X) = \int_{\tau^{-1}(X)} f(x)\mu(dx) = \int_X f(x)\mu(dx) = \|f\| < \infty$$

we deduce that ν is finite.
On the other hand, if $\mu(A) = 0$, then $\mu(\tau^{-1}(A)) = 0$ since τ is nonsingular, therefore by (2.4)

$$\nu(A) = \int_{\tau^{-1}(A)} f(x)\mu(dx) = 0$$

as integral over set of measure zero. We conclude that ν is absolutely continuous with respect to μ.
Hence, there exists by the RNT (Theorem 1.25) a unique integrable function denoted by $(P_\tau f)$ such that

$$\nu(A) \;=\; \int_A (P_\tau f)(x)\mu(dx); \quad A \in \mathcal{A} \tag{2.5}$$

Combining 2.4 and 2.5, we have also proved 2.3 for nonnegative $f \in L^1(X)$.
Second step: Let $f \in L^1(X)$. Since $f = f^+ - f^-$; $f^+, f^- \in L^1(X)$ and f^+, f^- are nonnegative functions, then $P_\tau f^+$ and $P_\tau f^-$ are well defined by Step 1. Define

$$P_\tau f = P_\tau f^+ - P_\tau f^- \tag{2.6}$$

By linearity of the integral, and by 2.6 we get for any $A \in \mathcal{A}$

$$\begin{aligned}
\int_A (P_\tau f)(x)\mu(dx) &= \int_A (P_\tau f^+ - P_\tau f^-)(x)\mu(dx) \\
&= \int_A (P_\tau f^+)(x)\mu(dx) - \int_A (P_\tau f^-)(x)\mu(dx) \\
&= \int_{\tau^{-1}(A)} f^+(x)\mu(dx) - \int_{\tau^{-1}(A)} f^-(x)\mu(dx) \\
&= \int_{\tau^{-1}(A)} (f^+ - f^-)(x)\mu(dx) \\
&= \int_{\tau^{-1}(A)} f(x)\mu(dx)
\end{aligned}$$

This proves formula 2.3 for any function $f \in L^1(X)$. □

Definition 2.9. The operator $P_\tau : L^1(X) \to L^1(X)$ defined implicitly by the formula

$$\int_A (P_\tau f)(x)\mu(dx) = \int_{\tau^{-1}(A)} f(x)\mu(dx); \quad A \in \mathcal{A}, f \in L^1(X) \tag{2.7}$$

is called the *Frobenius-Perron operator* or *transfer operator* defined by the nonsingular transformation τ.

Remarks 2.10. 1- If X is a Borel subset of $\mathbb{R}^d$, τ is invertible and τ^{-1} is differentiable then

$$P_\tau f(x) = f(\tau^{-1}(x))J^{-1}(x) \tag{2.8}$$

where J^{-1} is the jacobian of τ^{-1}. (2.8) is obtained from (2.7) by the variable change $y = \tau(x)$ and by Corollary 1.19.
2- In general, chaotic maps are not invertible, so that formula 2.8 is not valuable for such maps. However, there is an explicit formula where $X = [a, b]$ is an interval of $\mathbb{R}$: For any nonsingular transformation τ of X and any a.e. continuous function f defined on $[a, b]$

$$P_\tau f(x) = \frac{d}{dx} \int_{\tau^{-1}([a,x])} f(t)dt; \quad \forall x \in [a, b]. \tag{2.9}$$

Formula (2.9) is obtained by differentiation of (2.7) for $A = [a, x], x \in [a, b]$.
3- Formula (2.9) can be easily generalized for $d \geq 2$.

Examples 2.11. 1- Let $\tau(x) = x^2, x \in [0, 1]$, then by (2.8), we get

$$P_\tau f(x) \;=\; f(\sqrt{x})\frac{1}{2\sqrt{x}}$$

2- Let $\tau : [0, 1] \to [0, 1]$ be a tent map, that is

$$\tau(x) \;=\; \begin{cases} 2x & \text{if } \; 0 \leq x < \frac{1}{2}, \\ 2 - 2x & \text{if } \; \frac{1}{2} \leq x \leq 1. \end{cases}$$

In this case,

$$\tau^{-1}[0, x] \;=\; [0, \frac{x}{2}] \;\cup\; [1 - \frac{x}{2}, 1]$$

By application of (2.9), we get

$$\begin{aligned} P_\tau f(x) &= \frac{d}{dx} \int_{[0,\frac{x}{2}]\cup[1-\frac{x}{2},1]} f(t)dt \\ &= \frac{d}{dx} \int_0^{\frac{x}{2}} f(t)dt + \frac{d}{dx} \int_{1-\frac{x}{2}}^1 f(t)dt \\ &= \frac{1}{2}[f(\frac{x}{2}) \;+\; f(1 - \frac{x}{2})] \end{aligned}$$

Theorem 2.12. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, τ be a nonsingular transformation of X, and let P_τ be the associated transfer operator.*

1. *If $f \in L^1(X), f \geq 0$, then $P_\tau f \geq 0$.*

2. *For each $f \in L^1(X)$*

$$\int_X P_\tau f(x)\mu(dx) = \int_X f(x)\mu(dx) \tag{2.10}$$

   In particular, if $f \geq 0$, then $\|P_\tau f\| = \|f\|$.

3. *P_τ is a linear operator: For all $f_1 f_2 \in L^1(X); \lambda_1, \lambda_2 \in \mathbb{R}$*

$$P_\tau(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 P_\tau(f_1) + \lambda_2 P_\tau(f_2) \tag{2.11}$$

4. *P_τ is a contraction on $L^1(X)$: for all $f, g \in L^1(X)$*

$$\|P_\tau f - P_\tau g\| \leq \|f - g\| \tag{2.12}$$

   In particular, $\|P_\tau f\| \leq \|f\|$.

5. *If τ and γ are two nonsingular transformations of X, then $\tau \circ \gamma$ is a nonsingular transformation of X and*

$$P_{\tau\circ\gamma} = P_\tau \circ P_\gamma \tag{2.13}$$

6. *For any natural number $n \geq 1$:*

$$P_{\tau^n} = (P_\tau)^n \tag{2.14}$$

Proof. 1- If $f \geq 0$, then for all $A \in \mathcal{A}$:

$$0 \leq \int_{\tau^{-1}(A)} f(x)\mu(dx) = \int_A P_\tau f(x)\mu(dx)$$

Hence $P_\tau f$ is a positive function by Proposition 1.20.
2- Let $f \in L^1(X)$. If we apply (2.7) for $A = X$, we obtain (2.10) since

$\tau^{-1}(X) = X$.
3- Let $f_1 f_2 \in L^1(X); \lambda_1, \lambda_2 \in \mathbb{R}, A \in \mathcal{A}$. By the linearity of the integral

$$\begin{aligned}\int_A P_\tau(\lambda_1 f_1 + \lambda_2 f_2)\mu(dx) &= \int_{\tau^{-1}(A)} (\lambda_1 f_1 + \lambda_2 f_2)(x)\mu(dx) \\ &= \lambda_1 \int_{\tau^{-1}(A)} f_1(x)\mu(dx) + \lambda_2 \int_{\tau^{-1}(A)} f_2(x)\mu(dx) \\ &= \lambda_1 \int_A P_\tau f_1(x)\mu(dx) + \lambda_2 \int_A P_\tau f_2(x)\mu(dx) \\ &= \int_A \lambda_1 P_\tau(f_1) + \lambda_2 P_\tau(f_2)\mu(dx)\end{aligned}$$

Formula (2.11) is then a consequence of Corollary 1.19.
4- Let $f \in L^1(X)$. we prove first that $\|P_\tau f\| \leq \|f\|$: Since $|f| + f \geq 0$ and $|f| - f \geq 0$, then by linearity of P_τ and by the first result, we see easily $|P_\tau f| \leq P_\tau |f|$. Thus, the second result gives

$$\|P_\tau f\| = \int_X |P_\tau f|(x)\mu(dx) \leq \int_X P_\tau |f|(x)\mu(dx) = \|f\| \tag{2.15}$$

Now, let $f, g \in L^1(X)$. By linearity of P_τ and (2.15), we get

$$\|P_\tau f - P_\tau g\| = \|P_\tau(f - g)\| \leq \|f - g\|$$

5- Let τ and γ are two nonsingular transformations of X. Since $(\tau \circ \gamma)^{-1}(A) = \gamma^{-1}(\tau^{-1}(A))$, we see easily that $\tau \circ \gamma$ is a nonsingular transformations of X. Let $A \in \mathcal{A}$ and $f \in L^1(X)$. Using formula 2.7 many times, we get

$$\begin{aligned}\int_A P_{\tau\circ\gamma} f(x)\mu(dx) &= \int_{(\tau\circ\gamma)^{-1}(A)} f(x)\mu(dx) = \int_{\gamma^{-1}(\tau^{-1}(A))} f(x)\mu(dx) \\ &= \int_{\tau^{-1}(A)} P_\gamma f(x)\mu(dx) = \int_A P_\tau(P_\gamma f)(x)\mu(dx) \\ &= \int_A (P_\tau \circ P_\gamma) f(x)\mu(dx)\end{aligned}$$

