Kingdom of Saudi Arabia

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Locally Connected Space and Locally Path Connected Space

A research project in mathematics (mat 499)

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Contents

1. Preliminaries

In this preliminaries section, we will give some essential concepts that we will need in following sections of this project. We start with summary about topological space.

1.1 Topological Space

Definition 1.1.1:

Let X be a non empty set. A collection τ of subsets of X is called a topology on X if it satisfies the following :

- i. **X and $\emptyset$ belong to τ .**
- ii. **The intersection of finite collection of sets in τ belong to τ .**
- iii. **The union of any collection of sets in τ belong to τ .**

A pair (X,τ) , X is non-empty set and τ is topology on x is called a topological space.

Example 1.1.1 :

Let X = {a,b,c,d,e} and let

$\tau_1 = \{X,\emptyset,\{a\},\{a,c,d\},\{c,d\},\{b,c,d,e\}\}$
$\tau_2 = \{X,\emptyset,\{a\},\{c,d\},\{a,c,d\},\{b,c,d\}\}$
$\tau_3 = \{X,\emptyset,\{a\},\{c,d\},\{a,c,d\},\{a,b,d,e\}\}$

Then τ_1 is a topology on X, while τ_2 is not a topology on X since {a,c,d}U{b,c,d}={a,b,c,d) $\notin \tau_2$.

Also τ_3 is not a topology on X since $\{a,c,d\} \cap \{a,b,d,e\} = \{a,d\} \notin \tau_3$.

Example 2.1.1 :

Let X be any non-empty set , then :

i. **The collection of all subsets of X is a topology on X called the discrete topology and it is denoted by $\tau_d = P(X)$ is the power set of X.**
ii. **The collection τ_i = { , ∅} is a topology on X called the indiscrete topology or the trivial topology.**

Note that if τ is any topology on X, then $\tau_i \subseteq \tau \subseteq \tau_d$

2.1 Open Set and Closed Set

Definition 1.2.1 :

Let (X,τ) be a topological space :

i. A subset U ⊆ X is said to be open iff U∈ τ .

ii. A subset E of X is called closed set if its complement $E^c \in \tau$.

Example 1.2.1 :

Let X = { a,b,c,d,e } and τ = { X,∅,{a},{c,d},{a,c,d},{b,c,d,e}} , then (X,τ) is a topological space . Note that :
i. X,∅ ,{a},{c,d},{a,c,d},{b,c,d,e} are open sets .
ii. X,∅,{b,c,d,e},{a,b,e},{b,e},{a} are closed sets .

iii. There are subsets which are both open and closed sets as {b,c,d,e}.

iv. There are subsets which are neither open nor closed sets as {a,b} .

Remark 1.2.1 :

Let (X,τ) be a topological space . Then :

i. **X and ∅ are open sets.**
ii. **Intersection of any two open sets is also open .**
iii. **Union of the collection of open sets is also open.**

Therefore to give topology to X means to define open sets in X

Remark 2.1.1 :

Let (X,τ) be a topological space then the collection of closed sets G has the following properties :

i. **X and ∅ are closed sets.**
ii. **The intersection of the collection of closed sets is closed .**
iii. **The union of any tow closed sets is closed .**

1.1 Basis for a Topology

Definition 1.3.1 :

Let X be a non-empty set. A collection $\beta \subset P(X)$ is called a basis for a topology τ on X if it satisfies:

i. **For each $x \in X$, there is $B \in \beta$ such that $x \in B$.**
ii. **If $x \in B_1 \cap B_2$ and $B_1, B_2 \in \beta$, then there is $B_3 \in \beta$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.**

Definition 2.3.1 :

If β is a basis for a topology on X , the topology τ generated by β described as follows:
U ∈ τ ⇔ For each x ∈ U, there is B ∈ β such that x ∈ B and B ⊂ U.
That is :
U is open set ⇔ For each x ∈ U,there is B ∈ β, x ∈ B and B ⊂ U.

