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Fredholm Integral Equations

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بسم الله الرحمن الرحيم

شكر وعرفان

إلهي لا يطيب الليل إلا بشكرك.. ولا يطيب النهار إلا بطاعتك..
ولا تطيب اللحظات إلا بذكرك .. ولا تطيب الآخرة إلا بعفوك..
ولا تطيب الجنة إلا برؤيتك سبحانك

أقدم أسمى آيات الشكر والامتنان والتقدير:

إلى من كلله الله بالهيبة والوقار. .إلى من علمني العطاء دون انتظار..
إلى من أحمل اسمه بكل افتخار.. أرجو من الله أن يمد في عمرك لترى ثماراً قد حان قطافها بعد طول انتظار.. وستبقى كلماتك نجوما أهتدي بها طول الزمان
.. والدي العزيز...

إلى ملاكي في الحياة ..إلى معنى الحب
وإلى معنى الحنان والتفاني ..إلى بسمة الحياة وسر الوجود
إلى من كان دعاؤها سر نجاحي ..وحنانها بلسم جراحي
...أمي الحبيبة...

إلى الذي حمل أقدس رسالة في الحياة
إلى الذي مهد لي طريق العلم والمعرفة
إلى من علمني التفاؤل والمضي إلى الأمام
إلى من وقف إلى جانبي عندما ضللت الطريق
...د. لزهر بوقفه و د. عمار خنفر...

إلى القلوب الطاهرة الرقيقة والنفوس البريئة
إلى رياحين حياتي
...إخوتي...

محبتكم **:موضي البريكان**

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INTRODUCTION

The last century witnessed a revolution in physical sciences that caused a profound change in the concepts and tools of applied mathematics. The theory of integral equations was one of these tools that have emerged and played – and still playing – a major role in studying the behavior of solutions of various types of boundary value problems. As a result, an intensive amount of research has been carried on this area, and the majority of it was devoted to the Fredholm integral equations, because they offer a powerful tool in solving a wide spectrum of problems related to physics and other areas.

The theory has been developed by Fredholm in 1900, then flourished by the hands of Hilbert, Frechet, Riesz, Lebesgue, and other famous mathematicians. Today, the theory occupies a prominent place in the mathematical research, and attracts an increasing number of researches from different disciplines. This research project is a humble attempt to make a contribution to this theory and its applications.

The project consists of four chapters. The first chapter introduces important theorems, such as Banach contraction principle and existence and uniqueness theorem of linear and nonlinear Fredholm integral equations in the spaces $C([a,b])$ and L_2.

The second chapter deals with the Adomian decomposition method (ADM), and how to use it to find solutions of the linear and nonlinear Fredholm integral equations of second kind. Then we discuss the convergence analysis of the ADM for them.

In the third and fourth chapters, Fredholm integral equations of the first kind and Schlömilch's integral equation are transformed in such a manner that the ADM can be applied. We use the regularization method combined with the ADM to handle all forms of this type of integral equations. We also present theorems concerning the convergence of the ADM.

Chapter 1

Existence and Uniqueness Theorems for the Fredholm Integral Equation of the Second Kind

1.1 Fixed-Point Theorem and its Applications

Theorems concerning the existence and properties of fixed points are known as fixed-point theorems [6,11]. Such theorems are the most important tools for proving the existence and uniqueness of the solution to various mathematical models (differential integral and partial differential equation, etc.) representing phenomena arising in different fields, such as steady-state temperature distribution, chemical reaction, neutron transept theory, economic theories, epidemics and flow of fluids.

In this chapter, we state and prove the Banach contraction principle which is one of the simplest and most useful methods for the contraction of solution of linear and nonlinear integral equation.

<u>Definition (1.1)</u>
Let T be a mapping of a normed space X into itself.

$$T: X \to X,$$

then T is called a contraction mapping if there exists a constant M, $0 \leq M < 1$ such that

$$\|Tx - Ty\| \leq M\, \|x - y\|, \qquad \forall x, y \in X. \tag{1.1}$$

<u>Definition (1.2)</u>
Let X be a Banach space and $T: X \to X$, then a point $x \in X$ such that

$$Tx = x$$

is called a fixed point of T.

<u>Theorem (1.3):</u> Banach contraction principle
Let X be a Banach space and $T: X \to X$ is a contraction mapping, then it has a unique fixed point $x^* \in X,\ i.e. Tx^* = x^*$.

<u>Proof:</u>
We first show the existence of the fixed point.

Let x_0 be an arbitrary point in X and we define

$$x_1 = Tx_0,\ x_2 = Tx_1, \dots,\ x_n = Tx_{n-1}\,.$$

Thus,

$$x_2 = Tx_1 = T(Tx_0) = T^2x_0,$$

$$x_3 = Tx_2 = T(T^2x_0) = T^3x_0,$$

$$\vdots$$

$$x_n = Tx_{n-1} = T(T^{n-1}x_0) = T^n x_0$$

If $m > n$, say $m = n + p,\ p = 1, 2, 3, \ldots$

then

$$\|x_{n+p} - x_n\| = \|T^{n+p}x_0 - T^n x_0\| = \|T(T^{n+p-1}x_0) - T(T^{n-1}x_0)\|$$

$$\leq M\ \|T^{n+p-1}x_0 - T^{n-1}x_0\|.$$

Continuing this process $(n-1)$ times, we have:

$$\|x_{n+p} - x_n\| \leq M^n\ \|T^p x_0 - x_0\|,\ n = 0,1, \ldots. \qquad (1.2)$$

However,

$$\|T^p x_0 - x_0\| = \|T^p x_0 - T^{p-1}x_0 + T^{p-1}x_0 - T^{p-2}x_0 + T^{p-2}x_0 - \cdots + Tx_0 - x_0\|$$

$$\leq \|T^p x_0 - T^{p-1}x_0\| + \|T^{p-1}x_0 - T^{p-2}x_0\| + \cdots + \|Tx_0 - x_0\|.$$

Since

$$T^P x_0 = T^{P-1}x_1,\ T^{P-1}x_0 = T^{P-2}x_1, \ldots,\ Tx_0 = x_1,$$

we have

$$\|T^p x_0 - x_0\| \leq \|T^{p-1}x_1 - T^{p-1}x_0\| + \|T^{p-2}x_1 - T^{p-2}x_0\| + \cdots + \|x_1 - x_0\|.$$

By (1.2) we see that:

$$\|x_{n+p} - x_n\| \leq M^n[M^{p-1}\ \|x_1 - x_0\| + M^{p-2}\|x_1 - x_0\| + \cdots + \|x_1 - x_0\|],$$

or

$$\|x_{n+p} - x_n\| \leq M^n[M^{p-1}\ + M^{p-2} + \cdots + M + 1]\|x_1 - x_0\|$$

$$\leq M^n \frac{1}{1-M}\|x_1 - x_0\|.$$

As $n, m = n + p \to \infty$, we see that

$$\|x_{n+p} - x_n\| \to 0,$$

that is (x_n) is a Cauchy sequence in the Banach space X.

Hence, (x_n) converges to a limit $x^* \in X$.

Since T is continuous, we have

$$Tx^* = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Thus x^* is a fixed point of T.

Now, we prove the uniqueness of the fixed point.

Let x^* and y^* be two fixed points, $Tx^* = x^*$ and $Ty^* = y^*$, we also have

$$\|Tx^* - Ty^*\| \leq M\,\|x^* - y^*\|,$$

as T is a contraction.

But

$$\|Tx^* - Ty^*\| = \|x^* - y^*\|.$$

Thus

$$\|x^* - y^*\| \leq M\|x^* - y^*\|,$$

or

$$(1 - M)\|x^* - y^*\| \leq 0,$$

since $M < 1$, we have

$$\|x^* - y^*\| = 0 \Longrightarrow x^* = y^*.$$

This proves that the fixed point of T is unique.

■

In general, the condition that M is strictly less than one is needed for uniqueness and the existence of a fixed point. For example, if $X = \{0,1\}$ is the discrete metric space with metric determined by $d(0,1) = 1$, then T defined by $T(0) = 1$ and $T(1) = 0$ satisfies (1.1) with $M = 1$, but T does not have any fixed point.

It may happen that X is not complete in any metric for which one can prove that T is a contraction. This can be an indication that the solution of the fixed point problem does not belong to X.

1.2 Integral Equations

Definition (1.4):

An integral equation is an equation containing an unknown function under an integral operator and can be written as [13]:

$$\alpha(x)\,\varphi(x) = f(x) + \lambda \int_{a(x)}^{b(x)} k(x,t)\, F\big(\varphi(t)\big)dt, \tag{1.3}$$

where $f(x)$, $\alpha(x)$, $a(x)$ and $b(x)$ are given functions, $k(x,t)$ is called the kernel, λ is a parameter and $F\big(\varphi(t)\big)$ is nonlinear function.

- A linear Fredholm integral equation of the second kind for an unknown function $\varphi: [a,b] \to \mathbb{R}$ is an equation of the form [13]:

$$\varphi(x) = f(x) + \lambda \int_a^b k(x,t)\, \varphi(t)dt \tag{1.4}$$

- A Fredholm integral equation of the first kind is an equation of the form [13]:

$$f(x) = \lambda \int_a^b k(x,t)\, \varphi(t)dt \tag{1.5}$$

1.3 Theorems

1.3.1 Linear Fredholm Integral Equations of the Second Kind

Theorem (1.5):

Let $k(x,t)$ be defined in

$$A = \{(x,t): a \le x \le b, a \le t \le b\}$$

and

$$k: [a,b] \times [a,b] \to \mathbb{R}$$

such that

$$\int_a^b \int_a^b |k(x,t)|^2\, dx\, dt < \infty \text{ and } f(x) \in L_2[a,b].$$

Then the integral equation (1.4) has a unique solution $\varphi(x) \in L_2[a,b]$ for sufficiently small value of the parameter λ.