We conclude by Corollary 1.19 that $P_{\tau\circ\gamma} = P_\tau \circ P_\gamma$.
6- By induction: For $n = 1$, there is nothing to prove. For $n \geq 2$ suppose that

$$P_{\tau^{n-1}} = (P_\tau)^{n-1} \tag{2.16}$$

If we take $\gamma = \tau^{n-1}$ in (2.13), then (2.16) imply that

$$P_{\tau^n} = P_{\tau \circ \tau^{n-1}} = P_\tau \circ P_{\tau^{n-1}} = P_\tau \circ (P_\tau)^{n-1} = (P_\tau)^n$$

□

Remarks 2.13. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space.
1- A map $U : L^1(X) \to L^1(X)$ is called *Markov operator* if

1. $Uf \geq 0$ for any $f \in L^1(X), f \geq 0$;
2. $\|Uf\| = \|f\|$ for any $f \in L^1(X), f \geq 0$;
3. U is linear.

In Theorem 2.12, we have proved that any transfer operator P, associated to a nonsingular transformation of X, is a Markov operator.
2- From the properties of the Definition of a Markov operator, it can be proved that U is a contraction on $L^1(X)$ (The proof is direct and more easy for the P).
We deduce that a Markov operator is unitary, that is $\|U\| = 1$ where $\|U\| := \sup\{\|Uf\| : \|f\| \leq 1\}$.

2.3 Invariant measures

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, $\tau : X \to X$ be a nonsingular transformation of X, and P_τ be the transfer operator associated to τ.

Definition 2.14. A measure ν on $(X, \mathcal{A})$ is said to be τ-*invariant* if

$$\nu(\tau^{-1}(A)) \;=\; \nu(A); \quad A \in \mathcal{A} \tag{2.17}$$

Formula (2.17) means that the two measures $\nu\tau^{-1}$ and ν are equal.
The most important case of τ-invariant measures is when ν is absolutely continuous with respect to μ, that is $\nu = (f \cdot \mu)$ where $f \in L^1(X), f \geq 0$. In this case $\nu = (f \cdot \mu)$ is called *absolutely continuous invariant measure* (a.c.i.m) of τ.
We have the following important result.

Theorem 2.15. *Let $f \in L^1(X), f \geq 0$. The finite measure $(f \cdot \mu)$ is an a.c.i.m for τ if and only if f is invariant for P_τ, that is*

$$P_\tau f \;=\; f \tag{2.18}$$

Proof. For any $A \in \mathcal{A}$

$$(f \cdot \mu)(\tau^{-1}(A)) = \int_{\tau^{-1}(A)} f(x)\mu(dx) = \int_A P_\tau f(x)\mu(dx)$$

and

$$(f \cdot \mu)(A) = \int_A f(x)\mu(dx)$$

Hence $(f \cdot \mu)$ is an a.c.i.m for τ if and only if

$$\int_A P_\tau f(x)\mu(dx) \;=\; \int_A f(x)\mu(dx); \quad A \in \mathcal{A}$$

which is equivalent to $P_\tau f = f$ by Corollary 1.19. □

Remarks 2.16. 1- In fact, we look at of nonzero solution of $P_\tau f = f$ that is $\|f\| > 0$. Therefore, if we consider $g = \frac{f}{\|f\|}$ then $g \in L^1(X)$; $g \geq 0$ and $\|g\| = 1$. Moreover by linearity of P_τ, we get $P_\tau g = g$ and therefore, the probability measure $\lambda = (g \cdot \mu)$ is an a.c.i.m. by Theorem 2.15.
2- Since $\mu = (1 \cdot \mu)$ then by Theorem 2.15, μ is invariant if and only if $P_\tau 1 = 1$.
3- If $P_\tau f = f$, then $P_{\tau^n} f = (P_\tau)^n f = f$.
4- For the applications in the real life (physic, biology, economy, ...), we consider probability invariant measure. This suggests the following definition.

Definition 2.17. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space. A function $f : X \to \overline{\mathbb{R}}$ is called *density* if f is nonnegative, f is integrable, and $\|f\| = 1$. We denote by $\mathcal{D}$ the set of all densities defined on $(X, \mathcal{A}, \mu)$.

Remark 2.18. Let τ be a nonsingular transformation of X and let P_τ the associated transfer operator. Since P_τ is a Markov operator, then $P_\tau f$ is density if f is a density, that is $P_\tau(\mathcal{D}) \subset \mathcal{D}$

Definition 2.19. Any sequences $\{f_n\}$ of functions defined by

$$f_{n+1} = P_\tau f_n; \quad f_0 \in \mathcal{D} \tag{2.19}$$

is called *recurrent sequence* of densities associated with the transformation τ.

Remarks 2.20. 1- As mentioned in the introduction, recurrent sequences of densities are naturally associated with chaotic nonsingular transformations by considering f_0 the density defined by a large numbers of initial values

$$I_0 = \{x_0^1, x_0^2, ..., x_0^n\}$$

and f_n is density associated to

$$I_n = \{\tau^n(x_0^1), \tau^n(x_0^2), ..., \tau^n(x_0^n)\}$$

2- Note that $f_n = (P_\tau)^n f_0 = P_{\tau^n} f_0$.
3- If recurrent sequence of densities $\{f_n\}$ converges in $L^1(X)$ to some function f_*, then f_* is an invariant density for P_τ, that is $f_* \in \mathcal{D}$ and $P_\tau f_* = f_*$. This observation will discussed later with more details.
4-The determination of invariant densities of the transfer operator P_τ is an important problem for evolution systems modeled by chaotic maps. Indeed, the distribution f_n of I_n approaches the distribution f_* for big n. However, there is no general solution for the problem $f = P_\tau f$.
In the next chapter, we will solve this problem for some piecewise monotonic maps defined on an interval $X = [a, b]$. For the moment, let us only verify the existence of an invariant density is invariant for the transfer operator of an important example, namely the logistic map.

Example 2.21. Let $X = [0, 1]$ endowed with its Borel σ-algebra $\mathcal{A}$ and the Lebesgue measure μ. The Logistic map is defined on $[0, 1]$ by

$$\tau(x) = 4x(1-x); \quad x \in [0, 1]$$

The Logistic map is one of the most important example in chaos theory since it serves as demographic model in biology. τ is not one-to-one and the equation $\tau(y) = x$ admits 2 solutions

$$y = \frac{1}{2} - \frac{1}{2}\sqrt{1-x} \text{ and } z = \frac{1}{2} + \frac{1}{2}\sqrt{1-x} \tag{2.20}$$

Therefore $\tau^{-1}([0, x]) = [0, y] \cup [z, 1]$. Using Formulas (2.9) and (2.20) , we get for any a.e. continuous function $f \in L^1[0, 1]$ and $x \in [0, 1]$

$$\begin{aligned} P_\tau f(x) &= \frac{d}{dx}\int_{\tau^{-1}([0,x])} f(t)dt = \frac{d}{dx}(\int_0^y f(t)dt + \int_z^1 f(t)dt) \\ &= f(y)\frac{dy}{dx} - f(z)\frac{dz}{dx} \\ &= \frac{1}{4\sqrt{1-x}}[f(\frac{1}{2} - \frac{1}{2}\sqrt{1-x}) + f(\frac{1}{2} + \frac{1}{2}\sqrt{1-x})] \end{aligned}$$

Now, let $f_*(x) = \frac{1}{\pi}g(x)$ where

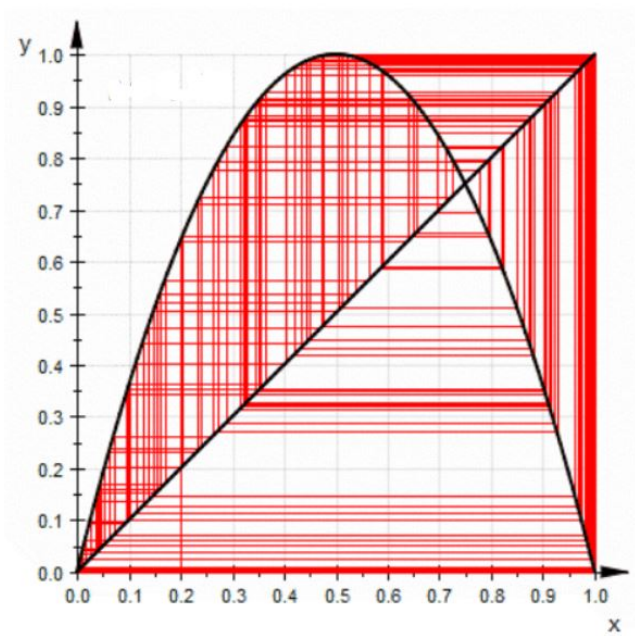
$$g(x) = \frac{1}{\sqrt{x(1-x)}};\ x \in (0,1)$$

