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Title

Derivation on a prime rings

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Contents

Abstract

In this research project, we present a derivations on prime rings, Jordan derivations on prime rings, $(\theta, \phi)-$derivations on prime rings and reverse derivations. based on the work of several authors.

Throughout the research project R will denote an associative ring with unity 1 and Z is called the center, R is called prime if $aRb = \{0\}$, where $x, y \in R$ implies that $a = 0$ or $b = 0$. As usual $[x, y]$ (resp. $x \circ y$) will denote the Lie product (resp. Jordan product) if $xy - yx$, (resp. $xy + yx$) $\forall\, x, y \in R$. An additive mapping $d : R \to R$ is called a derivation (resp. Jordan derivation) of a ring R if $d(ab) = d(a)b + ad(b)$, (resp. $d(a^2) = d(a)a + ad(a)$) for all $a, b \in R$. Suppose that θ, ϕ are endomorphisms of R. An additive mapping $d : R \to R$ is called a $(\theta, \phi)-$derivation if $d(xy) = d(x)\theta(y) + \phi(x)d(y)$, holds for all $x, y \in R$. An additive mapping $d : R \to R$ is called a reverse derivation if $d(xy) = d(y)x + yd(x)$ for all $x, y \in R$.

The main purpose of our research project, we get some results concerning the relationship between the commutativity of a prime ring and the existence of certain specific types of derivations, $(\theta, \phi)-$derivations and revers derivations of a prime ring R.

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Munirah Alsarami

Introduction

Ring theory is one of the most important concepts in abstract algebra, bringing together several branches of the subject and creating a powerful machine for the study of problems of considerable historical and mathematical importance.

Rings with derivations are not a popular kind of subject that undergoes tremendous revolutions. However, this has been studied by many authors in the last 50 years, specially the relationships between derivations and the structure of rings.

Many mathematicians of recent years studied derivations and commutativity in rings with keen interest and their investigations throw light on the study of different types of derivations like reverse derivations, Jordan derivations, Jordan left derivations, generalized Jordan triple derivations, generalized Jordan triple left derivations on rings. Among these mathematicians E.C. Posner, I.N. Herstein, H.E. Bell, M.N. Daif, W.S. Martindale, L.O. Chung, T.K. Lee, P.H. Lee, A. Laradji, A.B. Thaheem, Q. Deng, N. Argac, M. Bresar , J. Vukman, M.Ashraf, A. Ali, M.A. Choudhary, C. Lanski, N.R. Rehman, A. Nakajima, E.Albas, A.H. Majeed, M.J. Atteya, M. Samman, C. Haetinger, N. Alyamani and R.K. Sharma are the ones whose contributions to this field are outstanding.

This research project is an attempt to present the derivations on prime rings. And in a manner suitable for everybody who have some basic knowledge in ring theory. In this work, we present certain properties of derivations, $(\theta, \phi)-$derivations and reverse derivations in prime rings. We study the properties of derivations, $(\theta, \phi)-$derivations and revers derivations in prime rings with ideals and prove the commutativity of these prime rings.