Proof:

Let $X = L_2$ and consider the mapping $T: L_2[a,b] \to L_2[a,b]$

$$T\varphi = h$$

where

$$h(x) = f(x) + \lambda \int_a^b k(x,t)\, \varphi(t)dt.$$

This definition is valid for each $\varphi \in L_2[a,b]$ and $h \in L_2[a,b]$.

Since $f \in L_2[a,b]$ and $\lambda \in \mathbb{R}$ it is sufficient to show that

$$\int_a^b k(x,t)\ \varphi(t)dt \in L_2[a,b].$$

Let

$$\psi(x) = \int_a^b k(x,t)\ \varphi(t)dt,$$

then by the Cauchy-Schwartz inequality we have:

$$|\psi(x)| = \left|\int_a^b k(x,t)\ \varphi(t)dt\right| \leq \int_a^b |k(x,t)\ \varphi(t)|dt$$

$$\leq \left(\int_a^b |k(x,t)|^2\ dt\right)^{1/2} \left(\int_a^b |\varphi(t)|^2\ dt\right)^{1/2}\ .$$

Therefore,

$$|\psi(x)|^2 \leq \int_a^b |k(x,t)|^2\ dt \int_a^b |\varphi(t)|^2\ dt,$$

or

$$\int_a^b |\psi(x)|^2\ dx \leq \int_a^b \left(\int_a^b |k(x,t)|^2\ dt\right)\ dx \int_a^b |\varphi(t)|^2\ dt.$$

By the hypotheses,

$$\int_a^b \left(\int_a^b |k(x,t)|^2\ dt\right)\ dx < \infty \text{ and } \varphi \in L_2[a,b].$$

We have, $\psi(x) \in L_2[a,b]$.

Now, we show that T is a contraction, for any $\varphi_1, \varphi_2 \in L_2[a,b]$, we have

$$\|T\varphi_1 - T\varphi_2\| = \|h_1 - h_2\|,$$

where

$$h_1(x) = f(x) + \lambda \int_a^b k(x,t)\ \varphi_1(t)dt$$

and

$$h_2(x) = f(x) + \lambda \int_a^b k(x,t)\ \varphi_2(t)dt.$$

Thus

$$\|h_1 - h_2\| = \left\|\lambda \int_a^b k(x,t)\ [\varphi_1(t) - \varphi_2(t)]dt\right\|$$

$$= |\lambda| \left(\int_a^b \left|\int_a^b k(x,t)\ [\varphi_1(t) - \varphi_2(t)]dt\right|^2 dx\right)^{1/2}$$

$$\leq |\lambda| \left(\int_a^b \int_a^b |k(x,t)|^2 \, dx \, dt\right)^{1/2} \left(\int_a^b |\varphi_1(t) - \varphi_2(t)|^2 dt\right)^{1/2}.$$

Hence

$$\|T\varphi_1 - T\varphi_2\| \leq |\lambda| \left(\int_a^b \int_a^b |k(x,t)|^2 \, dx \, dt\right)^{1/2} \|\varphi_1 - \varphi_2\| \leq C \, \|\varphi_1 - \varphi_2\|,$$

where

$$C = |\lambda| \left(\int_a^b \int_a^b |k(x,t)|^2 \, dx \, dt\right)^{1/2}.$$

If we choose $C < 1$, that is

$$|\lambda| < \frac{1}{\left(\int_a^b \int_a^b |k(x,t)|^2 \, dx \, dt\right)^{1/2}}.$$

Thus,

T is a contraction and by Theorem (1.3), T has a unique fixed point, that is,

$$T\varphi^* = \varphi^*, \;\; \varphi^* \in L_2[a,b].$$

■

1.3.2 Nonlinear Fredholm Integral Equations of the Second Kind

The same technique can be applied to the nonlinear integral equation [13]:

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,t,\varphi(t)) dt, \tag{1.6}$$

where K and f are continuous functions, and for all $x, t \in [a,b]$ and $y \in \mathbb{R}$

$$|K(x,t,z_1) - K(x,t,z_2)| \leq M|z_1 - z_2|, \text{ where } M \text{ is constant.} \tag{1.7}$$

The integral equation (1.6) can be written as a fixed point equation $T\varphi = \varphi$, where the map T is defined by

$$T\varphi = f(x) + \lambda \int_a^b K(x,t,\varphi(t)) dt,$$

where

$$T: \mathcal{C}([a,b]) \to \mathcal{C}([a,b]).$$

We can prove in the same manner that T is a contraction.

We have for any $\varphi_1, \varphi_2 \in \mathcal{C}([a,b])$,

$$\|T\varphi_1 - T\varphi_2\|_\infty = \|h_1 - h_2\|_\infty = \left\|\lambda \int_a^b \Big(K\big(x,t,\varphi_1(t)\big) - K\big(x,t,\varphi_2(t)\big)\Big)\, dt\right\|_\infty$$

$$\leq |\lambda| \sup_{a\leq x\leq b} \int_a^b \left|K\big(x,t,\varphi_1(t)\big) - K\big(x,t,\varphi_2(t)\big)\right| dt$$

$$\leq |\lambda| \sup_{a\leq x\leq b} \int_a^b M|\varphi_1 - \varphi_2|\, dt \ \leq |\lambda|\, M(b-a)\|\varphi_1 - \varphi_2\|_\infty \ .$$

Hence,

$$\|T\varphi_1 - T\varphi_2\|_\infty \leq C\ \|\varphi_1 - \varphi_2\|_\infty,$$

where

$$C = |\lambda|\, M(b-a).$$

Consequently, the mapping T is a contraction if

$$|\lambda| < 1/M(b-a). \tag{1.8}$$

Hence, T has a unique fixed point, that is

$$T\varphi^* = \varphi^*, \varphi^* \in \mathcal{C}([a,b]).$$

Thus we have proved.

Theorem (1.6):

If λ satisfies (1.8) and K satisfies (1.7), then the integral equation (1.6) has a unique solution $\varphi(x) \in \mathcal{C}([a,b])$.

Chapter 2

ADM for Solving the Fredholm Integral Equation of the Second Kind

2.1 The Adomian Decomposition Method (ADM)

The Adomian decomposition method has been receiving much attention in recent years in applied mathematics in general, and in the area of series solutions in particular. The method was proved to be powerful, effective, and can easily handle a wide class of linear or nonlinear, ordinary or partial differential equations, and linear and nonlinear integral equations. The ADM demonstrates fast convergence of the solution and therefore provides several significant advantages. The method will be successfully used to handle most types of ordinary differential equations that appear in several physical models and scientific applications [1,2,3,7,8,9,10,12].

The ADM consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x),$$

where the components $u_0, u_1, u_2, \ldots$ are to be determined in a recursive manner. The ADM concerns itself with finding the components individually. The determination of these components can be achieved in an easy way through a recursive relation that usually involves simple integrals.

2.1.1 Solving Linear Ordinary Differential Equations by the ADM

We first consider the linear differential equation written in an operator form as

$$Lu + Ru = g, \qquad (2.1)$$

where L is the highest order derivative which is assumed to be invertible (L^{-1} exists), R is other linear differential operator and g is a source term.

We next apply the inverse operator L^{-1} to both sides of equation (2.1) and use the given condition to obtain

$$u = f - L^{-1}(Ru), \qquad (2.2)$$

where the function f represents the term arising from integrating the source term g and from using the given conditions that are assumed to be prescribed. As indicated before, ADM defines the solution u by an infinite series of components given by

$$u = \sum_{n=0}^{\infty} u_n, \qquad (2.3)$$

where the components $u_0, u_1, u_2, \ldots$ are usually recurrently determined.

Substituting equation (2.3) into both sides of equation (2.2) leads to

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R \sum_{n=0}^{\infty} u_n\right),$$

which can be rewritten as

$$u_0 + u_1 + u_2 + \cdots = f - L^{-1}\left(R(u_0 + u_1 + u_2 + \cdots)\right).$$

To construct the recursive relation needed for the determination of the components $u_0, u_1, u_2, \ldots$, it is important to note that the ADM suggests that the zeroth component u_0 is usually defined by the function f described above, i.e. by all terms, that are not included under the inverse operator L^{-1}, which arise from the initial data and from integrating the inhomogeneous term. Accordingly, the formal recursive relation is defined by

$$\begin{cases} u_0 = f, \\ u_{\mathrm{n}+1} = -L^{-1}\left(R(u_{\mathrm{n}})\right), \quad n \geq 0, \end{cases} \tag{2.4}$$

or equivalently

$$\begin{cases} u_0 = f, \\ u_1 = -L^{-1}\left(R(u_0)\right), \\ u_2 = -L^{-1}\left(R(u_1)\right), \\ \quad\quad . \\ \quad\quad . \\ \quad\quad . \\ u_{\mathrm{n}} = -L^{-1}\left(R(u_{\mathrm{n}-1})\right), \quad n \geq 0. \end{cases} \tag{2.5}$$

Consequently, the solution can be obtained in a series form.

Example 1.