Then f_* is nonnegative. Moreover, by variable changes, it can be verified that $\int_0^1 g(x)dx = \pi$, therefore f_* is a density on X=[0,1].
Let us prove that $P_\tau f_* = f_*$. By linearity of P_τ, it is enough to prove that $P_\tau g = g$. Using the expressions of y and z in (2.20), we see easily that

$$y(1-y) \ = \ z(1-z) \ = \ \frac{x}{4}$$

Hence

$$\begin{aligned} P_\tau g(x) &= \frac{1}{4\sqrt{1-x}}[g(\frac{1}{2} - \frac{1}{2}\sqrt{1-x}) + g(\frac{1}{2} + \frac{1}{2}\sqrt{1-x})] \\ &= \frac{1}{4\sqrt{1-x}}[g(y) + g(z)] = \frac{1}{4\sqrt{1-x}}[\frac{1}{\sqrt{\frac{x}{4}}} + \frac{1}{\sqrt{\frac{x}{4}}}] = g(x) \end{aligned}$$



Starting from $x_0 = 0.2$, the recurrent sequence $x_{n+1} = \tau(x_n)$ moves chaotically in the whole interval $[0,1]$.

Chapter 3

Invariant densities for piecewise monotonic maps

In this chapter, we consider the one dimensional case, namely transformations τ on compact interval $I = [a, b]$ which are piecewise monotonic. Such maps are always chaotic and the associated transfer operators have special representation. Under more convenient conditions, we prove that they admit an a.c.i.m with respect to the Lebesgue measure on I. The proof uses the space of functions of bounded variations.
For this chapter, we will refer to [1, 5].

3.1 Functions of bounded variation

Let $I = [a, b]$ be a compact interval of $\mathbb{R}$, let $\mathcal{A}$ be the Borel σ-algebra of I and let $\mu(dx) = dx$ be the Lebesgue measure on I.

Definition 3.1. A *partition* $\mathcal{P} = \{I_i : 1 \leq i \leq n\}$ of $I = [a, b]$ is defined by $I_i = [x_{i-1}, x_i); \quad 1 \leq i \leq n$ where $E = \{x_0, x_1, ..., x_n\}$ is a finite set of point of $[a, b]$ such that $a = x_0 < x_1 < ... < x_n = b$.
In this case the points of E are called *end-points* of the partition $\mathcal{P}$ and the partition is also denoted by $\mathcal{P}(x_0, x_1, ..., x_n)$.

Definition 3.2. A function $f : I \to \mathbb{R}$ is said to be of *bounded variation* (BV) on I, if there exists a positive real number M such that

$$S_n(f) = S(f, P_n) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \ \leq M \tag{3.1}$$

for any $n \in \mathbb{N}$ and any partition P_n of I.
In this case, the real number

$$V_I(f) = \sup\{S(f, P_n)\} \tag{3.2}$$

is called *total variation* of f or simply *variation* of f.
Notice that f is not of BV means that $V_I(f) = +\infty$.

Examples 3.3. Let $f : I = [a, b] \to \mathbb{R}$ be a function.
1- If f is constant, then f is of BV and $V_I(f) = 0$.
2- If f is increasing on I, then f is of BV and $V_I(f) = f(b) - f(a)$.
3- If f is a Lipschitz function on I, that is there exists some constant $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in I$, then f is of BV and $V_I(f) \leq C(b - a)$.
4- If f is a C^1-function on I then f is of BV on I and

$$V_I(f) \ = \ \int_a^b |f'(x)|dx \tag{3.3}$$

5- If f is the restriction to I of the characteristic function $1_{\mathbb{Q}}$ then f is not of BV.
6- If $I = [0, 2\pi]$ and

$$f(x) = \begin{cases} x\sin(\frac{1}{x});\ 0 < x \leq 2\pi \\ 0;\ x = 0 \end{cases}$$

then f is continuous on I but not of BV on I.
7- If $I = [0, 1]$ and

$$f(x) = \begin{cases} p \quad if\ x = \frac{1}{p}; p = 1, 2, ... \\ 0; \quad elsewhere \end{cases}$$

Then $f = 0$ a.e. but $V_I(f) = \infty$.

Lemma 3.4. *Let $f : I = [a, b] \to \mathbb{R}$ be a function of BV on I, then f is bounded on I and*

$$|f(x)| \leq |f(a)| + V_I(f); \quad x \in I \tag{3.4}$$

PROOF. Let $x \in I$. By considering the partition $\mathcal{P}_2 = \mathcal{P}(a, x, b)$, we get

$$||f(x)| - |f(a)|| \leq |f(x) - f(a)| \leq |f(x) - f(a)| + |f(b) - f(x)| \leq V_I(f)$$

This proves (3.4). □

Theorem 3.5. *Let $f : I = [a, b] \to \mathbb{R}$ be a function of BV on I such that $f \in L^1(I)$, then*

$$|f(x)| \leq \frac{\|f\|}{b-a} + V_I(f); \quad x \in I \tag{3.5}$$

PROOF. First step: we prove by contradiction that there exist $y \in I$ such that

$$|f(y)| \leq \frac{\|f\|}{b-a}$$

If not

$$\forall x \in I : \quad |f(x)| > \frac{\|f\|}{b-a}$$

Hence

$$\|f\| = \int_a^b |f(x)|dx > \int_a^b \frac{\|f\|}{b-a} dx = \|f\|$$

and we have a contradiction.
Second step: We use the same idea of Lemma 3.4

$$\begin{aligned} \forall x \in I : \quad |f(x)| &\leq |f(y)| + |f(x) - f(y)| \\ &\leq \frac{\|f\|}{b-a} + V_I(f) \end{aligned}$$

□

For the proof of the following two results, we will refer to Chapter 6 of [5] and Chapter 2 of [1].

Theorem 3.6. *Let $f : I = [a, b] \to \mathbb{R}$ be a function of BV on I.*

1. *If $C \in \mathbb{R}$ is a constant then Cf is of BV and $V_I(Cf) = |C|V_I(f)$.*

2. *If $J = [c, d] \subset I$, then f is of BV on J and $V_J(f) \leq V_I(f)$.*

3. *If $c \in (a, b)$, then f is of BV on $[a, b]$ if and only if f is of BV on $[a, c]$ and on $[c, d]$. In this case*

$$V_{[a,b]}(f) \;=\; V_{[a,c]}(f) \;+\; V_{[c,b]}(f) \tag{3.6}$$

Theorem 3.7. *Let $f, g : I = [a, b] \to \mathbb{R}$ be a function of BV on I, then*

1. *$(f + g)$ is of BV on I and*

$$V_I(f + g) \leq V_I(f) + V_I(g) \tag{3.7}$$

2. *(fg) is of BV on I and if $A = \sup |g|$ and $B = \sup |f|$, we have*

$$V_I(fg) \leq AV_I(f) + BV_I(g) \tag{3.8}$$

The space of bounded variation functions

Let $f, g \in L^1(I)$. If $f = g$ a.e., then f and g are identical as functions of $L^1(I)$ but f and g may have different variations (see 7. of Example 3.3). For this reason, let us define for $f \in L^1(I)$

$$V_I^*(f) \;=\; \inf_{f=ga.e.} V_I(g) \tag{3.9}$$

Definition 3.8. The space $BV(I)$ of integrable functions which are of bounded variation is defined by

$$BV(I) = \{f \in L^1(I) \;:\; V_I^*(f) < \infty\}$$

We define a norm on $BV(I)$ as follows: For $f \in BV(I)$

$$\|f\|_{BV} \;=\; \|f\| + V_I^*(f) \tag{3.10}$$

Since $C^1(I)$ is dense in $L^1(I)$ and $C^1(I) \subset BV(I)$, we deduce that $BV(I)$ is dense in $L^1(I)$.

The following result, known as Helly's Theorem, is fundamental in order to prove the existence of invariant densities for piecewise monotonic maps. For the prove,we will refer to [6].

Theorem 3.9. *Let $\{f_n\}$ be a bounded sequence in $BV(I)$, then there exist a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a function $f_* \in BV(I)$ such that $\{f_{n_k}\} \to f_*$ in $L^1(I)$.*

3.2 Representation of the transfer operator

Definition 3.10. Let $[c, d] \subset [a, b]$ and $\tau : [c, d] \to [a, b]$ is a map. τ is said to be *monotonic* on $[c, d]$ if

(i) τ is a C^1-function;

(ii) $|\tau'(x)| > 0$ for all $x \in (c, d)$.