Example 1.3.1 :

Consider the set of all real number R. The collection β = { (a,b) : a,b ∈ R a < b } where (a,b) = { x ∈ R : a < x < b } is a basis for a topology on R because :

i. **For each x ∈ R, x ∈ (x - 1, x - 1),(x – 1 , x + 1) ∈ β.**
ii. **If x ∈ (a,b) ∩ (c,d),then it is easy to find ∈ > 0 such that. x ∈ (x - ϵ, x + ϵ),(x - ϵ, x + ϵ) ⊂ (a,b) ∩ (c,d).**
So β is a basis for a topology on R . The topology τ generated by the basis β = { (a,b) : a , b ∈ R, a < b } is called the standard topology on R and denoted by R = (R , τ).

Example 2.3.1:

Consider X = R. Then, the collection β = { [a , b) : a , b ∈ R , a < b } where [a , b) = a ⩽ x < b } is a basis for a topology on R because:

i. **For each x ∈ R, x ∈ [x , x + 1),[x , x + 1) ∈ β.**
ii. **If x ∈ [a , b) ∩ [c , d), Then it easy find ϵ > 0 such that x ∈ [x , x +ϵ) and [x , x +ϵ) ⊂ [a , b) ∩ [c , d).**
So β is a basis for a topology on R. the topology τ_i generted by the basis β = { [a,b) : a , b ∈ R and a < b } is called the lower limit topology on R and denoted by R_l = (R , τ_l).

4.1 Topological Subspaces

Definition 1.4.1 :

Let (X,τ) be a topological space. Let Y be a non empty subset of X. A relative topology on Y is defined to be the class of all intersections of Y with open subsets of X. i.e τ_Y = { V = Y ∩ U : U ϵ τ }.
Then τ_Y is a topology on Y , (Y,τ_Y) is called a topological subspace of (X ,τ).

Example 1.4.1 :

Let X = { a,b,c,d,e } , and Y = { a,d,e } and τ = {X,∅,{a} , {c,d} , {a,c,d} , { b,c,d,e}}. Then (Y,τ_Y) is a topological subspace of (X ,τ) where τ_Y= { Y,∅,{a},{d},{a,d},{d,e}}.

Remark 1.4.1 :

i. **Every topological subspace of discrete topological space is discrete.**
ii. **Every topological subspace of indiscrete topological space is indiscrete.**

Theorem 1.4.1 :

If β is a basis for the topology of X and Y is a subspace of X, then the collection β_y = { Y ∩ Y: B ∈ β } is a basis for the topology on Y.

Theorem 2.4.1 :

Let Y be a subspace of X. If U is open set in Y and Y is open set in X, then U is open set in X .

5.1 Product of Topology Space

Definition 1.5.1 :

Let X and Y be any two sets. The Cartesian product, or simply product of X by Y is denoted by X × Y and defined as:

$$\mathbf{X \times Y = \{ (x,y) \; ; \; x \in X \text{ and } y \in Y \}.}$$

Definition 2.5.1 :

Let (X,τ) and (Y, τ') be two topological spaces. We say that the topology which has the base β = { U × V ; U ∈ τ and V ∈ τ' } is the product Topology on the set X × Y and denoted by $\tau_{x\times y}$ and (X × Y , $\tau_{x\times y}$) is called the Product Space of X by Y.

6.1 Hausdorff Space

Definition 1.6.1 :

A topological space X is called a Hausdorff space if for each pair x_1 , x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2 respectively such that $U_1 \cap U_2 = \emptyset$

Theorem 1.6.1 :

Any subspace of a Hausdorff space is a Hausdorff space

Theorem 2.6.1 :

The product of two Hausdorff spaces is a Hausdorff space.

7.1 Continuity in Topological Space

Definition 1.7.1 :

Let (X,τ_1) and (Y,τ_2) be topological spaces . A function f : X → Y is continuous relative to τ_1 and τ_2 , or $\tau_1 - \tau_2$ continuous,or simply continuous, iff the inverse image of every open set of Y is a τ_1 −open set of X. i.e.,

$$\forall \ V \in \tau_2 \ \Rightarrow f^{-1}(V) \in \tau_1 .$$

Example 1.7.1:

Consider the following topologies τ_1, τ_2 on X = {a,b,c,d} and Y = {x,y,z,w} respectively:

$$\tau_1 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\},$$
$$\tau_2 = \{Y, \emptyset, \{x\}, \{y\}, \{x, y\}, \{y, z, w\}\}$$

Define f: X → Y, g : X→Y
$f^{-1}(\tau_2) = \{f^{-1}(Y), f^{-1}(\emptyset), f^{-1}\{x\}, f^{-1}\{y\}, f^{-1}\{x,y\}, f^{-1}\{y,z,w\}\}$

= {X,∅,{a}}. Then the inverse image of every open set in τ_2 is open in X relative to τ_1 so f is a continuous function .