Use ADM to solve the following equation [12]

$$u'(x) = u(x), u(0) = A. \tag{2.6}$$

In an operator form equation (2.6) becomes

$$Lu = u, \tag{2.7}$$

where the differential operator L is given by

$$L = \frac{d}{dx},$$

and therefore the inverse operator L^{-1} is defined by

$$L^{-1}(.) = \int_0^x (.)\, dx.$$

Applying L^{-1} to both sides of equation (2.7) and using the initial condition, we obtain

$$u(x) - u(0) = L^{-1}(u),$$

or equivalently

$$u(x) = A + L^{-1}(u). \tag{2.8}$$

Substituting equation (2.3) into both sides of equation (2.8) gives

$$\sum_{n=0}^{\infty} u_n = A + L^{-1}\left(\sum_{n=0}^{\infty} u_n\right). \tag{2.9}$$

In view of equation (2.9), we obtain the following recursive relation

$$\begin{cases} u_0(x) = A \\ u_{n+1}(x) = L^{-1}\big(u_n(x)\big), \quad n \geq 0. \end{cases} \tag{2.10}$$

or

$$\begin{cases} u_0(x) = A, \\ u_1(x) = L^{-1}\big(u_0(x)\big) = Ax, \\ u_2(x) = L^{-1}\big(u_1(x)\big) = Ax^2/2\,, \\ u_3(x) = L^{-1}\big(u_2(x)\big) = {Ax^3}/{3!}\,, \\ \qquad \vdots \end{cases} \tag{2.11}$$

Consequently, the solution is given by

$$\begin{aligned} u(x) &= A + Ax + Ax^2/2 + \cdots \\ &= A\,(1 + x + x^2/2 + \cdots) \\ &= Ae^x. \end{aligned}$$

2.1.2 Solving the Nonlinear Ordinary Differential Equations by the ADM

To apply the ADM for solving nonlinear ordinary differential equations, we consider the equation

$$Lu + Ru + Fu = g, \tag{2.12}$$

where the differential operator L may be considered as the highest order derivative in the equation, R is the remainder of the differential operator, Fu is the nonlinear term such as $u^2, u^3, \sin u\,, e^u$,etc. , and g is an inhomogeneous term.

We next apply the inverse operator L^{-1} to both sides of equation (2.12) and use the given condition to obtain

$$u = f - L^{-1}(Ru) - L^{-1}(Fu), \qquad (2.13)$$

where the function f represents the terms arising from integrating the source term g and from using the given conditions that are assumed to be prescribed.

The nonlinear term can be expressed by an infinite series of the so-called Adomian polynomials A_n given in the form

$$Fu = \sum_{n=0}^{\infty} A_n, \qquad (2.14)$$

where A_n are the Adomian polynomials for the nonlinear term Fu, and can be evaluated by using the following expression [1,2,3,7,8,9,10,12]

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\alpha^n} F\left(\sum_{i=0}^{\infty} \alpha^i u_i\right)\right]_{\alpha=0} \quad , n = 0,1,2,\dots . \qquad (2.15)$$

where α is parameter and $F \in C^{\infty}([a,b])$.

Substituting equation (2.3) and equation (2.14) into equation (2.13), we obtain:

$$\sum_{n=0}^{\infty} u_n = f - L^{-1}\left(R\sum_{n=0}^{\infty} u_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right).$$

The various components u_n of the solution u can be easily determined by using the recursive relation

$$\begin{cases} u_0 = f, \\ u_{n+1} = -L^{-1}(Ru_n) - L^{-1}(A_n), \ \ n \geq 0. \end{cases} \qquad (2.16)$$

Having determined the components $u_n, n \geq 0$, the solution u in a series form follows immediately.

2.1.3 Adomian Polynomials [1,2,3,7,8,9,10,12]

In this section, we will calculate polynomials for several forms of nonlinearity that may arise in nonlinear ordinary equations.

1. $Fu = u^2$

   The Adomian polynomials are given by

$$\begin{cases} A_0 = {u_0}^2, \\ A_1 = 2u_0u_1, \\ A_2 = 2u_0u_2 + {u_1}^2, \\ A_3 = 2u_0u_3 + 2u_1u_2. \end{cases} \qquad (2.17)$$

2. $Fu = sinu$

The Adomian polynomials for this form of nonlinearity are given by

$$\begin{cases} A_0 = sinu_0, \\ A_1 = u_1 cosu_0, \\ A_2 = u_2 cosu_0 - \frac{1}{2!}{u_1}^2 sinu_0, \\ A_3 = u_3 cosu_0 - u_1 u_2 sinu_0 - \frac{1}{3!}{u_1}^3 cosu_0. \end{cases} \quad (2.18)$$

3. $Fu = e^u$
   The Adomian polynomials are given by

$$\begin{cases} A_0 = e^{u_0}, \\ A_1 = u_1 e^{u_0}, \\ A_2 = \left(u_2 + \frac{1}{2!}{u_1}^2\right) e^{u_0}, \\ A_3 = (u_3 + u_1 u_2 + \frac{1}{3!}{u_1}^3) e^{u_0}. \end{cases} \quad (2.19)$$

4. $Fu = ln\, u, u > 0$
   The A_n polynomials for logarithmic nonlinearity are given by

$$\begin{cases} A_0 = ln\, u_0, \\ A_1 = \frac{u_1}{u_0}, \\ A_2 = \frac{u_2}{u_0} - \frac{{u_1}^2}{2{u_0}^2}, \\ A_3 = \frac{u_3}{u_0} - \frac{u_1 u_2}{{u_0}^2} + \frac{{u_1}^3}{3{u_0}^3}. \end{cases} \quad (2.20)$$

Example 2.

Solve the first order nonlinear ordinary differential equation [12]

$$u' + u^2 = 0, u(0) = 1 \quad (2.21)$$

In an operator form, equation (2.21) can be written as

$$Lu = -u^2, \quad (2.22)$$

where L is a first order differential operator.

Applying L^{-1} to both sides of equation (2.22) and using the initial condition give

$$u(x) = 1 - L^{-1}(u^2). \quad (2.23)$$

The ADM suggests that the solution $u(x)$ be expressed by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad (2.24)$$

and the nonlinear terms u^2 be equated to

$$u^2 = \sum_{n=0}^{\infty} A_n. \tag{2.25}$$

Substituting equation (2.24) and equation (2.25) into equation (2.23) yields

$$\sum_{n=0}^{\infty} u_n(x) = 1 - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{2.26}$$

The Adomian polynomials A_n for the nonlinear term u^2 were determined before, where we found

$$\begin{cases} A_0 = {u_0}^2, \\ A_1 = 2u_0u_1, \\ A_2 = 2u_0u_2 + {u_1}^2, \\ A_3 = 2u_0u_3 + 2u_1u_2, \end{cases} \tag{2.27}$$

and so on.

Comparing the two sides of (2.26), we obtain

$$\begin{cases} u_0 = 1, \\ u_1 = -L^{-1}A_0 = -L^{-1}{u_0}^2 = -x, \\ u_2 = -L^{-1}A_1 = -L^{-1}(2u_0u_1) = x^2, \\ u_3 = -L^{-1}A_2 = -L^{-1}(2u_0u_2 + {u_1}^2) = -x^3, \\ \qquad\qquad\qquad\qquad . \\ \qquad\qquad\qquad\qquad . \\ \qquad\qquad\qquad\qquad . \end{cases} \tag{2.28}$$

Consequently, the solution is given by

$$\begin{aligned} u(x) &= 1 - x + x^2 - x^3 + \cdots \\ &= \frac{1}{1+x}. \end{aligned}$$

2.2 Linear Fredholm Integral Equations of the Second Kind

Consider the linear Fredholm integral equation of the second kind [13]

$$\varphi(x) = f(x) + \lambda \int_a^b k(x,t)\, \varphi(t)dt\, , a \le x \le b. \tag{2.29}$$

Following the ADM, the unknown solution of $\varphi(x)$ is assumed to be the decomposition

$$\varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x). \tag{2.30}$$

We begin by choosing the initial component $\varphi_0(x)$ to be the function $f(x)$ and where the remaining components $\varphi_n(x)$ we will be determined recursively by using ADM.

Therefore, this iterative method can be stated as follows

$$\begin{cases} \varphi_0(x) = f(x), \\ \varphi_{n+1}(x) = \lambda \int_a^b k(x,t)\, \varphi_n(t)dt\,, \quad n \geq 0. \end{cases} \tag{2.31}$$

Now, by assuming the first $(n+1)$ terms of equation (2.30), we obtain the n-th approximation to the solution as

$$\phi_n(x) = \sum_{i=0}^{n-1} \varphi_i(x), \tag{2.32}$$

or

$$\phi_n(x) = \varphi_0(x) + \sum_{i=1}^{n-1} \varphi_i(x). \tag{2.33}$$

By substitution of the recursive scheme equation (2.31) into equation (2.33), we conclude that the ADM for equation (2.29) can be converted into an equivalent problem, which we state as follows.

Lemma (2.1):
The ADM for equation (2.29) is equivalent to the following problem:

Find the sequence ϕ_n such that

$$\phi_n = \varphi_0 + \varphi_1 + \cdots + \varphi_{n-1} \text{ with } \phi_0 = 0,$$

and satisfies

$$\begin{cases} \varphi_0(x) = f(x), \\ \phi_n(x) = \varphi_0(x) + \lambda \int_a^b k(x,t)\, \phi_{n-1}(t)\, dt, \quad n \geq 1. \end{cases} \tag{2.34}$$

2.2.1 Convergence Analysis of the ADM
In this section, we will discuss the result concerning the convergence analysis of the ADM for equation (2.29).