In this case, τ is invertible and $\phi = \tau^{-1} : [a, b] \to [c, d]$. Moreover, for all $x \in (a, b)$

$$\phi'(x) = (\tau^{-1})'(x) = \frac{1}{\tau'(\tau^{-1}(x))} = \frac{1}{\tau'(\phi(x))}$$

Examples 3.11. Suppose that $[a, b] = [0, 1]$.
1- If $\tau(x) = x^2$ and $[c, d] = [0, 1]$, then $\phi(x) = \sqrt{x}$.
2- If $\tau(x) = 2x$ and $[c, d] = [0, 1/2]$, then $\phi(x) = \frac{x}{2}$.
3- If $\tau(x) = 2 - 2x$ and $[c, d] = [1/2, 1]$, then $\phi(x) = \frac{2-x}{2}$.

Lemma 3.12. *Let $[c, d] \subset [a, b]$ and $\tau : [c, d] \to [a, b]$ be monotonic, then by putting $\tau^{-1} = \phi$*

$$\int_{\tau^{-1}(A)} f(y)dy = \int_A f(\phi(x))|\phi'(x)|dx \tag{3.11}$$

for any $f \in L^1([a, b])$ and $A \in \mathcal{A}$. In particular

$$\mu(A) = 0 \;\Rightarrow\; \mu(\tau^{-1}(A)) = 0 \tag{3.12}$$

PROOF. Formula (3.11) is obtained by the variable change $y = \phi(x)$. Moreover, by taking $f(x) = 1$, then formula 3.11 implies

$$\mu(\tau^{-1}(A)) = \int_{\tau^{-1}(A)} dy = \int_A |\phi'(x)|dx$$

Hence $\mu(\tau^{-1}(A)) = 0$ whenever $\mu(A) = 0$. □

Remark 3.13. Let $\tau : [a, b] \to [a, b]$ be monotonic. From Lemma 3.12, we deduce immediately that τ is nonsingular and for $f \in L^1([a, b])$

$$P_\tau f(x) = f(\phi(x))|\phi'(x)| = \frac{f(\tau^{-1}(x))}{|\tau'(\tau^{-1}(x))|}$$

Definition 3.14. A transformation $\tau : [a,b] \to [a,b]$ is called *piecewise monotonic* if there exists a partition $\mathcal{P}$ $\mathcal{P}_q = \mathcal{P}(a_0, a_1, ..., a_q)$ of $[a,b]$ such that for all $1 \le i \le q$

(i) τ_i is a C^1-function which can be extended to a C^1-function $\tau_i : [a_{i-1}, a_i] \to [a,b]$;

(ii) $|\tau_i{}'(x)| > 0$ for all $x \in (a_{i-1}, a_i)$.

Remark 3.15. If τ is piecewise monotonic, then

$$\tau_i : [a_{i-1}, a_i] \to [a,b]; \quad \forall 1 \le i \le q$$

is monotonic in the sense of Definition 3.10. Its inverse

$$\phi_i = \tau_i^{-1} : [a,b] \to [a_{i-1}, a_i]; \quad \forall 1 \le i \le q$$

Examples 3.16. 1- Tent map: The transformation $\tau : [0,1] \to [0,1]$ defined by

$$\tau(x) = \begin{cases} 2x \quad if\ 0 \le x < \frac{1}{2} \\ 2 - 2x \quad if\ \frac{1}{2} \le x \le 1 \end{cases}$$

is a piecewise monotonic with $\mathcal{P}^1 = \{0, \frac{1}{2}, 1\}$, $\tau_1(x) = 2x; x \in [0, \frac{1}{2}]$ and $\tau_2(x) = 2 - 2x; x \in [\frac{1}{2}, 1]$.
2- Logistic map: The transformation $\tau : [0,1] \to [0,1]$ defined by $\tau(x) = 4x(1-x)$ is a piecewise monotonic with $\mathcal{P}^1 = \{0, \frac{1}{2}, 1\}$, $\tau_1(x) = 4x(1-x); x \in [0, \frac{1}{2}]$ and $\tau_2(x) = 4x(1-x); x \in [\frac{1}{2}, 1]$.
3- Tri-adic map: The transformation $\tau : [0,1] \to [0,1]$ defined by

$$\tau(x) = \begin{cases} 3x \quad if\ 0 \le x < \frac{1}{3} \\ 3x - 1 \quad if\ \frac{1}{3} \le x < \frac{2}{3} \\ 3x - 2 \quad if\ \frac{2}{3} \le x \le 1 \end{cases}$$

is a piecewise monotonic with $\mathcal{P}^1 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, $\tau_1(x) = 3x; x \in [0, \frac{1}{3}]$, $\tau_2(x) = 3x - 1; x \in [\frac{1}{3}, \frac{2}{3}]$, and $\tau_3(x) = 3x - 2; x \in [\frac{2}{3}, 1]$.]

Remark 3.17. If τ is piecewise monotonic, then τ^n is piecewise monotonic. Moreover, if q is the size of the partition $\mathcal{P}^1$ of τ, then q^n is the size of the partition $\mathcal{P}^n$ of τ^n, and $E^n \subset E^{n+1}$ where E^n is the set of end-points of the

partition $\mathcal{P}^n$.
For example, consider $\tau^2 : [0,1] \to [0,1]$ where τ is the tent map. Since the partition of τ is $\mathcal{P}^1 = P(0, 1/2, 1)$ and $\tau^{-1}(1/2) = \{1/4, 3/4\}$, then the partition of τ^2 is $\mathcal{P}^2 = \mathcal{P}(0, 1/4, 1/2, 3/4, 1)$, and τ^2 is duplication of τ. Therefore

$$\tau^2(x) = \begin{cases} 4x & if\ 0 \leq x < \frac{1}{4} \\ 2 - 4x & if\ \frac{1}{4} \leq x < \frac{1}{2} \\ 4x - 2 & if\ \frac{1}{2} \leq x < \frac{3}{4} \\ 4 - 4x & if\ \frac{3}{4} \leq x \leq 1 \end{cases}$$

In the same way, consider $\tau^3 : [0,1] \to [0,1]$ where τ is the tent map. using the same method, we find that $\mathcal{P}^3 = \mathcal{P}(0, 1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8, 1)$ is the partition of τ^3 and τ^3 is a duplication of τ^2.

Theorem 3.18. *Let $I = [a, b]$ and $\tau : I \to I$ be a piecewise monotonic map defined by a partition $P^1 = P(a_0, a_1, ..., a_q)$. Then*

(i) τ is nonsingular.

(ii) For all $f \in L^1(I)$ and $x \in I$

$$P_\tau f(x) = \sum_{i=1}^{i=q} f(\phi_i(x))|\phi_i'(x)| = \sum_{i=1}^{i=q} \frac{f({\tau_i}^{-1}(x))}{|\tau_i'({\tau_i}^{-1}(x))|} \tag{3.13}$$

where $\phi = {\tau_i}^{-1} : I \to (a_{i-1}, a_i)$ and τ_i is the restriction of τ on (a_{i-1}, a_i).

Proof. Notice that, for any $A \in \mathcal{A}$:

$$\tau^{-1}(A) = \cup_{i=1}^{q} {\tau_i}^{-1}(A) \quad \text{and} \quad \{{\tau_i}^{-1}(A) : 1 \leq i \leq q\} \text{ are disjoint} \tag{3.14}$$

Moreover $\tau_i : [a_{i-1}, a_i] \to I$ is monotonic in the sense of Definition 3.10.
If $\mu(A) = 0$, then

$$\mu(\tau^{-1}(A)) = \mu(\cup_{i=1}^{q} {\tau_i}^{-1}(A)) = \sum_{i=1}^{q} \mu({\tau_i}^{-1}(A))$$

by σ-additivity 1.1 of μ. Since $\tau_i : [a_{i-1}, a_i] \to I$ is monotonic, then Lemma 3.12 imply that $\mu(\tau_i^{-1}(A)) = 0$ for any $1 \leq i \leq q$. We conclude that

$\mu(\tau^{-1}(A)) = 0$.
Let $f \in L^1(I)$ and $x \in I$. Using Formulas (2.7), (3.14), and Lemma 3.12, we get for any $A \in \mathcal{A}$:

$$\begin{aligned}
\int_A (P_\tau f)(x)dx &= \int_{\tau^{-1}(A)} f(x)dx = \int_{\cup_{i=1}^q \tau_i{}^{-1}(A)} f(x)dx \\
&= \sum_{i=1}^q \int_{\tau_i{}^{-1}(A)} f(x)dx = \sum_{i=1}^q \int_A f(\phi_i(x))|\phi_i'(x)|dx \\
&= \int_A \sum_{i=1}^q f(\phi_i(x))|\phi_i'(x)|dx
\end{aligned}$$