Throughout the present research project, R will denote an associative ring with unity 1 having at least two elements. The symbol Z stand for the center of R. Recall that a ring R is called prime if $aRb = 0$ with $a, b \in R$ implies $a = 0$ or $b = 0$. Equivalently, the product of any two nonzero ideals of R is nonzero. A ring R is called semi-prime if $aRa = 0$ with $a \in R$ implies $a = 0$. Equivalently, it has no nonzero nilpotent ideals. For any $x, y \in R$, using its associative multiplication one can induce two new products viz. The Lie product $[x, y] = xy - yx$ and the Jordan product $x \circ y = xy + yx$. We use the identities $[xy, z] = x[y, z]+[x, z]y,\ [x, yz] = [x, y]z+y[x, z],\ [z, x+y] = [z, x]+[z, y]$, $[x+y, z] = [x, z]+[y, z]$, $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z)+[x, y]z$ and $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$ for all $x, y, z \in R$. Suppose that $\theta, \phi : R \to R$ be two homomorphisms. Define $(\theta, \phi)-$Lie product $[\ ,\ ]_{\theta,\phi}$ on R as follows $[x, y]_{\theta,\phi} = x\theta(y) - \phi(y)x,$ for all $x, y \in R$. Also, we use the identities $[xy, z]_{\theta,\phi} = x[y, z]_{\theta,\phi} + [x, \phi(z)]y = x[y, \theta(z)] + [x, z]_{\theta,\phi}y$, $[x, yz]_{\theta,\phi} = [x, y]_{\theta,\phi}\theta(z) + \phi(y)[x, z]_{\theta,\phi},\ [z, x + y]_{\theta,\phi} = [z, x]_{\theta,\phi} + [z, y]_{\theta,\phi}$ and $[x + y, z]_{\theta,\phi} = [x, z]_{\theta,\phi} + [y, z]_{\theta,\phi}$ for all $z, x, y \in R$. An additive mapping $d : R \to R$ is called a derivation (resp. Jordan derivation) of a ring R if $d(ab) = d(a)b + ad(b)$, (resp. $d(a^2) = d(a)a + ad(a)$) for all $a, b \in R$. Obviously every derivation is a Jordan derivation. But the converse need not true in general. Recall that $[a, xy] = x[a, y] + [a, x]y$. For a fixed $a \in R$, define $I_a : R \to R$ by $I_a(x) = [a, x]$ for all $x \in R$. The function I_a so defined can be easily checked to be additive and $I_a(xy) = [a, xy] = x[a, y] + [a, x]y = xI_a(y) + I_a(x)y$, for all $x, y \in R$. Thus, I_a is a derivation which is called inner derivation of R. It is obvious to see that every inner derivation on a ring R is a derivation. But one can find plenty of examples of derivations which are not inner. An additive mapping $d : R \to R$ is called $(\theta, \phi)-$derivation of a ring R, (resp. Jordan $(\theta, \phi)-$derivation), where $\theta, \phi : R \to R$ are two mappings of R, if $d(xy) = d(x)\theta(y) + \phi(x)d(y)$, (resp. $d(x^2) = d(x)\theta(x) + \phi(x)d(x)$) holds for all $x, y \in R$. Recall that $\theta, \phi : R \to R$ be two homomorphism. Recall that $[r, xy]_{\theta,\phi} = [r, x]_{\theta,\phi}\theta(y) + \phi(x)[r, y]_{\theta,\phi}$. For a fixed $r \in R$, define $I_r : R \to R$ by $I_r(x) = [r, x]_{\theta,\phi}$, for all $x \in R$. The function I_r so defined can be easily checked to be additive and $I_r(xy) = [r, xy]_{\theta,\phi} = [r, x]_{\theta,\phi}\theta(y) + \phi(x)[r, y]_{\theta,\phi} = I_r(x)\theta(y) + \phi(x)I_r(y)$, for all $x, y \in R$. Thus, I_r is $(\theta, \phi)-$derivation which is called $(\theta, \phi)-$inner derivation of R. It is obvious to see that every $(\theta, \phi)-$inner derivation on a ring R is a $(\theta, \phi)-$derivation. An additive mapping $d : R \to R$ is called a reverse derivation if $d(xy) = d(y)x + yd(x)$ for all $x, y \in R$, on the other hand we said that d is a homomorphism (resp. anti-homomorphism) if $d(xy) = d(x)d(y)$, (resp. $d(xy) = d(y)d(x)$) for all $x, y \in R$.

This research project consists of four chapters, in section one of each chapter we state some definitions which are basic in our study and we present some remarks, elementary properties and examples to explain our objective of each chapter. In the other sections, we present some of results and properties as following

The first chapter (**Basic Property of Algebraic Structures**) is devoted to present the necessary background.

Chapter 2, (**Derivation on Prime Rings**) contains some results on derivations in prime rings. *In section 2.2,* we prove Posner's First Theorem which are of great importance in the rest of our work, namely (2.2.1) in a prime ring of characteristics not 2, if the composition of two derivations is a derivation, then one of them is zero. Also, we consider the following Theorem (2.2.2) for a derivation $d : R \to R$, and let U is a non-zero right ideal of R. If d acts as a homomorphism or an anti-homomorphism on U, then $d = 0$ on R. *In section 2.3,* for R is a prime ring with char$R \neq 2$. We prove the following

Theorem (2.3.1) for d be a non-zero derivation on R. If $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative.