$g^{-1}(\tau_2)$ = {X, ∅, {a,b}, {c,d}}. But {c,d} is not open in τ_1 so g is not a continuous function.

Example 2.7.1 :

Let (X, τ_d) be discrete topological space, and (Y, τ)be any topological space.Then any function f : X → Y is continuous , since if V ∈ τ , then f^{-1}(V) is an open subset of X ,because all subsets of X is open set in the discrete topology .

Example 3.7.1 :

Let (X, τ)be any topological space and (Y, τ_i) be an indiscrete topological space, Then any function f : X → Y is continuous ,since f^{-1}(Y) = X and $f^{-1}(\emptyset) = \emptyset$ which are open .

Remark 1.7.1 :

Let f : X → Y where X and Y are topological space, and let β be a base for the topology on Y .Suppose for each member B $\in \beta$, f^{-1}(B) is an open subset of X . Then f is a continuous function .

Definition 2.7.1 :

i. **A function f : X → Y is called open function if the image of every open set in X is open set in Y.**
ii. **A function f : X → Y is called closed function if the image of every closed set in X is closed set in Y .**

8.1 Homeomorphic Spaces

Definition 1.8.1 :

Two topological spaces X and Y are called homeomorphic or topologically equivalent if there exists a bijective function : X → Y such that f and f^{-1} are continuous. The function f is called a homeomorphism function .

Theorem 1.8.1 :

For a 1-1 mapping f of a topological space X onto a topological space Y, the following conditions are equivalent :

1. **The mapping f is a homeomorphism .**
2. **The mapping f is closed and continuous .**
3. **The mapping f is open and continuous .**
4. **The set f(A) is closed in Y if and only if A is closed in X .**
5. **The set f(A) is open in Y if and only if A is open in X.**

Example 1.8.1 :

Let X = {1,2,3,4}, τ = { X ,∅,{1,2},{3,4}} and Y = {a,b,c,d}, τ_2 = {Y,∅ , {a,d} ,{b,c}}. Then f : X → Y , which is defined as follows :
f(1) = a,f(4) = b, f(3) = c, f(2) = d , is homeomorphism because :

i. **Clearly f is one to one and onto .**

ii. **f is continuous since**
f^{-1}(Y) = X , f^{-1}(∅) = ∅ , f^{-1}{a,b}={1,2} , f^{-1}{b,c} = {3,4}.

iii. **f^{-1} : Y → X is defined by**
f^{-1}(a)=1 , f^{-1}(b) = 4 , f^{-1}(c) = 3 , f^{-1}(d) = 2
Then f^{-1} is continuous , since
$(f^{-1})^{-1}$(X) = Y , $(f^{-1})^{-1}$(∅) = ∅, $(f^{-1})^{-1}${1,2} = {a,b},
$(f^{-1})^{-1}${3,4} ={b,c} .

So f is a homeomorphism .

Remark 1.8.1:

Let f : R → R be a function defined by f(X) = x ,∀ x ∈ R where (R,τ) is a standard topology. Then f is homeomorphism, Let (a,b) be an open interval of R . Then f^{-1} (a,b) = (a,b) which is open set so f is continuous.
Moreover , f is 1-1 ,onto and $f = f^{-1}$, thus f is a homeomorphism.
so Every identity map of (X, τ_x) is a homeomorphism .