Theorem (2.2):
A sufficient condition for ϕ_n to be convergent is that

$$|\lambda| \sup_{a \leq x \leq b} \int_a^b |k(x,t)|\ dt < 1.$$

The sequence ϕ_n defined by (2.34) is convergent and has a limit ϕ solution of

$$\phi(x) = \varphi_0(x) + \lambda \int_a^b k(x,t)\, \phi(t)\, dt.$$

Proof:
We first show that ϕ_n is a Cauchy sequence.

$$\|\phi_{n+2} - \phi_{n+1}\|_\infty = \sup_{a \leq x \leq b} \left| \lambda \int_a^b k(x,t)\ [\phi_{n+1}(t) - \phi_n(t)]dt \right|$$

$$\leq |\lambda| \sup_{a\leq x\leq b} \int_a^b |k(x,t)|\ |\phi_{n+1}(t) - \phi_n(t)|\ dt$$

$$\leq C\ \|\phi_{n+1} - \phi_n\|_\infty,$$

where

$$C = |\lambda| \sup_{a\leq x\leq b} \int_a^b |k(x,t)|\ dt\ < 1.$$

If $m > n, m = n + p, p = 1,2, \ldots$:

$$\left\|\phi_{n+p} - \phi_n\right\|_\infty \leq \left\|\phi_{n+p} - \phi_{n+p-1}\right\|_\infty + \left\|\phi_{n+p-1} - \phi_{n+p-2}\right\|_\infty \ldots + \|\phi_{n+1} - \phi_n\|_\infty$$

$$\leq C^n(1 + C + C^2 + \cdots + C^{p-2} + C^{p-1})\|\phi_1 - \phi_0\|_\infty$$

$$\leq C^n \frac{1}{1-C} \|\phi_1 - \phi_0\|_\infty.$$

As $n, m = n + p \to \infty$, we see that:

$$\left\|\phi_{n+p} - \phi_n\right\|_\infty \to 0.$$

Hence, ϕ_n is a Cauchy sequence in the complete space $\mathcal{C}([a,b])$, that is

$$\lim_{n\to\infty} \phi_n(x) = \phi(x).$$

It remains to show that $\phi(x)$ is a solution of the Fredholm integral equation of the second kind:

$$\left\|\phi - f - \lambda \int_a^b k(x,t)\ \phi(t)\ dt\right\|_\infty \leq \|\phi - \phi_n\|_\infty + \left\|\lambda \int_a^b k(x,t)\ [\phi_{n-1}(t)\right.$$

$$\left. -\phi(t)]\ dt\right\|_\infty + \left\|\phi_n - f - \lambda \int_a^b k(x,t)\ \phi_{n-1}(t)\ dt\right\|_\infty.$$

As $n \to \infty$, we get

$$\phi(x) = f(x) + \lambda \int_a^b k(x,t)\ \phi(t)\ dt\ .$$

Therefore, ϕ is a solution of the Fredholm integral equation.

■

<u>Example 3:</u>

Solve the following Fredholm integral equation

$$\varphi(x) = e^x - 1 + \int_0^1 t\ \varphi(t) dt. \qquad (2.35)$$

The ADM assumes that the solution $\varphi(x)$ has a series form given by (2.30). Substituting the decomposition series (2.30) into both sides of equation (2.35) gives

$$\varphi_0(x) + \varphi_1(x) + \varphi_2(x) + \cdots = \; e^x - 1 + \; \int_0^1 t\,[\varphi_0(t) + \varphi_1(t) + \varphi_2(t) + \cdots]dt.$$

Therefore, we obtain

$$\begin{cases} \varphi_0(x) = \; e^x - 1, \\ \varphi_1(x) = \int_0^1 t\,\varphi_0(t)dt = \int_0^1 t\,[\,e^t - 1]dt = \frac{1}{2}, \\ \varphi_2(x) = \int_0^1 t\,\varphi_1(t)dt = \int_0^1 t\,[1/2]dt = \frac{1}{4}, \\ \varphi_3(x) = \int_0^1 t\,\varphi_2(t)dt = \int_0^1 t\,[1/4]dt = \frac{1}{8}, \end{cases}$$

and so on. Using equation (2.30) gives the series solution

$$\varphi(x) = \; e^x - 1 + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right).$$

Since $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is a geometric series, that its sum is given by

$$S = \frac{1/2}{1 - 1/2} = 1.$$

Thus

$$\varphi(x) = \; e^x.$$

2.3 Nonlinear Fredholm Integral Equation of the Second Kind

Consider the nonlinear Fredholm integral equation of the second kind [13]

$$\varphi(x) = f(x) + \lambda \int_a^b k(x,t)\; F(\varphi(t))dt. \tag{2.36}$$

The same procedure can be applied to resolve this nonlinear Fredholm integral equation.

Proceeding as before, we substitute the expansion (2.30) into (2.36) yields

$$\sum_{n=0}^{\infty} \varphi_n(x) = f(x) + \lambda \int_a^b k(x,t)\; F(\sum_{n=0}^{\infty} \varphi_n(t))dt. \tag{2.37}$$

We define $F(\varphi)$ by

$$F(\varphi) = \sum_{n=0}^{\infty} A_n(\varphi_0, \varphi_1, \dots, \varphi_n), \tag{2.38}$$

where A_n are the Adomian polynomials [12] and depend on $\varphi_0, \varphi_1, \dots, \varphi_n$.

Thus,

$$\sum_{n=0}^{\infty} \varphi_n(x) = f(x) + \lambda \int_a^b k(x,t)\; \sum_{n=0}^{\infty} A_n(t)\, dt. \tag{2.39}$$

We get the scheme

$$\begin{cases} \varphi_0(x) = f(x), \\ \varphi_{n+1}(x) = \lambda \int_a^b k(x,t)\, A_n(t)dt \quad , n \geq 0. \end{cases} \tag{2.40}$$

Modified ADM:

A modified recurrence relation is usually used, where $f(x)$ is decomposed into two components $f_1(x)$ and $f_2(x)$, such that

$$f(x) \; = \; f_1(x) + f_2(x).$$

In this case the modified recurrence relation becomes in the form

$$\begin{cases} \varphi_0(x) = f_1(x), \\ \varphi_1(x) = f_2(x) + \lambda \int_a^b k(x,t)\, A_0(t)dt, \\ \varphi_{n+1}(x) = \lambda \int_a^b k(x,t)\, A_n(t)dt \quad , n \geq 1. \end{cases} \tag{2.41}$$

As before, we conclude that the ADM for equation (2.36) can be converted to be the following problem.

Lemma (2.3):

The ADM for equation (2.36) is equivalent to the following problem:

Find the sequence ϕ_n such that

$$\phi_n = \varphi_0 + \varphi_1 + \cdots + \varphi_{n-1} \text{ with } \phi_0 = 0,$$

and satisfies

$$\begin{cases} \varphi_0(x) = f(x), \\ \phi_n(x) = \varphi_0(x) + \lambda \int_a^b k(x,t)\, F(\phi_{n-1}(t))\, dt, \;\; n \geq 1. \end{cases} \tag{2.42}$$

2.3.1 Convergence Analysis of the ADM

In this section, we will discuss the result concerning the convergence analysis of the ADM for equation (2.36).

Theorem (2.4):

Let us assume that the operator $F(\phi)$ satisfies Lipchitz condition. The sequence ϕ_n defined by (2.42) is convergent and has a limit ϕ solution of

$$\phi(x) = \varphi_0(x) + \lambda \int_a^b k(x,t)\, F(\phi(t))\, dt.$$

Proof:

We will show that ϕ_n is a Cauchy sequence.

$$\|\phi_{n+2}-\phi_{n+1}\|_\infty = \sup_{a\le x\le b}\left|\lambda\int_a^b k(x,t)\ [F(\phi_{n+1}(t)) - F(\phi_n(t))]dt\right|$$

$$\le |\lambda| \sup_{a\le x\le b}\int_a^b |k(x,t)|\ |F(\phi_{n+1}(t)) - F(\phi_n(t))|\ dt$$

$$\le |\lambda| \sup_{a\le x\le b}\int_a^b |k(x,t)|\ M\ |\phi_{n+1}(t) - \phi_n(t)|\ dt$$

$$\le C\ \|\phi_{n+1}-\phi_n\|_\infty,$$

where

$$C = |\lambda|\ M \sup_{a\le x\le b}\int_a^b |k(x,t)|\ dt\ < 1.$$

If $m > n, m = n+p, p = 1,2,\ldots$:

$$\left\|\phi_{n+p}-\phi_n\right\|_\infty \le \left\|\phi_{n+p}-\phi_{n+p-1}\right\|_\infty + \cdots + \|\phi_{n+1}-\phi_n\|_\infty$$

$$\le C^n(1 + C + C^2 + \cdots + C^{p-2} + C^{p-1})\|\phi_1-\phi_0\|_\infty$$

$$\le C^n\frac{1}{1-C}\|\phi_1-\phi_0\|_\infty.$$

As $n, m = n+p \to \infty$, we see that:

$$\left\|\phi_{n+p}-\phi_n\right\|_\infty \to 0.$$

Hence, ϕ_n is a Cauchy sequence in the complete space $C([a,b])$, that is

$$\lim_{n\to\infty}\phi_n(x) = \phi(x).$$

It remains to show that $\phi(x)$ is a solution of the Fredholm integral equation of the second kind:

$$\left\|\phi - f - \lambda\int_a^b k(x,t)\,F\big(\phi(t)\big)\,dt\right\|_\infty \le \|\phi-\phi_n\|_\infty + \left\|\lambda\int_a^b k(x,t)\,[F(\phi_{n-1}(t))\right.$$

$$\left. - F(\phi(t))]\ dt\right\|_\infty + \left\|\phi_n - f - \lambda\int_a^b k(x,t)\ F(\phi_{n-1}(t))\,dt\right\|_\infty.$$

As $n\to\infty$, we get

$$\phi(x) = f(x) + \lambda\int_a^b k(x,t)\ F(\phi(t))\,dt.$$

Therefore, ϕ is a solution of the Fredholm integral equation.