Hence

$$\int_A (P_\tau f)(x)dx = \int_A \sum_{i=1}^q f(\phi_i(x))|\phi_i'(x)|dx$$

We conclude by Corollary 1.19 that 3.13 holds. □

Examples 3.19. By application of Theorem 3.18, we get
1- Tent map:
$\tau_1(x) = 2x, x \in [0, 1/2]$ hence $\phi_1(x) = x/2, x \in [0, 1]$ and $|\phi_1'(x)| = 1/2$.
$\tau_2(x) = 2-2x, x \in [1/2, 1]$ hence $\phi_2(x) = 1-x/2, x \in [0, 1]$ and $|\phi_2'(x)| = 1/2$.
Hence

$$P_\tau f(x) = \frac{1}{2}[f(\frac{x}{2}) + f(1 - \frac{x}{2})] \tag{3.15}$$

This Formula was obtained in Example 2.11 by using (2.9).
2- Logistic map: (See Example 2.21) $\tau_1(x) = 4x(1 - x), x \in [0, 1/2]$ hence $\phi_1(x) = \frac{1}{2} - \frac{1}{2}\sqrt{1-x}, x \in [0, 1]$ and $|\phi_1'(x)| = \frac{1}{4\sqrt{1-x}}$.
$\tau_1(x) = 4x(1 - x), x \in [1/2, 1]$ hence $\phi_1(x) = \frac{1}{2} + \frac{1}{2}\sqrt{1-x}, x \in [0, 1]$ and $|\phi_2'(x)| = \frac{1}{4\sqrt{1-x}}$.
Hence

$$P_\tau f(x) = \frac{1}{4\sqrt{1-x}}[f(\frac{1}{2} - \frac{1}{2}\sqrt{1-x}) + f(\frac{1}{2} + \frac{1}{2}\sqrt{1-x})]$$

3- Di-adic map:

$$P_\tau f(x) = \frac{1}{2}[f(\frac{x}{2}) + f(\frac{1}{2} + \frac{x}{2})] \tag{3.16}$$

4- Tri-adic map:

$$P_\tau f(x) = \frac{1}{3}[f(\frac{x}{3}) + f(\frac{1}{3} + \frac{x}{3}) + f(\frac{2}{3} + \frac{x}{3})]$$

5- τ^2 where τ is the tent map:

$$P_{\tau^2} f(x) = \frac{1}{4}[f(\frac{x}{4}) + f(\frac{1}{2} - \frac{x}{4}) + f(\frac{1}{2} + \frac{x}{4}) + f(1 - \frac{x}{4})]$$

3.3 Existence of invariant densities

Definition 3.20. A transformation $\tau : I \to I$ is said to be *piecewise expanding* if

1. τ is piecewise monotonic on I;
2. There exist a constant $0 < \alpha < 1$ such that

$$g(x) = \frac{1}{|\tau'(x)|} \le \alpha, \quad x \in I \tag{3.17}$$

   (At the end-points, g is defined by the appropriate one-sided derivative).

3. The function g is of BV on I.

Notice that the Tent map, the Di-adic map and the Tri-adic map are expanding but the Logistic map is not expanding.

Remarks 3.21. Let $\tau : I \to I$ be piecewise expanding and defined by a partition $\mathcal{P}^1 = \mathcal{P}(a_0, a_1, ..., a_q)$ and let $g = \frac{1}{|\tau'|}$.
1- Using g, Formula (3.13) becomes: For all $f \in L^1(I)$ and $x \in I$

$$P_\tau f(x) \;=\; \sum_{i=1}^{i=q} f(\phi_i(x)) g(\phi_i(x)) \tag{3.18}$$

where $\phi = {\tau_i}^{-1} : I \to (a_{i-1}, a_i)$ and τ_i is the restriction of τ on (a_{i-1}, a_i).
2- Let $n \ge 2$. Since $\tau^n = \tau^{n-1} \circ \tau = \tau \circ \tau^{n-1}$ and

$$(\tau^n)'(x) \;=\; \tau'(\tau^{n-1}(x)) \cdot (\tau^{n-1})'(x) \;=\; (\tau^{n-1})'(\tau(x)) \cdot \tau'(x)$$

we deduce by induction that τ^n is expanding on I,

$$g_n(x) = \frac{1}{|(\tau^n)'(x)|} \leq \alpha^n < \alpha < 1, \quad n \geq 1, \quad x \in I \tag{3.19}$$

by putting $g_1 = g$ and

$$g_{n+1}(x) = g_n(\tau(x)) \cdot g_1(x), \quad n \geq 1, \quad x \in I \tag{3.20}$$

Lemma 3.22. *Let $I = [a, b]$ and $\tau : I \to I$ be a piecewise expanding transformation on I. Then for any $f \in BV(I)$*

$$V_I(P_\tau f) \leq AV_I(f) + B\|f\| \tag{3.21}$$

where $A = \alpha + V_I(g)$ and $B = \frac{1}{b-a}V_I(g)$.

Proof. Let $\mathcal{P}^1 = \mathcal{P}(a_0, a_1, ..., a_q)$ is the partition defining τ and let $Q(x_0, x_1, ..., x_n)$ be any partition of I (not related to $\mathcal{P}^1$). Then

$$\begin{aligned}
s_n(P_\tau f) &= \sum_{j=1}^{n} |P_\tau f(x_j) - P_\tau f(x_{j-1})| \\
&= \sum_{j=1}^{n} |\sum_{i=1}^{q} g(\phi_i(x_j)) f(\phi_i(x_j)) - \sum_{i=1}^{q} g(\phi_i(x_{j-1})) f(\phi_i(x_{j-1}))| \text{ by 3.18} \\
&\leq \sum_{j=1}^{n} \sum_{i=1}^{q} |g(\phi_i(x_j)) f(\phi_i(x_j)) - g(\phi_i(x_{j-1})) f(\phi_i(x_{j-1}))| \\
&\leq \sup_I |f| \sum_{j=1}^{n} \sum_{i=1}^{q} |g(\phi_i(x_j)) - g(\phi_i(x_{j-1}))| \\
&\quad + \sup_I |g| \sum_{j=1}^{n} \sum_{i=1}^{q} |f(\phi_i(x_j)) - f(\phi_i(x_{j-1}))| \text{ by 3.8} \\
&\leq \sup_I |f| \sum_{i=1}^{q} V_{I_i}(g) + \sup_I |g| \sum_{i=1}^{q} V_{I_i}(f) \\
&= \sup_I |f| V_I(g) + \sup_I |g| V_I(f) \text{ by Theorem 3.6} \\
&\leq \left(\frac{\|f\|}{(b-a)} + V_I(f)\right) V_I(g) + \alpha V_I(f) \text{ by (3.11), (3.17)} \\
&= (\alpha + V_I(g)) V_I(f) + \frac{V_I(g)}{(b-a)} \|f\|
\end{aligned}$$

□

Lemma 3.23. *Let $\tau : I \to I$ be a piecewise expanding, then for any $n \geq 1$*

$$V_I(g_n) \;\leq\; n\alpha^{n-1}V_I(g_1) \tag{3.22}$$

where $g_n = \frac{1}{|(\tau^n)'|}$ such that $g_1 < \alpha < 1$.

Proof. We proceed by induction: The relation 3.22 is obviously true for $n = 1$.
Suppose that relation (3.22) is true at the order n and let $Q(x_0, x_1, ..., x_k)$ be any partition of I. Using (3.1), (3.20) and (3.8) we get

$$\begin{aligned}
s_k(g_{n+1}) &= \sum_{j=1}^{k} |g_{n+1}(x_j) - g_{n+1}(x_{j-1})| \\
&= \sum_{j=1}^{k} |g_n(\tau(x_j))g_1(x_j) - g_n(\tau(x_{j-1}))g_1(x_{j-1})| \\
&\leq \sup_I |g_1| \sum_{j=1}^{k} |g_n(\tau(x_j)) - g_n(\tau(x_{j-1}))| + \sup_I |g_n| \sum_{j=1}^{k} |g_1(x_j) - g_1(x_{j-1})| \\
&\leq \alpha V_I(g_n) + \alpha^n V_I(g_1) \quad \text{by Definition 3.20} \\
&\leq \alpha n\alpha^{n-1}V_I(g_1) + \alpha^n V_I(g_1) \quad \text{by induction hypothesis}
\end{aligned}$$

Hence $s_k(g_{n+1}) \leq (n+1)\alpha^n V_I(g_1)$ for any $k \geq 2$. Letting $k \to \infty$, we get $V_I(g_{n+1}) \leq (n+1)\alpha^n V_I(g_1)$. □