2. Connected Spaces and Path Connected Space

1.2 Connected Space

The definition of connectedness for a topological space is a quite natural one. One says that a space can be a (separated) if it can be broken up into two (globs) disjoint open sets . Otherwise , one says that it is connected . From this simple idea we get the following definition:

Definition 1.1.2 :

Let (X, τ) be a topological space. Two subsets U, V of X are said to be separated if U, V are disjoint non-empty open subsets of X whose union is X. i.e.,

i. **U and V are open sets in X**
ii. **$U \neq \emptyset$ and $V \neq \emptyset$.**
iii. **$U \cap V = \emptyset$, U , V$\in$ τ**
iv. **$X = U \cup V$.**

Remark 1.1.2 :

i. **If U,V is a separation of a topological space X, then $U = X \setminus V$ and $V = X \setminus U$. So U and V are both open and closed sets in X.**
ii. **If $U \subset X$ is both open and closed set in X and $U \neq \emptyset$, $U \neq X$ then U,V = $X \setminus U$ is a separation of X.**
iii. **If X is connected and $U \subset X$ is both open and closed set, then either $U = \emptyset$ or $U = X$.**

Definition 2.1.2 :

i. **A topological space X is said to be connected if there is no separation of X.**
ii. **A topological space X is said to be disconnected if there is separation of X.**

Example 1.1.2 :

(R , τ_l) is disconnected space where τ_l is the lower limit topology because U = ($-\infty$, 0) , V = [0 , ∞) is separation of (R , τ_l).

Example 2.1.2 :

i. **R_l = (R , τ_l) is disconnected space because for example U = [0,1) ⊂ R_l is both open and closed set in R_l and U ≠ ∅ U ≠ R .**
ii. **(X , τ_d) is disconnected space for any nonempty set with the number of elements of X is bigger than 1 where τ_d is the discrete topology.**
iii. **(X , τ_i) is connected space for any nonempty set X where τ_i is the trivial topology.**

Examples 3.1.2 :

Let X = {1,2,3,4,5} , τ_1 = {X,∅,{1},{1,2}}, τ_2 = {X,∅,{1,2},{3,4,5}. Then (X,τ_1) is connected and (X,τ_2) is disconnected.

Theorem 1.1.2 :

The Euclidean space R is connected space .

Proof : Assume R is disconnected space and U , V is a separation of R, then R = U ∪ V where U and V are both open and closed subsets of R . Take a ∈ U and b ∈ V then a ≠ b. Suppose a < b and let W = [a , b] ∩ U. Note that W is closed and bounded subset of R , so Sub W exist say c and c ∈ W . Clearly c < b and c ∈ W ⊂ U. Note that (c , b] ∩ U = ∅, so (c , d] ⊂ V thus (c,d] ⊂ V = V and hence [c,b] ⊂ V , so c ∈ V, so c ∈ U ∩ V which is a contradiction. So R is connected space

Corollary 1.1.2 :

A subspace E of the Euclidean space R is connected if and only if E is an interval.

Theorem 2.1.2 :

The image of a connected space under a continuous map is connected space.
Proof:
Let X be a connected space. We need prove that f(X) is connected space. Assume by contradiction f(X) is disconnected. Then, there is a separation of f(X). Let U,V be a separation f(X), then note that :

i. **f^{-1}(U) , f^{-1}(V) are open sets since f is continuous map and U , V are open sets.**
ii. **$f^{-1}(U) \neq \emptyset$, $f^{-1}(V) \neq \emptyset$ since $V \neq \emptyset$, $U \neq \emptyset$**
iii. **$f^{-1}(U) \cap f^{-1}(V) = \emptyset$ then $U \cap V = \emptyset$.**
iv. **$f^{-1}(U) \cup f^{-1}(V) = X$ since $U \cup V = f(X)$**

So f^{-1}(U) , f^{-1}(V) is a separation of X and thus X is disconnected which is contradiction with the given that X is connected space and therefor f(X) is connected space.

Theorem 3.1.2 :

Let U, V be a separation of the topological space X. If $Y \subset X$ is connected space, then either $Y \subset U$ or $Y \subset V$.

Proof:
Take $U_1 = U \cap Y$, $V_1 = V \cap Y$, then U1 and V_1 are open subsets of Y, $U_1 \cap V_1 = \emptyset$ and $Y = U_1 \cup V_1$. As Y is connected , either $U_1 = \emptyset$ or $V_1 = \emptyset$. If $U_1 = \emptyset$, Then $Y \subset V$ and if $V_1 = \emptyset$, then $Y \subset U$.

Theorem 4.1.2 :

If A is a connected subspace of a topological space X and $A \subseteq B \subseteq \bar{A}$, then B is connected.

Proof:
Suppose B is not connected and C, D be a separation of B.

Then by theorem 1.1.2, either $A \subset C$ or $A \subset D$.