■

Example 4:

Use the ADM to solve the nonlinear Fredholm integral equation

$$\varphi(x) = x^2 - \frac{1}{12} + \frac{1}{2}\int_0^1 t\, \varphi^2(t)dt. \tag{2.43}$$

Substituting the series (2.30) and (2.38) into (2.43), we find

$$\sum_{n=0}^{\infty} \varphi_n(x) = x^2 - \frac{1}{12} + \frac{1}{2}\int_0^1 t\, \sum_{n=0}^{\infty} A_n(t)\, dt,$$

where A_n are the Adomian polynomials for φ^2 as shown previous. Using the modified ADM, we set

$$\begin{cases} \varphi_0(x) = x^2, \\ \varphi_1(x) = -\frac{1}{12} + \frac{1}{2}\int_0^1 t\, A_0(t)dt = 0, \\ \quad\vdots \\ \varphi_{n+1}(x) = \frac{1}{2}\int_0^1 t\, A_n(t)dt = 0, \quad n \geq 1. \end{cases}$$

This in turn gives the exact solution

$$\varphi(x) = x^2.$$

Chapter 3

Regularization Method and ADM for Solving the Fredholm Integral Equation of the First Kind

3.1 Linear Fredholm integral Equations of the First Kind

Consider the linear Fredholm integral equation of the first kind [13]:

$$\int_a^b k(x,t)\varphi(t)dt = f(x),\ \ x \in [a,b], \tag{3.1}$$

where

$$\varphi \in L_2[a,b] \text{ and } f(x) \in L_2[a,b]$$

and suppose that

$$\int_a^b \int_a^b k^2(x,t)\, dx\, dt < \infty.$$

This equation appears in the theory of elasticity and certain problems of the mechanics of continuous media.

As it was pointed out in Cherruault [5], the general difficulty is how the ADM can be applied to solve these types of integral equations.

The present chapter is to overcome this general difficulty.

3.1.1 The Regularization Technique for Linear Fredholm Integral Equations of the First Kind

In this section, we develop a new iterative procedure using the regularization technique, where the integral equations of the first kind are recast into a canonical form from suitable for ADM. More precisely, we consider the approximated integral equation

$$\varepsilon\, \varphi_\varepsilon(x) + \int_a^b k(x,t)\varphi_\varepsilon(t)dt = f(x), \tag{3.2}$$

where ε is a fixed positive number.

It can proved that the solution $\varphi_\varepsilon(x)$ of (3.2) converges to the solution $\varphi(x)$ of equation (3.1) when $\varepsilon \to 0$.

Lemma (3.1):

Suppose that the integral operator of equation (3.2) is continuous and coercive in the Hilbert space $H = L_2[a,b]$ where f, φ_ε and φ are defined, then:

- $\|\varphi_\varepsilon\|$ is bounded independently of ε.
- $\|\varphi_\varepsilon - \varphi\|$ tends to 0 when $\varepsilon \to 0$.

Proof:

From equation (3.2), we deduce

$$\varepsilon\, \|\varphi_\varepsilon\| = \left\| -\int_a^b k(x,t)\varphi_\varepsilon(t)dt + f \right\| \geq -\|f\| + \left\| \int_a^b k(x,t)\varphi_\varepsilon(t)dt \right\|. \quad (3.3)$$

The coercivity of the integral operator implies:

$$\left\| \int_a^b k(x,t)\, \varphi_\varepsilon(t)dt \right\| \geq \beta \|\, \varphi_\varepsilon\|, \quad (3.4)$$

where β is the coercivity constant,

$$\left\| \int_a^b k(x,t)\, \varphi_\varepsilon(t)dt \right\|^2 = \int_a^b [\int_a^b k(x,t)\varphi_\varepsilon(t)dt]^2 dx$$

and

$$\|\, \varphi_\varepsilon\|^2 = \int_a^b \varphi^2{}_\varepsilon(t)dt.$$

From equations (3.3) and (3.4), we have

$$\varepsilon\, \|\varphi_\varepsilon\| \geq -\|f\| + \beta \|\, \varphi_\varepsilon\| \quad (3.5)$$

and therefore

$$(\beta - \varepsilon)\|\varphi_\varepsilon\| \leq \|f\|,\ \beta \gg \varepsilon.$$

So, $\|\varphi_\varepsilon\|$ is bounded independently of ε.

We now prove the second part.

By using equations (3.1) and (3.2), we have

$$\varepsilon\, \varphi_\varepsilon(x) = -\int_a^b k(x,t)\varphi_\varepsilon(t)dt + \int_a^b k(x,t)\, \varphi(t)\, dt. \quad (3.6)$$

Thus,

$$\varepsilon\, \varphi_\varepsilon(x) = -\int_a^b k(x,t)\, [\varphi_\varepsilon(t) - \varphi(t)]dt.$$

It follows that

$$-\varepsilon\, (\varphi_\varepsilon(x) - \varphi(x)) - \varepsilon\, \varphi(x) = \int_a^b k(x,t)\, [\varphi_\varepsilon(t) - \varphi(t)]dt. \quad (3.7)$$

Taking the norm of both sides of the above equation, and using the coercivity property implies

$$\beta\|\varphi_\varepsilon - \varphi\| \leq \|\varepsilon\, (\varphi_\varepsilon - \varphi) + \varepsilon\, \varphi\| \leq \varepsilon\, \|(\varphi_\varepsilon - \varphi)\| + \varepsilon\|\varphi\|. \quad (3.8)$$

Finally, we have

$$(\beta - \varepsilon)\|\varphi_\varepsilon - \varphi\| \leq \varepsilon\|\varphi\|,\ \beta - \varepsilon > 0,$$

and therefore $\|\varphi_\varepsilon - \varphi\| \to 0$ when $\varepsilon \to 0$.

■

3.1.2 Adomian Decomposition Method

To develop a new iterative method, consider equation (3.2), which is expressed in the canonical form.

Following the ADM, the unknown solution $\varphi_\varepsilon(x)$ is assumed to be the decomposition series of the form

$$\varphi_\varepsilon(x) = \sum_{n=0}^{\infty} \varphi_{\varepsilon,n}(x). \tag{3.9}$$

Therefore, this new iterative method can be stated as follows

$$\begin{cases} \varphi_{\varepsilon,0}(x) = \frac{1}{\varepsilon} f(x), \\ \varphi_{\varepsilon,n+1}(x) = -\frac{1}{\varepsilon}\int_a^b k(x,t)\, \varphi_{\varepsilon,n}(t)dt\,, \;\; n \geq 0. \end{cases} \tag{3.10}$$

By assuming the first $n+1$ terms of equation (3.9), we obtain the n-th approximation to the solution as

$$\phi_{\varepsilon,n}(x) = \sum_{i=0}^{n-1} \varphi_{\varepsilon,i}(x), \tag{3.11}$$

or

$$\phi_{\varepsilon,n}(x) = \varphi_{\varepsilon,0}(x) + \sum_{i=1}^{n-1} \varphi_{\varepsilon,i}(x). \tag{3.12}$$

By substitution of the recursive scheme (3.10) into (3.12), we conclude that the ADM for equation (3.2) can be converted into an equivalent problem, which we state as follows.

Lemma (3.2):

The ADM for equation (3.2) is equivalent to the following problem:

Find the sequence $\phi_{\varepsilon,n}$ such that

$$\phi_{\varepsilon,n} = \varphi_{\varepsilon,0} + \varphi_{\varepsilon,1} + \cdots + \varphi_{\varepsilon,n-1} \text{ with } \phi_{\varepsilon,0} = 0,$$

and satisfies

$$\begin{cases} \varphi_{\varepsilon,0}(x) = \frac{1}{\varepsilon} f(x), \\ \phi_{\varepsilon,n}(x) = \varphi_{\varepsilon,0}(x) - \frac{1}{\varepsilon}\int_a^b k(x,t)\, \phi_{\varepsilon,n-1}(t)\, dt, \;\; n \geq 1. \end{cases} \tag{3.13}$$

Now, we will discuss the result concerning the convergence analysis of the ADM for equation (3.13).

We first rewrite equation (3.2) in the semi-linear equation

$$\varphi_\varepsilon = \frac{1}{\varepsilon} f + A\varphi_\varepsilon,$$

where

$$A\varphi_\varepsilon = -\frac{1}{\varepsilon}\int_a^b k(x,t)\,\varphi_\varepsilon(t)\,dt.$$

Therefore,

$$\begin{cases} \varphi_{\varepsilon,0} = \frac{1}{\varepsilon} f, \\ \varphi_{\varepsilon,n} = A\varphi_{\varepsilon,n-1}, \;\; n \geq 1. \end{cases} \tag{3.14}$$

Thus,

$$\begin{cases} \varphi_{\varepsilon,0} = \frac{1}{\varepsilon} f, \\ \phi_{\varepsilon,n} = \varphi_{\varepsilon,0} + A\phi_{\varepsilon,n-1}, \;\; n \geq 1. \end{cases} \tag{3.15}$$

Theorem (3.3):

A sufficient condition for $\phi_{\varepsilon,n}$ to be convergent is that $\|A\| < \delta < 1$.