Chapter 3, ($(\theta, \phi)-$**Derivation on Prime Ring**) we discuss $(\theta, \phi)-$derivation on prime rings. It contains some results on $(\theta, \phi)-$derivations in prime rings. *In section 3.2,* for $(\theta, \phi)-$derivation $d : R \to R$, we prove the following results (3.2.1) if d is a homomorphism on R then $d = 0$. (3.2.2) if d is a an anti-homomorphism on R then $d = 0$. *In section 3.3,* for $(\theta, \phi)-$derivation $d : R \to R$, we prove the following Theorem (3.3.3) if $d(xy) = d(yx)$, for all $x, y \in R$ then R is commutative.

Chapter 4, (**Reverse Derivation of Prime Ring**) is devoted to present some results on reverse derivation in prime rings. *In section 4.2,* we prove that result(4.2.1) if a reverse derivation d acts as homomorphism or an anti-homomorphism on a non-zero right ideal U of a prime ring R, then $d = 0$. *In section 4.3,* for R is a prime ring with char$R \neq 2$ and U is a non-zero right ideal of R. We prove the following Theorems (4.3.2) let d be a non-zero reverse derivation of R. If $[d(x), x] = 0$ for all $x \in U$, then R is commutative. (4.3.3) let d be a non-zero reverse derivation of R. If $[d(x), d(y)] = 0$ for all $x, y \in U$, then R is commutative.

Finally, let us say that the research project is partially based on work of several authors, One more time, the interested readers can consult the innumerable references cited in the end.

Chapter 1

Basic Property of Algebraic Structures

In this chapter includes some basic notions and important terminology which we shall need for the development of the susequent chapters of our research project. Also, we will present examples and necessary remarks are given at proper places to make the exposition self contained as much as possible.

1.1 Deinitions and Examples

Definition 1.1.1.

1. *A ring R is a set together with two binary operations (usually denoted as addition $(+)$ and multiplication $(\cdot)$) satisfying the following conditions:*

   i. $(R, +)$ *is an abelian group,*

   ii. *associative multiplication:* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ *for all* $a, b, c \in R$,

   iii. *the distributive laws hold in* R *: for all* $a, b, c \in R$
   $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$ *and* $a \cdot (b \cdot c) = (a \cdot b) + (a \cdot c)$.

2. *If $a \cdot b = b \cdot a$ for all $a, b \in R$, then R is called a commutative ring.*

3. *If R contains an element 1 such that $1 \cdot a = a \cdot 1 = a$ for all $a \in R$, then R is called a ring with unity 1.*

We shall usually with simply ab rather than $a \cdot b$ for $a, b \in R$. The additive identity of R will always be denoted by 0 and the additive inverse of the ring element a will be denoted by $-a$.

Examples 1.1.1.

1. *$(\mathbb{Z}, +, \cdot)$ is a commutative ring with unity under usual addition and multiplication, where $\mathbb{Z}$ is the set of all integers,*

2. *$(\mathbb{E}, +, \cdot)$ is a commutative ring under usual addition and multiplication, where $\mathbb{E}$ is the set of all even numbers,*

3. *$(M_n(\mathbb{Z}), +, \cdot)$ is a ring with unity I_n (I_n be an identity matrix) under addition and multiplication of matrices, where $M_n(\mathbb{Z})$ is the set of all $n \times n$ matrices with entries in $\mathbb{Z}$.*

Definition 1.1.2.
A subset S of a ring R is a subring of R if S is itself a ring with the same operations of R.

Theorem 1.1.1.
A non-empty subset S of a ring R is a subring if S is closed under subtraction and multiplication that is, if $a \in S$ and $b \in S$ imply that $a - b$ and ab are in S.

Examples 1.1.2.

1. *If R is a ring, then it contains two trivial subrings, R and $\{0\}$,*

2. *the set of all integers $\mathbb{Z}$ is a subring of the ring $\mathbb{R}$ the set of all real numbers,*

3. *the set of all even numbers* $\mathbb{E}$ *is a subring of the ring* $\mathbb{Z}$.

Definition 1.1.3.
The characteristic of a ring R *is the least positive integer* n *such that* $nx = 0$ *for all* x *in* R*, and denoted by* $charR$. *If no such integer exists, we say that charR=0.*

Examples 1.1.3.