Suppose $A \subset C$, then $\bar{A} \subset \bar{C}$, so $\bar{A} \cap B \subset \bar{C} \cap B$, so $B \subset C$

and thus $D = \emptyset$ which is a contradiction. So B is connected.

Theorem 5.1.2 :

X and Y are connected space if and only if the product space $X \times Y$ is connected space.

Remark 2.1.2 :

i) **R^n is connected space.**

Proof: **Not that $R^n = \underbrace{R \times R \times \dots . \times R}_{n-times}$. So by we see it is connected space.**

ii) **$R \times R_l$ is disconnected space.**

Proof: **Not that R_l is disconnected space (see example 2.1.2). So, by theorem 5.1.2 we see $R \times R_l$ is disconnected space.**

iii) **The general linear group $GL(n, R) = \{A \in M(n, R) | \det A \neq 0\}$ as a subspace of $M(n, R) = R^{n^2}$ is disconnected.**

Proof: **Define the following function:**

$$f: M(n, R) \longrightarrow R$$

$$f(A) = \det A$$

Not that f is a continuous function. Assume $GL(n, R)$ is connected space , then by theorem 2.1.2 $f\big(GL(n, R)\big) = (-\infty, 0) \cup (0, \infty)$ is connected which is a contradiction. Therefore $GL(n, R)$ is disconnected space.

2.2 Path Connected

Definition 1.2.2 :

Let X be a topological space and p, q ∈X. A path in X from p to q is a continuous map f:[a, b] → X such that f(a) = p and f(b) = q.

Definition 2.2.2 :

A topological space X is said to be path connected if for any p , q ∈ X, there is a path in X from p to q.

Example 1.2.2 :

Consider the Euclidean space R^n , n≥1 For any p, q ϵ R^n, the function f:[0,1] → R^n, $f(t) = tq + (1 - t)p$ .is continuous map satisfies $f(0) = p, f(1) = q$.

So, f is a path from p to q and therefore R^n is path connected space.

Remark 1.2.2 :

Let X be a topological space and p , q and r ∈ X. Let ∝ , β : [0, 1] → X be two paths such that $\propto(0) = p$, $\propto(1) = q$, $\beta(0) = q$, $\beta(1) = r$.

Now, define γ: [0, 1] → X by:

$$\gamma(t) = \begin{cases} \propto(2t), & t \in \left[0, \frac{1}{2}\right] \\ \beta(2t - 1), & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

As $\propto(1) = (0) = q$, γ: [0, 1] → X is continuous map. So, γ is a path from $\gamma(0) = p$ to $\beta(1) = r$

Remark 2.2.2 :

Let X be a topology space and $\propto$: [0,1] → X be a path from p to q that is $\propto$ (0) = p and $\propto$ (1) = q. Then, the function β:[0,1] → X which is defined by $\beta(t) = \propto (1-t)$is path from $\beta(0) = q$ to $\beta(1) = p$ therefore x is disconnected space.

Theorem 1.2.2 :

Any path connected space is connected space.
Proof:
Let X be a path connected space. Assume by contradiction, X is disconnected space. Then there is a separation U and V of X. Let p ∈ U, q ∈ V. As X is path connected space, there is a path (continuous map) f : [0,1] → X such that f(0)=p , f(1)=q. As [0,1] is connected and f is continuous map, so f([0,1]) ⊂ U or f([0,1]) ⊂ V (since f is continuous and [0,1] is connected) but f(0)=p ∈ U , f(1)=q ∈ V which is a contradiction so X is connected space.

Theorem 2.2.2 :

Every continuous image of path connected space is path connected.
Proof:
Let X be path connected space and f : X → Y continues map. Let p,q∈ f(x). Take x ∈ f^{-1}(p) and y ∈ f^{-1}(q) Then also we have f : [0,1] → such that f(0) = x , f(1) = y. Now g o f (0) = f(x) = p, g o f (1) = f(y) = q, so g o f : p → q is path connected.

Theorem 3.2.2 :

A continuous image of path connected space is path connected.

Proof: If X and Y are two topological spaces such that X is path connected space.

Let f: X → Y be a surjective continuous map.

For any p, q ∈ Y, there exist p', q' ∈ X such that $f(p') = p$, $f(q') = q$.