Lemma 3.24. *If $\tau : I \to I$ is piecewise expanding then there two constants $0 < C < 1$ and $R > 0$ such that*

$$\forall f \in BV(I): \quad \|P_\tau^n f\|_{BV} \leq C\|f\|_{BV} + R\|f\| \tag{3.23}$$

Proof. Let us apply (3.21) to τ^n instead of τ. By using (2.14), Remark 3.21, and (3.22) we get

$$\begin{aligned}
V_I({P_\tau}^n f) &\leq (\alpha^n + V_I(g_n))V_I(f) + \frac{V_I(g_n)}{b-a}\|f\| \\
&\leq (\alpha^n + n\alpha^{n-1}V_I(g_1))V_I(f) + \frac{n\alpha^{n-1}V_I(g_1)}{(b-a)}\|f\|
\end{aligned}$$

Since $0 < \alpha < 1$, then $\alpha^n \to 0$ and $n\alpha^{n-1}V_I(g_1) \to 0$ as $n \to \infty$, then there exist $n_0 \in \mathbb{N}, 0 < A_1, B_1 < 1$ such that for any $n \geq n_0$

$$V_I({P_\tau}^n f) \leq A_1 V_I(f) + B_1\|f\| \tag{3.24}$$

Starting from τ^{n_0} instead of τ, we may suppose that (3.24) holds for any $n \geq 1$. Now

$$\begin{aligned}
V_I^*(P_\tau{}^n f) &= \inf_{h=f a.e.} V_I(P_\tau{}^n h) \\
&\leq \inf_{h=f a.e.} (A_1 V_I(h) + B_1\|h\|) \\
&= \inf_{h=f a.e.} (A_1 V_I(h) + B_1\|f\|) \quad \text{since } f = h \ a.e. \\
&= A_1 (\inf_{h=f a.e.} V_I(h)) + B_1\|f\| \\
&\leq A_1 V_I^*(f) + B_1\|f\|
\end{aligned}$$

Hence

$$\begin{aligned}
\|P_\tau{}^n f\|_{BV} &= \|P_\tau{}^n f\| + V_I^*(P_\tau{}^n f) \\
&\leq \|f\| + A_1 V_I^*(f) + B_1\|f\| \\
&\leq A_1\|f\|_{BV} + (1 - A_1 + B_1)\|f\|
\end{aligned}$$

□

Theorem 3.25. *Let $\tau : I \to I$ be a piecewise expanding transformation. Then τ admits an absolutely continuous invariant measure whose density is of BV.*

Proof. By Theorem 2.15 , we have to prove that there exist $f_* \in BV(I)$ such that $P_\tau f_* = f_*$. We apply Lemma 3.24 to $f = 1$: There exist two constants $0 < C < 1, R > 0$ such that

$$\|P_\tau^n 1\|_{BV} \leq C\|1\|_{BV} + R\|1\| \leq C + R, \quad n \geq 1 \tag{3.25}$$

Relation (3.25) means that the sequence of function $\{P_\tau^n 1\}$ is a bounded subset of BV(I). For $n \geq 1$, let

$$f_n = \frac{1}{n}\sum_{j=1}^{n} P_\tau^j 1 = \frac{1}{n}(P_\tau 1 + P_\tau^2 1 + ... + P_\tau^n 1)$$

Then

$$\|f_n\|_{BV} = \|\frac{1}{n}\sum_{j=1}^{n} P_\tau^j 1\|_{BV} \leq \frac{1}{n}\sum_{j=1}^{n} \|P_\tau^j 1\|_{BV} \leq \frac{1}{n}\sum_{j=1}^{n}(C + R) = C + R$$

Hence the sequence $\{f_n\}$ is also bounded subset of BV(I). By Helly's Theorem 3.9 there exist a subsequence $\{f_{n_k}\}$ that converge in $L^1(I)$ to a function $f_* \in BV(I)$. Next, we will prove that $P_\tau f_* = f_*$.

$$\begin{aligned}
\|P_\tau f_* - f_*\| &\leq \|P_\tau f_* - P_\tau f_{n_k} + P_\tau f_{n_k} - f_{n_k} + f_{n_k} - f_*\| \\
&\leq \|P_\tau f_* - P_\tau f_{n_k}\| + \|P_\tau f_{n_k} - f_{n_k}\| + \|f_{n_k} - f_*\| \\
&\leq \|f_* - f_{n_k}\| + \|P_\tau f_{n_k} - f_{n_k}\| + \|f_{n_k} - f_*\|
\end{aligned}$$

Since $f_{n_k} \to f_*$ in $L^1(I)$ then $\|f_* - f_{n_k}\| \to 0$ as $k \to \infty$. On the other hand

$$\begin{aligned}
\|P_\tau f_{n_k} - f_{n_k}\| &= \|P_\tau(\frac{1}{n_k}\sum_{j=1}^{n_k} P_\tau^j 1) - \frac{1}{n_k}\sum_{j=1}^{n_k} P_\tau^j 1\| \\
&= \frac{1}{n_k}\|P_\tau^2 1 + ... + P_\tau^{n_k} 1 + P_\tau^{n_k+1} 1 - (P_\tau 1 + P_\tau^2 1 + ... + P_\tau^{n_k} 1)\| \\
&= \frac{1}{n_k}\|P_\tau^{n_k+1} 1 - P_\tau 1\| \leq \frac{1}{n_k}\|P_\tau^{n_k+1} 1\| + \|P_\tau 1\| \\
&\leq \frac{1}{n_k}(1+1) = \frac{2}{n_k} \to 0 \ as \ k \to \infty
\end{aligned}$$

Hence $\|P_\tau f_* - f_*\| \leq \alpha_k \to 0$ as $k \to \infty$. So, $P_\tau f_* = f_*$. □

Examples 3.26. 1- Tend map: We know that τ is piecewise expanding on $I = [0,1]$ Hence, there exist by Theorem 3.25, $f_* \in BV(I)$ such that $P_\tau f_* = f_*$.
On the other hand, we know from Example 3.19 that

$$P_\tau f(x) = \frac{1}{2}[f(\frac{x}{2}) + f(1 - \frac{x}{2})] \tag{3.26}$$

We deduce easily from 3.26 that the constant function $f_* = 1$ satisfies $P_\tau f_* = f_*$.
2- Let $\tau(x) = \begin{cases} (x+1)^2 - 1 & if\ 0 \leq x < \sqrt{2} - 1 \\ \frac{x+1-\sqrt{2}}{2-\sqrt{2}} & if\ \sqrt{2} - 1 \leq x \leq 1 \end{cases}$ Then

$$g(x) = \frac{1}{|\tau'(x)|} = \begin{cases} \frac{1}{2(x+1)} & if\ 0 \leq x < \sqrt{2} - 1 \\ 2 - \sqrt{2} & if\ \sqrt{2} - 1 \leq x \leq 1 \end{cases}$$

$g_1(x) = \frac{1}{2(x+1)}$ is of BV on $[0, \sqrt{2} - 1]$ as C^1-function.
$g_2(x) = 2 - \sqrt{2}$ is of BV on $[\sqrt{2} - 1, 1]$ as constant function.

Hence g is of BV on $[0, \sqrt{2}-1] \cup [\sqrt{2}-1, 1] = [0, 1]$.
We conclude that τ is expanding on $I = [0, 1]$. By Theorem 3.25, there exists $f_* \in BV(I)$ such that $P_\tau f_* = f_*$.
On the other hand,By Theorem 3.18, we can easily find that

$$P_\tau f(x) = \frac{1}{2\sqrt{x+1}} f(\sqrt{x+1}-1) + (2-\sqrt{2}) f((2-\sqrt{2})x + \sqrt{2} - 1)$$

However, there is no apparent function f_* such that $P_\tau f_* = f_*$ as for tent map.

Remarks 3.27. 1- The condition "τ piecewise expanding" is not necessary to obtain an invariant density. It is only a sufficient condition. For example, the logistic map is not expanding but it admits an invariant density.
2- Theorem 3.25 says only that f_* exists. For the determination of f_*, recurrent sequences $f_{n+1} = P_\tau f_n$ are always used to approximate f_*.

Chapter 4

Invariant densities for random maps

In this chapter, we introduce the concepts of random map and the associated transfer operator. Moreover, we study the existence of a.c.i.m. when a random map is generated by piecewise monotonic maps.
For this chapter, we will refer to [3, 4, 7].

4.1 Transfer operators for random maps

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space.

Definition 4.1. A *random map* T on X is defined by

$$T = (T_1, T_2, ..., T_k; q_1, q_2, ..., q_k) = (T_k; q_k : 1 \leq k \leq K)$$

where

- $T_1, T_2, ..., T_K$ are K transformations of X that is $T_k : X \to X$ is non-singular for $1 \leq k \leq K$.
- $\{q_1, q_2, ..., q_K\}$ is a probability vector that is $0 < q_k < 1$ for any $1 \leq k \leq K$ and $\sum_{k=1}^{K} q_k = 1$.