Proof:

We first show that $\phi_{\varepsilon,n}$ is a Cauchy sequence.

$$\begin{aligned} \|\phi_{\varepsilon,n+2} - \phi_{\varepsilon,n+1}\| &= \|A\,[\phi_{\varepsilon,n+1} - \phi_{\varepsilon,n}]\| \\ &\leq \;||A||\,||\phi_{\varepsilon,n+1} - \phi_{\varepsilon,n}|| \\ &< \delta\,\|\phi_{\varepsilon,n+1} - \phi_{\varepsilon,n}\|. \end{aligned}$$

If $m > n, m = n + p, p = 1,2, \ldots,$

$$\begin{aligned} \|\phi_{\varepsilon,n+p} - \phi_{\varepsilon,n}\| &\leq \|\phi_{\varepsilon,n+p} - \phi_{\varepsilon,n+p-1}\| + \cdots + \|\phi_{\varepsilon,n+1} - \phi_{\varepsilon,n}\| \\ &< \delta^n(1 + \delta + \delta^2 + \cdots + \delta^{p-2} + \delta^{p-1})\|\phi_{\varepsilon,1} - \phi_{\varepsilon,0}\| \\ &< \delta^n \frac{1}{1-\delta}\|\phi_{\varepsilon,1} - \phi_{\varepsilon,0}\|. \end{aligned}$$

As $n, m = n + p \to \infty$, we see that:

$$\|\phi_{\varepsilon,n+p} - \phi_{\varepsilon,n}\| \to 0.$$

Hence $\phi_{\varepsilon,n}$ is a Cauchy sequence in the Banach space, that is

$$\lim_{n\to\infty} \phi_{\varepsilon,n}(x) = \phi_\varepsilon(x).$$

It remains to show that $\phi_\varepsilon(x)$ is a solution of the Fredholm integral equation of the second kind:

$$\begin{aligned} \left\|\phi_\varepsilon - \frac{1}{\varepsilon} f - A\phi_\varepsilon\right\| \leq \|\phi_\varepsilon - \phi_{\varepsilon,n}\| + \left\|\phi_{\varepsilon,n} - \frac{1}{\varepsilon} f - A\,\phi_{\varepsilon,n-1}\right\| \\ + \|A\,[\phi_{\varepsilon,n-1} - \phi_\varepsilon]\|. \end{aligned}$$

As $n \to \infty$, we get

$$\phi_\varepsilon = \frac{1}{\varepsilon} f + A\phi_\varepsilon.$$

Therefore, ϕ_ε is a solution of the Fredholm integral equation.

■

3.1.3 Examples

In order to demonstrate the feasibility and efficiency of this method, some examples with a priori known exact solution are studied in detail.

1. Let

$$\int_0^1 k(x,t)\varphi(t)dt = \sin \pi x,$$

where

$$k(x,t) = \begin{cases} (1-x)t, & 0 \le t \le x, \\ (1-t)x, & x \le t \le 1. \end{cases}$$

We solve this equation by the above recursive schemes and obtain

$$\begin{cases} \varphi_{\varepsilon,0}(x) = \frac{1}{\varepsilon} \sin \pi x, \\ \varphi_{\varepsilon,1}(x) = -\frac{1}{\pi^2 \varepsilon^2} \sin \pi x, \\ \varphi_{\varepsilon,2}(x) = \frac{1}{\pi^4 \varepsilon^3} \sin \pi x, \end{cases} \tag{3.16}$$

and so on.

So,

$$\varphi_\varepsilon(x) = \left(1 - \frac{1}{\pi^2 \varepsilon} + \frac{1}{\pi^4 \varepsilon^2} - \cdots\right) \frac{1}{\varepsilon} \sin \pi x.$$

Consequently,

$$\varphi_\varepsilon(x) = \left(\frac{\pi^2 \varepsilon}{1+\pi^2 \varepsilon}\right) \frac{1}{\varepsilon} \sin \pi x = \frac{\pi^2}{1+\pi^2 \varepsilon} \sin \pi x.$$

Setting $\varepsilon \to 0$, we obtain

$$\varphi(x) = \lim_{\varepsilon \to 0} \varphi_\varepsilon(x) = \pi^2 \sin \pi x,$$

which is equivalent to the exact solution.

2. Let

$$\int_0^\pi k(x,t)\varphi(t)dt = \frac{1}{2} \sin 2x,$$

where

$$k(x,t) = \begin{cases} \dfrac{t\,(\pi - x)}{\pi}, & 0 \le t \le x, \\ \dfrac{x\,(\pi - t)}{\pi}, & x \le t \le \pi. \end{cases}$$

Then the recursive scheme for the approximate equation be expressed as

$$\begin{cases} \varphi_{\varepsilon,0}(x) = \frac{1}{2\varepsilon}\sin 2x, \\ \varphi_{\varepsilon,1}(x) = -\frac{1}{8\varepsilon^2}\sin 2x, \\ \varphi_{\varepsilon,2}(x) = \frac{1}{32\varepsilon^3}\sin 2x, \\ \varphi_{\varepsilon,3}(x) = \frac{1}{128\varepsilon^4}\sin 2x, \end{cases} \tag{3.17}$$

and so on.

So,

$$\varphi_\varepsilon(x) = \left(1 - \frac{1}{4\varepsilon} + \frac{1}{16\varepsilon^2} - \frac{1}{64\varepsilon^3} + \cdots\right)\frac{1}{2\varepsilon}\sin 2x.$$

Consequently,

$$\varphi_\varepsilon(x) = \left(\frac{4\varepsilon}{1+4\varepsilon}\right)\frac{1}{2\varepsilon}\sin 2x = \frac{2}{1+4\varepsilon}\sin 2x.$$

Setting $\varepsilon \to 0$, we obtain

$$\varphi(x) = \lim_{\varepsilon\to 0}\varphi_\varepsilon(x) = 2\sin 2x,$$

which is the exact solution.

3. Let

$$\int_0^1 k(x,t)\varphi(t)dt = \frac{1}{9}\cos 3\pi x,$$

where

$$k(x,t) = \begin{cases} \dfrac{x^2+t^2}{2} + \dfrac{1}{3} - x, & 0 \le t \le x, \\ \dfrac{x^2+t^2}{2} + \dfrac{1}{3} - t, & x \le t \le 1. \end{cases}$$

We solve the approximate equation by the above recursive schemes and obtain

$$\begin{cases} \varphi_{\varepsilon,0}(x) = \frac{1}{9\varepsilon}\cos 3\pi x, \\ \varphi_{\varepsilon,1}(x) = -\frac{1}{9\varepsilon^2}\frac{\cos 3\pi x}{9\pi^2}, \\ \varphi_{\varepsilon,2}(x) = \frac{1}{9\varepsilon^3}\frac{\cos 3\pi x}{9^2\pi^4}, \end{cases} \tag{3.18}$$

and so on.

So,

$$\varphi_\varepsilon(x) = \left(1 - \frac{1}{9\pi^2\varepsilon} + \frac{1}{9^2\pi^4\varepsilon^2} - \cdots\right)\frac{1}{9\varepsilon}\cos 3\pi x.$$

Consequently,

$$\varphi_\varepsilon(x) = \left(\frac{9\pi^2\varepsilon}{1+9\pi^2\varepsilon}\right)\frac{1}{9\varepsilon}\cos 3\pi x = \frac{\pi^2}{1+9\pi^2\varepsilon}\cos 3\pi x.$$

Setting $\varepsilon \to 0$, we obtain

$$\varphi(x) = \lim_{\varepsilon\to 0}\varphi_\varepsilon(x) = \pi^2\cos 3\pi x,$$

which is the exact solution.

3.2 Nonlinear Fredholm Integral Equations of the First Kind

Consider the nonlinear Fredholm integral equation of the first kind [13]

$$\int_a^b k(x,t)F(\varphi(t))dt = f(x), \; x \in [a,b] \tag{3.19}$$

In this section, we will use another iterative method that has been used in [4].

We observe that equation (3.19) can be replaced by a suitable expression

$$\varphi(x) = f(x) + \varphi(x) - \int_a^b k(x,t)F(\varphi(t))dt. \tag{3.20}$$

The reason of this is to express equation (3.20) in the canonical from in order to employ the ADM.

Substituting the expansion equation

$$\sum_{n=0}^{\infty}\varphi_n(x) = f(x) + \sum_{n=0}^{\infty}\varphi_n(x) - \int_a^b k(x,t)F\left(\sum_{n=0}^{\infty}\varphi_n(t)\right)dt, \tag{3.21}$$

we define $F(\varphi(x))$ by

$$F(\varphi(x)) = \sum_{n=0}^{\infty} A_n(\varphi_0, \varphi_1, \dots, \varphi_n),$$

where A_n are the Adomian polynomials and dependent only on $\varphi_0, \varphi_1, \dots, \varphi_n$.

Thus

$$\sum_{n=0}^{\infty} \varphi_n(x) = f(x) + \sum_{n=0}^{\infty} \varphi_n(x) - \int_a^b k(x,t) \sum_{n=0}^{\infty} A_n \, dt, \tag{3.22}$$

we get the scheme

$$\begin{cases} \varphi_0(x) = f(x), \\ \varphi_{n+1}(x) = \varphi_n(x) - \int_a^b k(x,t)\, A_n(t) dt\,, \;\; n \geq 0. \end{cases} \tag{3.23}$$

<u>Lemma (3.4):</u>
The ADM for equation (3.20) is equivalent to the following problem:

Find the sequence ϕ_n such that

$$\phi_n = \varphi_0 + \varphi_1 + \cdots + \varphi_{n-1} \text{ with } \phi_0 = 0,$$

and satisfies

$$\begin{cases} \varphi_0(x) = f(x), \\ \phi_{n+1}(x) = \varphi_0(x) + \phi_n(x) - \int_a^b k(x,t)\, F\big(\phi_n(t)\big)\, dt, \;\; n \geq 0. \end{cases} \tag{3.24}$$

Proceeding as before, equation (3.20) can be written in the semi-nonlinear equation

$$\varphi = f + (I - N)\varphi,$$

where

$$N\varphi = \int_a^b k(x,t) F\big(\varphi(t)\big) dt.$$

Therefore,

$$\begin{cases} \varphi_0 = f, \\ \phi_{n+1} = \varphi_0 + (I - N)\phi_n, \;\; n \geq 0. \end{cases} \tag{3.25}$$

Thus.