1. *char* $\mathbb{Z}$ *is* 0,

2. *char* $\mathbb{Z}_n$ *is* n*, where* $\mathbb{Z}_n$ *is the set of all congruence classes modulo* n.

Definition 1.1.4.
A ring R *is said to be* n*-torsion free, where* $n \neq 0$ *is an integer, if whenever* $nx = 0$, $x \in R$ *then* $x = 0$.

Definition 1.1.5.
The center Z *of a ring* R *is the set of all those elements of* R *which commute with each element of* R *that is,*

$$Z = \{x \in R \mid xr = rx, \text{ for all } r \in R\},$$

Examples 1.1.4.

1. *The center of any commutative ring is itself,*

2. *the center of a ring* $M_n(\mathbb{Z})$ *is the set of all diagonal matrices whose entries in* $\mathbb{Z}$.

Definition 1.1.6.
Let A *be a non-empty subset of ring* R. *The centralizer of* A *in* R *is the set*

$$C_R(A) = \{r \in R \mid ar = ra, \text{ for all } a \in A\}.$$

Example 1.1.1.
Let $\mathbb{Z}$ *be the set of all integers and* $\mathbb{E}$ *be the set of all even numbers. Then,* $C_{\mathbb{Z}}(\mathbb{E})$ *is* $\mathbb{Z}$

Definition 1.1.7.
A ring homomorphismq f from a ring R to a ring S is a mapping from R to S that preserves the two ring operations; that is, for all a, b in R,

$$f(a+b) = f(a) + f(b) \quad \text{and} \quad f(ab) = f(a)f(b).$$

Remark 1.1.1. (***Types of homomorphisms***)
A homomorphism $f : R \to S$ of rings R and S is called

- *a monomorphism, if f is injective ,*
- *an epimorphism, if f is surjective,*
- *an isomorphism, if f is bijective ,*
- *an endomorphism, if $R = S$,*
- *an automorphism, if $R = S$ and f is an isomorphism.*

Examples 1.1.5.

1. *Let $R[x]$ denote the ring of all polynomials with real coefficients. The mapping $f(x) \to f(1)$ is a ring homomorphism from $R[x]$ onto R,*
2. *the mapping $f : Z \to Z_2$ defined in by sending an even integer to 0 and an odd integer to 1 is a ring homomorphism.*

Definition 1.1.8.
If f is a homomorphism from the ring R to the ring S, then the kernel of f is the following set

$$kerf = \{x \in R \mid f(x) = 0\}.$$

Definition 1.1.9.
The subring I of a ring R is an ideal (two-sided) of R if $x \in I$ and $r \in R$ imply that xr and rx are in I.

Remark 1.1.2.

- *A right ideal of R is a subring I of R such that $xr \in I$ for all $x \in I$, $r \in R$,*
- *a left ideal of R is a subring I of R such that $rx \in I$ for all $x \in I$, $r \in R$,*
- *the subrings $I = \{0\}$ and $I = R$ are always ideals of a ring R. These ideals are called trivial,*
- *if R is a ring with unity 1 and I is an ideal of R that contains 1, then it can be shown that $I = R$.*

Example 1.1.2.
For a fixed integer n, $n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z}$.

Definition 1.1.10.
Let a is a fixed element of the commutative ring R with unity, the ideal $(a) = \{ar \mid r \in R\}$, which consists of all multiples of a by elements r of R, is called the principal ideal generated by a in R.

Remark 1.1.3.
In the ring $\mathbb{Z}$ of all integers, every ideal is a principal ideal

Definition 1.1.11.
An ideal P of a commutative ring R is a prime ideal if $P \neq R$ and if $ab \in P$, where $a, b \in R$ implies either $a \in P$ or $b \in P$.

Example 1.1.3.
An ideal (10) is a prime ideal of $\mathbb{E}$.

Definition 1.1.12.
An ideal M of a ring R is called a maximal ideal if $M \neq R$ and the only ideals containing M are M and R.

Remark 1.1.4.

- *If $M \neq R$ is a maximal ideal of R then for every ideal A of R, $M \subseteq A \subseteq R$ holds only when either $A = M$ or $A = R$,*
- *every maximal ideal in a commutative ring with unity 1 is prime. However, the converse of this statement is not valid*

The following example shows that unity in the ring is essential for the validity of the above statement.