The randomness of T is given by

$$T = T_k \quad \text{with probability } q_k; \quad 1 \leq k \leq K \tag{4.1}$$

(See [3] for more details). By iteration of 4.1, we get

$$T^2 = (T_{k_1} \circ T_{k_2}) \quad \text{with probability } q_{k_1} q_{k_2};\ 1 \leq k_1, k_2 \leq K$$

In general, we have

$$T^n = (T_{k_n} \circ \cdots \circ T_{k_1}) \quad \text{with probability } q_{k_n} \cdots q_{k_1};\ 1 \leq k_1, ..., k_n \leq K$$

Examples 4.2. 1- Consider $T = (T_1, T_2; \frac{1}{4}, \frac{3}{4})$ where $T_1 : [0,1] \to [0,1]$ is the logistic map and $T_2 : [0,1] \to [0,1]$ is the tent map. We can easily verify that T^2 is also random map and

$$T^2 = (T_1 \circ T_2, T_2 \circ T_1, T_1^2, T_2^2; \frac{3}{16}, \frac{3}{16}, \frac{1}{16}, \frac{9}{16})$$

2- Consider $T = (T_k, q_k : 1 \leq k \leq K)$ such that $T_1 = T_2 = \cdots = T_K = \tau$, then $T = \tau$, and there is no randomness.
It is in this sense that random map generalize single maps.

Remark 4.3. In general, if $T = (T_k, q_k : 1 \leq k \leq K)$ and $S = (S_j, r_j : 1 \leq j \leq J)$ are random maps on X, then $T \circ S$ is a random map on X and

$$T \circ S = (T_k \circ S_j; q_k r_j : 1 \leq k \leq K, 1 \leq j \leq J) \tag{4.2}$$

In particular

$$T^2 = (T_k \circ T_j; q_k, q_j : 1 \leq k, j \leq K)$$

4.2 means that

$$(T \circ S) = (T_k \circ S_j) \text{ with probability } 1 \leq k \leq K, 1 \leq j \leq J$$

Definition 4.4. Let $T = (T_1, T_2, ..., T_K; q_1, q_2, ..., q_K)$ be a random map on X. A measure ν on $(X, \mathcal{A})$ is said to be T-*invariant* if

$$\nu(A) \ = \ \sum_{k=1}^{K} q_k \ \nu(T_k^{-1}(A));\ A \in \mathcal{A} \tag{4.3}$$

Next, we consider the important case where ν absolutely continuous with with respect to μ.

Theorem 4.5. *Let $T = (T_1, T_2, ..., T_K; q_1, q_2, ..., q_K)$ be a random map and let $f \in L^1(X), f \geq 0$. Then the measure $\nu = (f \cdot \mu)$ is T-invariant if and only if*

$$f \;=\; \sum_{k=1}^{K} q_k P_{T_k} f \tag{4.4}$$

where P_{T_k} is the transfer operator of the single map T_k for $1 \leq k \leq K$.

PROOF. Let $A \in \mathcal{A}$. Using 4.3, we get

$$\begin{aligned}(f \cdot \mu)(A) &= \sum_{k=1}^{K} q_k (f \cdot \mu)(T_k^{-1}(A)) \\ &= \sum_{k=1}^{K} q_k \int_{T_k^{-1}(A)} f(x)\mu(dx) \\ &= \sum_{k=1}^{K} q_k \int_A P_{T_k} f(x)\mu(dx) \\ &= \int_A \sum_{k=1}^{K} q_k P_{T_k} f(x)\mu(dx)\end{aligned}$$

Since $(f \cdot \mu)(A) = \int_A f(x)\mu(dx)$, we deduce that $(f \cdot \mu)$ is T-invariant if and only if

$$\int_A f(x)\mu(dx) = \int_A \sum_{k=1}^{K} q_k P_{T_k} f \mu(dx); \quad A \in \mathcal{A}$$

We conclude by Corollary 1.19 that 4.4 holds. □

Definition 4.6. The operator $P_T : L^1(X) \to L^1(X)$ defined by

$$P_T f = \sum_{k=1}^{K} q_k P_{T_k} f; \quad f \in L^1(X) \tag{4.5}$$

is called *transfer operator* for the random map T.

Corollary 4.7. *Let $f \in L^1(X), f \geq 0$. The measure $(f \cdot \mu)$ is T-invariant if and only if f is an invariant function for P_T, that is $P_T f = f$.*

Example 4.8. Let $T_1 : [0,1] \to [0,1]$ be the tent map, we have seen in Example 3.19 that

$$P_T f(x) = \frac{1}{2}[f(\frac{x}{2}) + f(1 - \frac{x}{2})]$$

Let $T_2 : [0,1] \to [0,1]$ be the di-adic map, we have seen in Example 3.19 that

$$P_T f(x) = \frac{1}{2}[f(\frac{x}{2}) + f(\frac{1}{2} + \frac{x}{2})]$$

Consider the random map $T = (T_1, T_2; \frac{1}{3}, \frac{2}{3})$, then by Definition 4.6

$$P_T f(x) = \frac{1}{2} f(\frac{x}{2}) + \frac{1}{6} f(1 - \frac{x}{2}) + \frac{1}{3} f(\frac{1}{2} + \frac{x}{2})$$

In the next two results, we prove that the transfer operator associated to a random map, has similar properties as the transfer operator associated to a single map.

Theorem 4.9. *Let $T = (T_1, T_2, ..., T_K; q_1, q_2, ..., q_K)$ be a random map on$X$ and let P_T be the associated transfer operator. Then*

1. *P_T is a linear operator: For all $f_1 f_2 \in L^1(X); \lambda_1, \lambda_2 \in \mathbb{R}$*

$$P_T(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 P_T(f_1) + \lambda_2 P_T(f_2)$$

2. *P_T is a positive operator: If $f \in L^1(X), f \geq 0$, then $P_T f \geq 0$.*

3. *P_T preserves the integral: For each $f \in L^1(X)$*

$$\int_X P_T f(x) \mu(dx) = \int_X f(x) \mu(dx)$$

4. *P_T is a contraction: For any $f \in L^1(X)$*

$$\|P_\tau f\| \leq \|f\|$$

Proof. For the proof, we use essentially that for each T_i the associated transfer operator P_{T_i} satisfies these properties by Theorem 2.12: 1- Let $f_1 f_2 \in$

$L^1(X); \lambda_1, \lambda_2 \in \mathbb{R}$, then by (4.5)

$$\begin{aligned} P_T(\lambda_1 f_1 + \lambda_2 f_2) &= \sum_{k=1}^{K} q_k P_{T_k}(\lambda_1 f_1 + \lambda_2 f_2) \\ &= \sum_{k=1}^{K} q_k (\lambda_1 P_{T_k} f_1 + \lambda_2 P_{T_k} f_2) \\ &= \lambda_1 \sum_{k=1}^{K} q_k P_{T_k} f_1 + \lambda_2 \sum_{k=1}^{K} q_k P_{T_k} f_2 \\ &= \lambda_1 P_T(f_1) + \lambda_2 P_T(f_2) \end{aligned}$$

2- If $f \in L^1(X), f \geq 0$ then Theorem 2.12, $P_{T_k} f \geq 0$ for each $1 \leq k \leq K$. Hence $P_T f \geq 0$ as sum of positive terms.
3- Let $f \in L^1(X)$ then

$$\begin{aligned} \int_X P_T f(x)\mu(dx) &= \int_X \sum_{k=1}^{K} q_k P_{T_k} f(x)\mu(dx) = \sum_{k=1}^{K} q_k \int_X P_{T_k} f(x)\mu(dx) \\ &= \sum_{k=1}^{K} q_k \int_X f(x)\mu(dx) = \int_X f(x)\mu(dx) \cdot \sum_{k=1}^{K} q_k = \int_X f(x)\mu(dx) \end{aligned}$$

4- Let $f \in L^1(X)$ then

$$\|P_T f\| = \|\sum_{k=1}^{K} q_k P_{T_k} f\| \leq \sum_{k=1}^{K} q_k \|P_{T_k} f\| \leq \sum_{k=1}^{K} q_k \|f\| = \|f\| \sum_{k=1}^{K} q_k = \|f\|$$

□

Theorem 4.10. *1. Let T and S be two random maps on X, then*

$$P_{T \circ S} = P_T \circ P_S$$

2. Let T be random map on X, then $P_{T^n} = (P_T)^n$ for all $n \geq 1$.

Proof. 1- Let $T = (T_k, q_k : 1 \leq k \leq K)$ and $S = (S_j, r_j : 1 \leq j \leq J)$ and $f \in L^1(X)$. By (4.2), we get

$$P_{T \circ S} f = \sum_{j=1}^{J} \sum_{k=1}^{K} q_k r_j P_{T_k \circ T_j} f = \sum_{j=1}^{J} \sum_{k=1}^{K} q_k r_j P_{T_k} \circ P_{T_j} f$$

On the other hand, using (4.5) for P_T and for P_S, we obtain

$$\begin{aligned} P_T \circ P_S f &= P_T(P_S f) = P_T(\sum_{j=1}^{J} r_j P_{T_j} f) = \sum_{j=1}^{J} r_j P_T(P_{T_j} f) \\ &= \sum_{j=1}^{J} r_j \sum_{k=1}^{K} q_k P_{T_k}(P_{T_j} f) = \sum_{j=1}^{J} \sum_{k=1}^{K} q_k r_j P_{T_k} \circ P_{T_j} f \end{aligned}$$

2- The proof is obtain by induction as in Theorem 2.12. □

4.2 Representation of the transfer operator

In this section, we consider random maps on compact intervals such that all its components are piecewise monotonic by the same partition. We give the representation of the associated transfer operator and we study the existence of invariant densities.