As X is path connected, there is a path $\propto: [a, b] \to X$ such that

$\propto(a) = p', \propto(b) = q'$.

Now, note that $\beta = f \circ \propto: [a, b] \to Y$ is a path from

$\beta(0) = f(\propto(0)) = p$ to $\beta(1) = f(\propto(1)) = q$. So, Y is path connected space.

Example 2.2.2 :

Consider the unit disk $D^n = \{x \in R^{n+1} / \|x\| \leq 1\}$, as a subspace of the Euclidean space R^n where $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^2}$.

For any $p, q \in D^n$, the function $f: [0, 1] \to D^n, f(t) = (1 - t)p + tq$ is well define since for any $t \in [0, 1]$, we have:

$$\|f(t)\| \leq (1 - t)\|p\| + t\|q\| \leq 1$$

and f is continuous map satisfies $f(0) = p$ *and* $f(1) = q$.

So, f is a path from p to q and therefore D^n is path connected space.

Example 3.2.2 :

Consider the space $X = R^n \setminus \{0\}, n > 1$ as a subspace of the Euclidean space R^n .Take $p, q \in X$, then we have two cases:

Case 1: If the straight line joining p and q does not pass through to the origin $0 = (0, 0, \ldots, 0)$, then the function:

$$\propto: [0, 1] \to X$$

$$\propto(t) = (1 - t)p + tq$$

a path in X from p to q.

Case 2: If the straight line joining p and q pass through the origin 0 = (0, 0, …, 0), then choose a point $x \in X$ not on the line joining p and q. Note that the functions : $\propto, \beta: [0, 1] \to X$, given by:

$$\propto(t) = (1 - t)p + tx$$

$$\beta(t) = (1 - t)x + tq$$

are paths in X. So, the function $\gamma : [0,1] \to X$, defined by:

$$\gamma(t) = \begin{cases} \propto (2t), & t \in \left[0, \frac{1}{2}\right] \\ \beta\,(2t - 1), & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

is path in X from p, q. So, X = $R^n \setminus \{0\}$, n >1 is path connected space.

Example 4.2.2 :

Consider the unit sphere $R^{n-1} = \{x \in R^n / \|x\| = 1\}$ as subspace of

the Euclidean space R^n, n >1 where $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

Note that the function: $f = R^n \setminus \{0\} \to S^{n-1}, f(x) = \frac{x}{\|x\|}$ is surjective

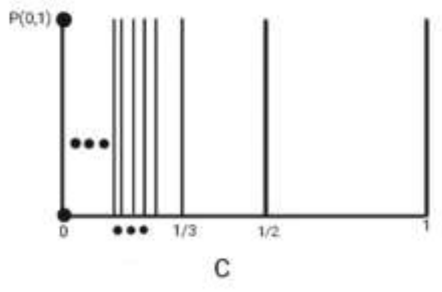
continuous map.

As $R^n \setminus \{0\}$, $n > 1$ is path connected space, S^{n-1} , $n > 1$ is path connected space (by theorem 1.2.2).

Example 5.2.2 :

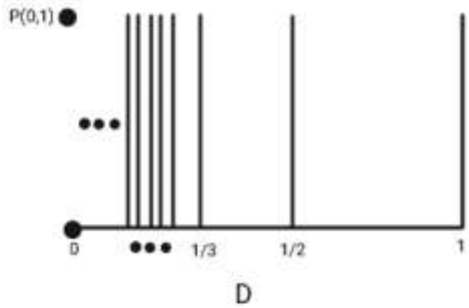
Let K = $\{\frac{1}{n} : n \in N\}$ and consider subspace C of the Euclidean space R^2 which is defined by:

$$C = ([0,1] \times \{0\}) \cup (K \times [0,1]) \cup (\{0\} \times [0,1]).$$



The space C is called the comb space and it is clear that this space is path connected and therefore it is connected space.

Let D = C \ ({0} × (0,1)) as subspace of the Euclidean space R^2. This space is called the deleted comb space.