Example 1.1.4.
The ideal (4) in $\mathbb{E}$ is maximal, but certainly not prime. Indeed,

$$2 \cdot 2 \in (4). \ \textit{but } 2 \notin (4)$$

Definition 1.1.13.
A ring R in which the set R^* of non-zero elements is a group with respect to the multiplication in R is called a division ring. Equivalently, R is a division ring if every non-zero element of R has a multiplicative inverse in R.

Examples 1.1.6.
The rings $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are some examples of division rings.

Definition 1.1.14.
A non-zero ring R is said to be a simple ring if R has no (two-sided) ideals other than $\{0\}$ and R.

Examples 1.1.7.

1. *Division rings are simple,*
2. *matrix rings over division rings are simple.*

Definition 1.1.15.
A zero divisor is a non-zero element x of a commutative ring R such that there is a non-zero element $y \in R$ with $xy = 0$.

Definition 1.1.16.

An integral domain is a commutative ring with unity and no zero divisors.

Example 1.1.5.

The ring $\mathbb{Z}_p$ *of all integers modulo* p *where* p *is a prime, is an integral domain.*

Definition 1.1.17.

A field is a commutative ring with unity in which every non-zero element has a multiplicative inverse.

Definition 1.1.18.

An element a *of a ring* R *is said to be a nilpotent element if* $a^n = 0$ *for some positive integer* n.

Example 1.1.6.

Let $M_2(\mathbb{Z})$ *be a ring and let* $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in M_2(\mathbb{Z})$.

Indeed, $A \cdot A = A^2 = \boldsymbol{0}$.

Then, A *is a nilpotent element of* $M_2(\mathbb{Z})$.

Definition 1.1.19.

An ideal I *of a ring* R *is said to be a nil ideal if every element of* I *is nilpotent.*

Definition 1.1.20.

An ideal I *of a ring* R *is said to be a nilpotent if there exists a positive integer* n *such that* $I^n = \{0\}$.

Remark 1.1.5.

Every nilpotent ideal is a nil ideal.

1.2 Prime and Semi-prime Rings

Definition 1.2.1.

A ring R *is called prime if* $a, b \in R$ *such that* $aRb = \{0\}$ *implies that* $a = 0$ *or* $b = 0$.

The definition 1.2.1 just given is equivalent to:
A ring R is called prime if whenever, $I_1 \neq \{0\}$ and $I_2 \neq \{0\}$ are ideals of R, then $I_1 I_2 \neq \{0\}$.

Examples 1.2.1.

1. *Any integral domain is a prime ring,*

2. *any simple ring is a prime ring,*

3. *any matrix ring over an integral domain is a prime ring. In particular, the ring $M_2(\mathbb{Z})$ is a prime ring.*

Definition 1.2.2.
A ring R is called semi-prime if $a \in R$ such that $aRa = \{0\}$ implies that $a = 0$.

The definition 1.2.2 just given is equivalent to:
$I^2 = \{0\}$ implise that $I = \{0\}$ for every ideal I of R.

Some Properties of Prime and Semi-prime Ring

1. A commutative ring is a prime ring if and only if it is an integral domain,

2. a ring is prime if and only if its zero ideal is a prime ideal,

3. the ring of matrices over a prime ring is again a prime ring,

4. the class of semi-prime rings includes prime rings.

1.3 Lie and Jordan Rings

Definition 1.3.1.
Let R be an associative ring, we can induce on R using its operations two structures as follows

1. *For all* $x, y \in R$, *the Lie product* $[x, y] = xy - yx$,

2. *for all* $x, y \in R$, *the Jordan product* $x \circ y = xy + yx$.

Lemma 1.3.1.
Let R *be a ring the following identities hold, for all* $x, y, z \in R$.

1. $[x, yz] = y[x, z] + [x, y]z$,

2. $[xy, z] = x[y, z] + [x, z]y$,

3. $[x + y, z] = [x, z] + [y, z]$,

4. $[x, y + z] = [x, y] + [x, z]$,

5. $[[x, y], z] + [[y, z], x] + [z, x], y] = 0$, *This is known as Jacobi identity,*

6. $[x, x] = 0$,

7. $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$,

8. $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$.