Let $T = (T_k, ; q_k : 1 \leq k \leq K)$ be a random map on $I = [a, b]$ such that there exist a partition (common for $T_1, T_2, ..., T_K$) $\mathcal{P} = \mathcal{P}(a_0, a_1, ..., a_N)$ for which each T_k is piecewise monotonic: For all $1 \leq i \leq N, 1 \leq k \leq K$

1. the restriction $T_{k,i}$ of T_k over (a_{i-1}, a_i) is a C^1-function which can be extended to a C^1-function $T_{k,i} : [a_{i-1}, a_i] \to I$.
2. $|T_k'(x)| > 0$ for all $x \in (a_{i-1}, a_i)$.

In this case, it is known that $T_{k,i}$ is monotonic and $T_{k,i}^{-1} : I \to [a_{i-1}, a_i]$.
With this notations, we have the following

Theorem 4.11. *For any $f \in L^1(I)$*

$$(P_T f)(x) = \sum_{k=1}^{K} q_k \sum_{i=1}^{N} \frac{f(T_{k,i}^{-1}(x))}{|T_k'(T_{k,i}^{-1}(x))|} \tag{4.6}$$

Proof. By definition 4.6, we have

$$P_T f = \sum_{k=1}^{K} q_k P_{T_k} f \tag{4.7}$$

By Theorem 3.18, applied to each T_k, we have

$$(P_{T_k}f)(x) = \sum_{i=1}^{N} \frac{f(T_{k,i}^{-1}(x))}{|T_k'(T_{k,i}^{-1}(x))|} \tag{4.8}$$

By combining 4.7 and 4.8 we deduce 4.6. $\square$

Example 4.12. Let $T_1 : [0,1] \to [0,1]$ be the logistic map, we have seen that in Example 3.19 that

$$P_\tau f(x) = \frac{1}{4\sqrt{1-x}}[f(\frac{1}{2} - \frac{1}{2}\sqrt{1-x}) + f(\frac{1}{2} + \frac{1}{2}\sqrt{1-x})]$$

Let $T_2 : [0,1] \to [0,1]$ defined by $T_2(x) = \begin{cases} 1-2x & if\ 0 \le x < \frac{1}{2} \\ 2x-1 & if\ \frac{1}{2} \le x \le 1 \end{cases}$ Then

$$P_T f(x) = \frac{1}{2}[f(\frac{1-x}{2}) + f(\frac{1+x}{2})]$$

Consider the random map $T = (T_1, T_2; \frac{1}{4}, \frac{3}{4})$, then by Theorem 4.11, we get

$$P_T f(x) = \frac{1}{16\sqrt{1-x}}[f(\frac{1}{2} - \frac{1}{2}\sqrt{1-x}) + f(\frac{1}{2} + \frac{1}{2}\sqrt{1-x})] \\ + \frac{3}{8}[f(\frac{1-x}{2}) + f(\frac{1+x}{2})]$$

Theorem 4.13. *Let $T = (T_1, T_2, ..., T_K; q_1, q_2, ..., q_K)$ be a random map on $I = [a,b]$ such that there exists a partition $\mathcal{P}$ of I such that each T_k is piecewise expanding with respect to $\mathcal{P}$. Then there exists $f_* \in BV(I)$ such that $P_T f_* = f_*$.*

The proof is similar to the proof of Theorem 3.25. We will refer to [7] for all details.

Remarks 4.14. 1- Combining Theorem 4.13, Theorem 4.5 and Definition 4.6, we conclude by Theorem 4.13 that $\nu = (f \cdot \mu)$ is T-invariant.
2- By Theorem 3.25, for each $1 \le k \le K$ there exist $f_{*,k} \in BV(I)$ such that $P_{T_k} f_{*,k} = f_{*,k}$. But a priori there are no apparent relations between the $f_{*,k}$ and f_* only in the trivial case $f_{*,1} = f_{*,2} = ... = f_{*,k}$ we obtain $f_* = f_{*,1}$.

Examples 4.15. 1- Let $T = (T_1, T_2; q_1, q_2)$ where $T_1 : [0,1] \to [0,1]$ is the tent map, $T_2 : [0,1] \to [0,1]$ is the di-adic map and let (q_1, q_2) be any probability vector.
We have seen in Remark 2.16 that the constant function $f_* = 1$ is T_1-invariant and T_2-invariant. Hence 1 is T-invariant.
2- Let $T = (T_1, T_2; \frac{3}{5}, \frac{2}{5})$ where

$$T_1(x) = \begin{cases} (x+1)^2 - 1 & if\ 0 \le x < \sqrt{2} - 1 \\ \frac{x+1-\sqrt{2}}{2-\sqrt{2}} & if\ \sqrt{2} - 1 \le x \le 1 \end{cases} \quad and$$

$$T_2(x) = \begin{cases} \frac{x}{\sqrt{2}-1} & if\ 0 \le x < \sqrt{2} - 1 \\ \frac{1-x}{2-\sqrt{2}} & if\ \sqrt{2} - 1 \le x \le 1 \end{cases}$$

Then T_1 and T_2 are piecewise expanding with respect to the common partition $P(0, \sqrt{2} - 1, 1)$.
We have seen in Example 3.26 that

$$P_{T_1} f(x) = \frac{1}{2\sqrt{x+1}} f(\sqrt{x+1} - 1) + (2 - \sqrt{2}) f((2 - \sqrt{2})x + \sqrt{2} - 1)$$

For T_2,

$$P_{T_2} f(x) = \frac{1}{\sqrt{2} - 1} f((\sqrt{2} - 1)x) + \frac{1}{2 - \sqrt{2}} f(1 - (2 - \sqrt{2})x)$$

By Theorem 4.11,

$$\begin{aligned} P_T f(x) &= \frac{2}{5} \frac{1}{2\sqrt{x+1}} f(\sqrt{x+1} - 1) \\ &+ \frac{2(2-\sqrt{2})}{5} (2 - \sqrt{2}) f((2 - \sqrt{2})x + \sqrt{2} - 1) \\ &+ \frac{3}{5} \frac{1}{\sqrt{2} - 1} f((\sqrt{2} - 1)x) + \frac{3}{5(2 - \sqrt{2})} \frac{1}{2 - \sqrt{2}} f(1 - (2 - \sqrt{2})x) \end{aligned}$$

By Theorem 4.13, Then there exist $f_* \in BV([0,1])$ such that $P_T f_* = f_*$. But a priori there are no apparent relations between the three densities $f_{*,1}$,$f_{*,2}$ and f_*.

Bibliography

[1] Boyarsky A., Gora P.: Laws of Chaos, Invariant Measures and Dynamical Systems in One Dimension, Springer Science and Business Media, Birkhaueser (1997)

[2] Ding J., Zhou A.: Statistical properties of deterministic systems, Springer-Verlag (2009)

[3] Hmissi M., Mokchaha-Hmissi F.: On some random densities for random maps, Journal of Difference Equations and Applications, Vol. 24, No. 1, 127-137 (2018)

[4] Islam M.S., Swishchuk A.: Random Dynamical Systems in Finance, CRC Press, Taylor and Francis Group, (2013)

[5] Lasota A., Mackey M.C.: Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics, Second Edition, Springer-Verlag (1994)

[6] Natanson I.P.: Theory of Functions of a Real Variable, Frederick Ungar Publ. Co.(1955)

[7] Pelikan S.: Invariant densities for random maps of the intervals , Trans. Am. Math. Soc. 281, 813-825 (1984)

[8] Royden H. L., Fitzpatrick P.M.: Real Analysis, Fourth Edition, Pearson Education (2010)