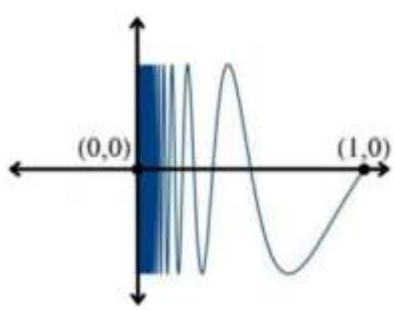
Note that $D\backslash\{p\}$, $p = (0, 1)$ is path connected space and therefore $D\backslash\{p\}$ is connected space. As p is limit point of D, D is connected space but it is clear that D is not path connected space.

Example 6.2.2 :

Let A={ (x, sinπ/ x): $x \in (0, 1]$ } and B = {{p = (0, 0) } as a subspace of the Euclidean space R^2.

Note that A is path connected space and so it is a connected space.

It is clear that p = (0, 0) is a limit point of A and therefore T = A U B is connected space but it is not path connected space because there is no path in T from p to any point q in A.



Conclusion 1.2.2 :

Any path connected space is connected space (see theorem 1.2.2) but the converse is not true (see the previous example).

3. Locally Connected Space and Locally Path Connected Space

1.3 Locally Connected Space

Definition 1.1.3 :

Let X be a topological space and $x \in X$ we say that X is locally connected at x if every neighborhood of x contains a connected neighborhood of x. If X is locally connected at every point $x \in X$ then we say that X is locally connected.

Example 1.1.3 :

R with standard topology is connected space by Theorem 2.1 and locally connected space because for any point $x \in R$ and any neighborhood U of x, we can choose $\epsilon > 0$ such that $x \in (x - \epsilon, x + \epsilon) \subset U$.

Example 2.1.3 :

Each interval and each ray in the real line is both connected and locally connected. The subset [-1,0) U (0,1] of $\mathbb{R}$ is not connected but it is locally connected.

Example 3.1.3 :

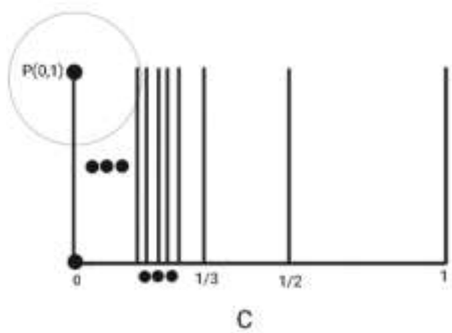
The space $\mathbb{Q}$ of rational numbers is neither connected nor locally connected.

Example 4.1.3 :

The discrete topological space for any non-empty set X is locally connected space because for every point $x \in X$ and any neighborhood U of x there is a connected neighborhood { x } of x and we note that { x}⊂U.

Example 5.1.3 :

The comb space is connected space but it is not locally connected at p = (0 , 1) .



Conclusion 1.1.3 :

By the previous example, we conclude that local connectedness and connectedness of a space are not related to one another.

2.3 Locally Path Connected

Definition 1.2.3:

A space X is said to be locally path connected at x if for every neighborhood U of x, there is a path-connected neighborhood V of x such that $V \subset U$. If X is locally path connected at each its points, then it is said to be locally path connected space.

Example 1.2.3 :

R with standard topology is both path connected and locally path connected.

Example 2.2.3 :

Let $X = (0,1) \cup (3,4)$ as subspace of the Euclidean space R. Then X is neither path connected nor connected but it is both locally connected and locally path connected.

Example 3.2.3 :

The set of rational numbers Q as subspace of the Euclidean space R is neither path connected nor locally path connected.

Example 4.2.3 :

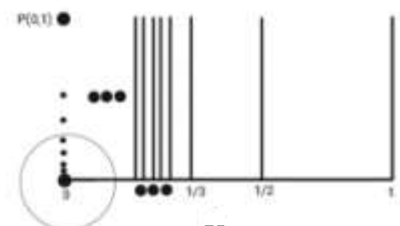
The comp space is path connected space but it is not locally path connected space .

Conclusion 1.2.3 :

By the previous examples, we conclude the local path connectedness and path connectedness of a space are not related to one another.

Example 5.2.3 :

Consider the space $\mathbf{Y = D \cup \{(0, \frac{1}{n}) \in R^2 : n \in Z^+\}}$ as subspace of the Euclidean space $\mathbf{R^2}$ where D is the deleted comb space. Then, Y is locally connected at the origin but not locally path connected of the origin.