<u>Theorem (3.5):</u>
Let us assume that

$$\|I - N\| < \delta < 1,$$

then the sequence ϕ_n defined by (3.25) converges to the solution ϕ of the equation

$$\phi = \varphi_0 + (I - N)\phi.$$

Proof:

We first show that ϕ_n is a contractive sequence, we have

$$\|\phi_{n+2}-\phi_{n+1}\| = \|(I-N)(\phi_{n+1}-\phi_n)\|$$
$$\leq \|I-N\|\|\|\phi_{n+1}(t)-\phi_n(t)\|$$
$$< \delta\|\phi_{n+1}(t)-\phi_n(t)\|.$$

If $m > n, m = n+p, p = 1,2,\ldots,$

$$\|\phi_{n+p}-\phi_n\| \leq \|\phi_{n+p}-\phi_{n+p-1}\| + \|\phi_{n+p-1}-\phi_{n+p-2}\| + \cdots + \|\phi_{n+1}-\phi_n\|$$
$$< \delta^n(1+\delta+\delta^2+\cdots+\delta^{p-2}+\delta^{p-1})\|\phi_1-\phi_0\|$$
$$< \delta^n\frac{1}{1-\delta}\|\phi_1-\phi_0\|.$$

As $n, m = n+p \to \infty$, we see that:

$$\|\phi_{n+p}-\phi_n\| \to 0.$$

Hence $\phi_{\varepsilon,n}$ is a Cauchy sequence in the Banach space, that is

$$\lim_{n\to\infty}\phi_n(x) = \phi(x).$$

It remains to show that $\phi(x)$ is a solution of the Fredholm integral equation of the second kind:

$$\|\phi-\varphi_0-(I-N)\phi\| \leq \|\phi-\phi_n\| + \|\phi_n-\varphi_0-(I-N)\phi_{n-1}\| + \|(I-N)[\phi_{n-1}-\phi]\|.$$

As $n \to \infty$, we get

$$\phi = \varphi_0 + (I-N)\phi.$$

Therefore, ϕ is a solution of the Fredholm integral equation.

■

3.2.1 Integral Equations of the Form $\int_a^b k(x,t)\varphi(x)\varphi(t)dt = f(x)$

Consider the nonlinear integral equation of the form

$$\int_a^b k(x,t)\varphi(x)\varphi(t)dt = f(x), x \in [a,b]. \quad (3.26)$$

We write equation (3.26) in the form

$$\varphi(x) = \varphi(x) + f(x) - \int_a^b k(x,t)\varphi(x)\varphi(t)dt. \quad (3.27)$$

We have

$$\begin{cases} \varphi_0(x) = f(x), \\ \varphi_{n+1}(x) = \varphi_n(x) - \int_a^b k(x,t)A_n dt\,,\ n \geq 1, \end{cases} \tag{3.28}$$

where

$$A_n = \sum_{k=0}^{n} \varphi_k(x)\varphi_{n-k}(t). \tag{3.29}$$

Example 1:

Consider equation (3.26) with $a = 0,\ b = 1,\ k(x,t) = t, f(x) = 3x$ and

$$A_0 = \varphi_0(x)\varphi_0(t), A_1 = \varphi_0(x)\varphi_1(t) + \varphi_1(x)\varphi_0(t), ...$$

Then, we have

$$\begin{cases} \varphi_0(x) = 3x, \\ \varphi_1(x) = 0, \\ \quad . \\ \quad . \\ \quad . \\ \varphi_n(x) = 0, n \geq 1. \end{cases}$$

Thus,

$$\varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x) = 3x.$$

Example 2:

Consider equation (3.26) with $a = 0,\ b = \frac{\pi}{2},\ k(x,t) = 2sint$ and $f(x) = cosx$ and

$$A_0 = \varphi_0(x)\varphi_0(t), A_1 = \varphi_0(x)\varphi_1(t) + \varphi_1(x)\varphi_0(t), ...$$

Then, we have

$$\begin{cases} \varphi_0(x) = cosx, \\ \varphi_1(x) = 0, \\ \quad . \\ \quad . \\ \quad . \\ \varphi_n(x) = 0, n \geq 1. \end{cases}$$

Thus,

$$\varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x) = cosx.$$

3.2.2 Integral Equations of the Form $\int_a^b k(x,t)F(\varphi(t))dt = f(x)$

We assume that F is invertible (F^{-1} exists), then we can set

$$F\big(\varphi(x)\big) = v(x), \varphi(x) = F^{-1}\big(v(x)\big). \tag{3.30}$$

The equation (3.19) becomes

$$\int_a^b k(x,t)v(t)dt = f(x), \qquad (3.31)$$

which is a linear equation with respect to v.

The problem is then to find $\varphi(x)$ from $v(x)$.

Example 3:
Let

$$\int_0^1 4xt\ \varphi^2(t)dt = x. \qquad (3.32)$$

- The Adomian polynomials for $F\big(\varphi(t)\big) = \varphi^2(t)$ are

$$A_0 = {\varphi_0}^2, A_1 = 2\varphi_0\varphi_1, A_1 = 2\varphi_0\varphi_2 + {\varphi_1}^2, \ldots.$$

We solve this equation by the above recursive scheme and obtain

$$\begin{cases} \varphi_0(x) = x, \\ \varphi_1(x) = 0, \\ \varphi_2(x) = 0, \\ \quad . \\ \quad . \\ \varphi_n(x) = 0, n \geq 1. \end{cases}$$

Thus, the first solution is given by

$$\varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x) = x.$$

- We first set

$$v(x) = \varphi^2(x), \varphi(x) = \pm\sqrt{v(x)}.$$

The equation (3.32) becomes

$$\int_0^1 4xt\ v(t)dt = x$$

Then, we get

$$\begin{cases} v_0(x) = x, \\ v_1(x) = -\dfrac{x}{3}, \\ v_2(x) = \dfrac{x}{9}, \\ v_3(x) = -\dfrac{x}{27}, \end{cases}$$

and so on.

Thus,

$$v(x) = \sum_{n=0}^{\infty} v_n(x) = \frac{3}{4}x.$$

Another solution is given by

$$\varphi(x) = \pm\frac{\sqrt{3x}}{2}.$$

Chapter 4

Regularization Method and ADM for Solving Schlömilch's Integral Equation

4.1 Linear Schlömilch's Integral Equation

The Schlömilch's integral equation is a related integral equation of the first kind, which is also found in some problem of mathematical physics such that the derivation of the electron density profile from the ionospheric for oblique incidence for the quasi-transverse approximation [14].

The linear Schlömilch integral equation reads:

$$f(x) = \frac{2}{\pi} \int_0^{\pi/2} \varphi(x \sin t) dt, \quad (4.1)$$

where $f(x)$ is a continuous differential coefficient for $-\pi \le x \le \pi$.

This equation has one solution given by [14]

$$\varphi(x) = f(0) + x \int_0^{\pi/2} f'(x \sin t) dt, \quad (4.2)$$

where f' is the derivative of f with respect to the argument $\xi = x \sin t$.

We will use the combined regularization-Adomian method [14] to handle the linear and nonlinear Schlömilch integral equations.

The combined regularization-Adomian method is proved to be reliable and efficient.

4.1.1 The Method of Regularization

The method of regularization converts the linear Schlömilch's integral equation (4.1) into the Schlömilch's integral equation of the second kind in the form

$$\varepsilon \varphi_\varepsilon(x) = f(x) - \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \varphi_\varepsilon(x \sin t) dt, \quad (4.3)$$

where ε is a small positive parameter, called the regularization parameter.

The same procedure that used that used in the previous chapter can be applied to show that the solution φ_ε of (4.3) converges to the solution $\varphi(x)$ of equation (4.1) as $\varepsilon \to 0$, that is

$$\lim_{\varepsilon \to 0} \varphi_\varepsilon(x) = \varphi(x).$$

Consequently, we can apply the ADM for solving the Schlömilch's integral equations of the second kind, and obtain the recursive scheme

$$\begin{cases} \varphi_{\varepsilon,0}(x) = \frac{1}{\varepsilon} f(x), \\ \varphi_{\varepsilon,n+1}(x) = -\frac{2}{\pi\varepsilon} \int_0^{\frac{\pi}{2}} \varphi_{\varepsilon,n}(x \sin t) dt, \ n \ge 0. \end{cases} \quad (4.4)$$

4.1.2 Examples

The scheme that we presented will be illustrated by the following examples.

Example 1:

Consider the linear Schlömilch's integral equation

$$1 + x = \frac{2}{\pi}\int_0^{\pi/2} \varphi(x\, sin\, t)\, dt,\ \ -\pi \le x \le \pi. \qquad (4.5)$$

Using the regularization method, equation (4.5) becomes

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}(1 + x) - \frac{2}{\varepsilon\pi}\int_0^{\pi/2} \varphi_\varepsilon(x\, sin\, t)\, dt.$$

Then the recursive scheme can be expressed as

$$\begin{cases} \varphi_{\varepsilon,0}(x) = \dfrac{1}{\varepsilon}(1 + x), \\ \varphi_{\varepsilon,1}(x) = -\dfrac{2}{\varepsilon^2\pi}\left(\dfrac{\pi}{2} + x\right), \\ \varphi_{\varepsilon,2}(x) = \dfrac{4}{\varepsilon^3\pi^2}\left(\dfrac{\pi^2}{4} + x\right), \\ \varphi_{\varepsilon,3}(x) = -\dfrac{8}{\varepsilon^4\pi^3}\left(\dfrac{\pi^3}{8} + x\right), \end{cases}$$

and so on.

So,

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}(1 + x) - \frac{2}{\varepsilon^2\pi}\left(\frac{\pi}{2} + x\right) + \frac{4}{\varepsilon^3\pi^2}\left(\frac{\pi^2}{4} + x\right) - \frac{8}{\varepsilon^4\pi^3}\left(\frac{\pi^3}{8} + x\right) + \cdots$$

$$= x\left(\frac{1}{\varepsilon} - \frac{2}{\varepsilon^2\pi} + \frac{4}{\varepsilon^3\pi^2} - \frac{8}{\varepsilon^4\pi^3} + \cdots\right) + \frac{1}{\varepsilon}\left(1 - \frac{1}{\varepsilon} + \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon^3} + \cdots\right).$$

Consequently,

$$\varphi_\varepsilon(x) = \frac{\pi x}{2 + \varepsilon\pi} + \frac{1}{1 + \varepsilon}.$$

Setting $\varepsilon \to 0$, we obtain

$$\varphi(x) = \lim_{\varepsilon\to 0} \varphi_\varepsilon(x) = \frac{\pi}{2}x + 1,$$

which is the exact solution of the given equation.

Example 2:

Let

$$x^2 = \frac{2}{\pi}\int_0^{\pi/2} \varphi(x\, sin\, t)\, dt,\ \ -\pi \le x \le \pi. \qquad (4.6)$$

Using the regularization method, equation (4.6) becomes

$$\varphi_\varepsilon(x) = \tfrac{1}{\varepsilon}x^2 - \tfrac{2}{\varepsilon\pi}\int_0^{\pi/2} \varphi_\varepsilon(x\,sin\,t)\,dt.$$

Then the recursive scheme can be expressed as

$$\begin{cases} \varphi_{\varepsilon,0}(x) = \dfrac{1}{\varepsilon}x^2, \\ \varphi_{\varepsilon,1}(x) = -\dfrac{x^2}{2\varepsilon^2}, \\ \varphi_{\varepsilon,2}(x) = \dfrac{x^2}{4\varepsilon^3}, \\ \varphi_{\varepsilon,3}(x) = -\dfrac{x^2}{8\varepsilon^4}, \end{cases}$$

and so on.

So,

$$\varphi_\varepsilon(x) = x^2\left(\frac{1}{\varepsilon} - \frac{1}{2\varepsilon^2} + \frac{1}{4\varepsilon^3} - \frac{1}{8\varepsilon^4} + \cdots\right).$$

Consequently,

$$\varphi_\varepsilon(x) = \frac{2x^2}{1+2\varepsilon}.$$

Setting $\varepsilon \to 0$, we obtain

$$\varphi(x) = \lim_{\varepsilon\to 0}\varphi_\varepsilon(x) = 2x^2,$$

which is the exact solution.

Example 3:

Let

$$1 + x^2 = \tfrac{2}{\pi}\int_0^{\pi/6} \varphi(x\,sin\,3t)\,dt,\ -\pi \le x \le \pi. \tag{4.7}$$

Using the regularization method, equation (4.7) becomes

$$\varphi_\varepsilon(x) = \tfrac{1}{\varepsilon}(1+x^2) - \tfrac{2}{\varepsilon\pi}\int_0^{\pi/6} \varphi_\varepsilon(x\,sin\,3t)\,dt.$$

Then the recursive scheme can be expressed as

$$\begin{cases} \varphi_{\varepsilon,0}(x) = \dfrac{1}{\varepsilon}(1+x^2), \\ \varphi_{\varepsilon,1}(x) = -\dfrac{1}{3\varepsilon^2}\left(1+\dfrac{x^2}{2}\right), \\ \varphi_{\varepsilon,2}(x) = \dfrac{1}{9\varepsilon^3}\left(1+\dfrac{x^2}{4}\right), \\ \varphi_{\varepsilon,3}(x) = -\dfrac{1}{27\varepsilon^4}\left(1+\dfrac{x^2}{8}\right), \end{cases}$$

and so on.

So,

$$\varphi_\varepsilon(x) = x^2\left(\frac{1}{\varepsilon} - \frac{1}{6\varepsilon^2} + \frac{1}{36\varepsilon^3} - \frac{1}{216\varepsilon^4} + \cdots\right) + \frac{1}{\varepsilon}\left(1 - \frac{1}{3\varepsilon} + \frac{1}{9\varepsilon^2} - \frac{1}{27\varepsilon^3} + \cdots\right).$$

Consequently,

$$\varphi_\varepsilon(x) = \frac{6x^2}{1+6\varepsilon} + \frac{3}{1+3\varepsilon}.$$

Setting $\varepsilon \to 0$, we obtain

$$\varphi(x) = \lim_{\varepsilon \to 0} \varphi_\varepsilon(x) = 6x^2 + 3.$$

which is the exact solution of the given equation.

4.2 Nonlinear Schlömilch's Integral Equation

Consider the nonlinear Schlömilch's integral equation of the form [14]

$$f(x) = \frac{2}{\pi}\int_0^{\pi/2} F(\varphi(x\,sin\,t))dt, \tag{4.8}$$

where $F(\varphi(x\,sin\,t))$ is a nonlinear function of $\varphi(x\sin t)$ and $f(x)$ is a continuous differential coefficient for $-\pi \le x \le \pi$.

To handle this nonlinear equation, we will follow the same analysis presented earlier for linear equations.

To achieve this goal, we should first transform this nonlinear equation to a linear form.

To transform equation (4.8) to a linear form of the first kind, we first use the transformation

$$F\big(\varphi(x \sin t)\big) = v(x \sin t), \tag{4.9}$$

such that

$$\varphi(x \sin t) = F^{-1}\big(v(x \sin t)\big), \tag{4.10}$$

which will transform equation (4.8) to

$$f(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} v(x \sin t) dt. \tag{4.11}$$

The method of regularization transforms the linear Schlömilch’s integral equation of the first kind (4.11) to the Schlömilch’s integral equation of the second kind given by

$$\varepsilon v_\varepsilon(x) = f(x) - \frac{2}{\pi} \int_0^{\frac{\pi}{2}} v_\varepsilon(x \sin t) dt, \tag{4.12}$$

where ε is a small positive parameter.

Consequently, we can apply the ADM for solving the equation (4.12), and obtain

$$\begin{cases} v_{\varepsilon,0}(x) = \frac{1}{\varepsilon} f(x), \\ v_{\varepsilon,n+1}(x) = -\frac{2}{\pi\varepsilon} \int_0^{\frac{\pi}{2}} v_{\varepsilon,n}(x \sin t) dt, \ n \geq 0. \end{cases} \tag{4.13}$$

4.2.1 Examples

The scheme that we presented will be illustrated by the following examples.

<u>**Example 1:**</u>

Consider the nonlinear Schlömilch’s integral equation

$$5x^6 = \frac{2}{\pi} \int_0^{\pi/2} \varphi^2(x \sin t)\, dt, \ -\pi \leq x \leq \pi. \tag{4.14}$$

Using the transformation $v = \varphi^2$, which transforms the equation (4.14) to a linear equation given by

$$5x^6 = \frac{2}{\pi} \int_0^{\pi/2} v(x \sin t)\, dt. \tag{4.15}$$

Using the regularization method, equation (4.15) becomes

$$v_\varepsilon(x) = \frac{5}{\varepsilon} x^6 - \frac{2}{\varepsilon\pi} \int_0^{\pi/2} v_\varepsilon(x \sin t)\, dt.$$

Then the recursive scheme can be expressed as

$$
\begin{cases}
v_{\varepsilon,0}(x) = \dfrac{5}{\varepsilon} x^6, \\
v_{\varepsilon,1}(x) = -\dfrac{25\, x^6}{16\, \varepsilon^2}, \\
v_{\varepsilon,2}(x) = \dfrac{125\, x^6}{256\, \varepsilon^3}, \\
v_{\varepsilon,3}(x) = -\dfrac{625\, x^6}{4096\, \varepsilon^4},
\end{cases}
$$

and so on.

So,

$$v_\varepsilon(x) = \frac{5}{\varepsilon} x^6 \left(1 - \frac{5}{16\, \varepsilon} + \frac{25}{256\, \varepsilon^2} - \frac{125}{4096\, \varepsilon^3} + \cdots \right).$$

Consequently,

$$v_\varepsilon(x) = \frac{80\, x^6}{5 + 16\, \varepsilon}.$$

Setting $\varepsilon \to 0$, we obtain

$$v(x) = \lim_{\varepsilon \to 0} v_\varepsilon(x) = 16\, x^6.$$

Hence the exact solution is given by

$$\varphi(x) = \pm 4x^3.$$

<u>**Example 2:**</u>

Let

$$\frac{35}{8} x^8 = \frac{2}{\pi} \int_0^{\pi/2} \varphi^4(x \sin t)\, dt, \; -\pi \le x \le \pi. \tag{4.16}$$

Using the transformation $v = \varphi^4$, which transforms the equation (4.16) to a linear equation given by

$$\frac{35}{8} x^8 = \frac{2}{\pi} \int_0^{\pi/2} v(x \sin t)\, dt. \tag{4.17}$$

Using the regularization method, equation (4.17) becomes

$$v_\varepsilon(x) = \frac{35}{8\varepsilon} x^8 - \frac{2}{\varepsilon\pi} \int_0^{\pi/2} v_\varepsilon(x \sin t)\, dt.$$

Then the recursive scheme can be expressed as

$$\begin{cases} v_{\varepsilon,0}(x) = \dfrac{35}{8\,\varepsilon} x^8, \\ v_{\varepsilon,1}(x) = -\dfrac{1225\,x^8}{1024\;\varepsilon^2}, \\ v_{\varepsilon,2}(x) = \dfrac{42875\,x^8}{131072\,\varepsilon^3}, \\ v_{\varepsilon,3}(x) = -\dfrac{1500625\,x^8}{16777216\,\varepsilon^4}, \end{cases}$$

and so on.

So,

$$v_\varepsilon(x) = \frac{35}{8\,\varepsilon} x^8 \left(1 - \frac{35}{128\,\varepsilon} + \frac{1225}{16384\,\varepsilon^2} - \cdots\right).$$

Consequently,

$$v_\varepsilon(x) = \frac{560\,x^8}{35 + 128\,\varepsilon}.$$

Setting $\varepsilon \to 0$, we obtain

$$v(x) = \lim_{\varepsilon \to 0} v_\varepsilon(x) = 16\,x^8.$$

Hence the exact solution is given by

$$\varphi(x) = \pm 2x^2.$$

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