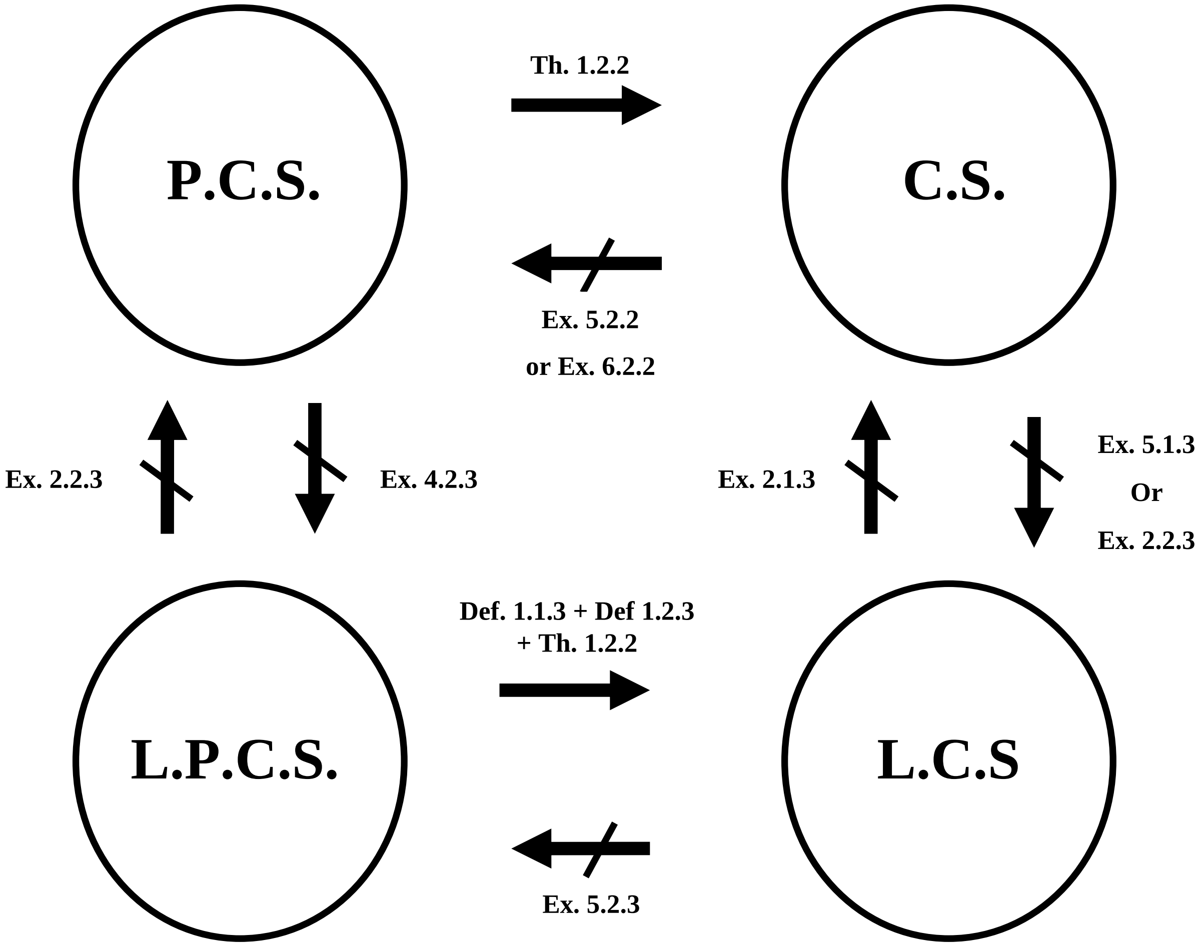
Y

Conclusion 2.2.3 :

It is obvious that every locally path connected is locally connected but the converse is not true (see example 5.2.3) .

Remark 1.2.3 :

Note that the following:



where :

P.C.S. : Path Connected Space .

C.S. : Connected Space .

L.P.C.S : Locally Path Connected Space.

L.C.S : Locally Connected Space.

References:

1. James R. Munkres, Topology; First Course, Prentice Hall, Inc. Englewood Cliffs, New Jersey, 1975.

2. Fred H.Croom, Principles of Topology, Bada Barkha Nath Printers, India, 2008.