Now we can prove some identities easily as follows

Proof. We shall prove (1), and (2) is similar

$$
\begin{aligned}
[x, yz] &= x(yz) - (yz)x \\
&= yxz - (yz)x + x(yz) - yxz \\
&= y(xz) - y(zx) + (xy)z - (yx)z \\
&= y(xz - zx) + (xy - yx)z \\
&= y[x, z] + [x, y]z.
\end{aligned}
$$

Now, we will prove (3), and (4) is similar

$$\begin{aligned} [x+y,z] &= (x+y)z - z(x+y) \\ &= xz + yz - zx - zy \\ &= xz - zx + yz - zy \\ &= [x,z] + [y,z]. \end{aligned}$$

Finally, we will prove (7), and (8) is similar

$$\begin{aligned} x \circ (yz) &= x(yz) + (yz)x \\ &= x(yz) + yxz - yxz + (yz)x \\ &= (xy + yx)z - y(xz - zx) \\ &= (x \circ y)z - y[x,z]. \end{aligned} \tag{i}$$

And,

$$\begin{aligned} x \circ (yz) &= x(yz) + (yz)x \\ &= (yz)x + yxz + x(yz) - yxz \\ &= y(zx + xz) + (yx - xy)z \\ &= y(z \circ x)z + [y,x]z. \end{aligned} \tag{ii}$$

Therefore, from i and ii we get

$$x \circ (yz) = (x \circ y)z - y[x,z] = y(x \circ z) + [x,y]z.$$

□

Definition 1.3.2.

Let R be a ring. We can define on R, the Lie operation by defining the product in this ring to be $[a,b] = ab - ba$ for all $a, b \in R$, then R is called a Lie ring

Definition 1.3.3.
Let R be a ring. We can define on R, the Jordan operation by defining the product in this ring to be $a \circ b = ab + ba$ for all $a, b \in R$, then R is called a Jordan ring.

Definition 1.3.4.
An additive subgroup A of R is called a Lie subring of R if whenever $a, b \in A$ then $[a, b]$ is also in A.

Definition 1.3.5.
An additive subgroup A of R is called a Jordan subring of R if whenever $a, b \in A$ then $(a \circ b)$ is also in A.

Definition 1.3.6.
An additive subgroup U of R is called a Lie ideal of R if whenever $u \in U$ and $r \in R$, then $[u, r]$ is also in U.

Definition 1.3.7.
An additive subgroup U of R is called a Jordan ideal of R if whenever $u \in U$ and $r \in R$, then $(u \circ r)$ is also in U.

Chapter 2

Derivation on Prime Rings

Throughout the present chapter R will denote an associative ring with unity 1, $Z = \{x \in R \mid xr = rx, \text{ for all } r \in R\}$ is called the center of R. Recall that R is prime if $aRb = \{0\}$ implies that $a = 0$ or $b = 0$. As usual $[x, y] = xy - yx$ will denote the Lie product, and the Jordan product $x \circ y = xy + yx$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$, $r \in R$. An additive mapping $d : R \to R$ is called a derivation (resp. Jordan derivation) of a ring R if $d(ab) = d(a)b + ad(b)$, (resp. $d(a^2) = d(a)a + ad(a)$) for all $a, b \in R$. Obviously every derivation is a Jordan derivation. But the converse need not true in general.

2.1 Definitions and Examples

Definition 2.1.1.
Let R be a ring. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$.

Definition 2.1.2.
Let R be a ring. An additive mapping $d : R \to R$ is called a Jordan derivation if $d(x^2) = d(x)x + xd(x)$, for all $x \in R$.

Example 2.1.1.

Let R be a ring of 2×2 matrices with respect to usual addition and multiplication in matrices, where

$$R = \left\{ \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \mid \quad a, b \in \mathbb{Z} \right\} \subseteq M_2(\mathbb{Z}).$$

Let $d : R \to R$ be a map defined by $d\left(\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, *for all* $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \in R$. *Then, d is a derivation of R.*

Suppose that, $A = \begin{bmatrix} 0 & a_1 \\ 0 & b_1 \end{bmatrix} \in R$ *and* $B = \begin{bmatrix} 0 & a_2 \\ 0 & b_2 \end{bmatrix} \in R$, *where* $a_1, a_2, b_1, b_2 \in \mathbb{Z}$.

Then, we will prove that $\quad d(A + B) = d(A) + d(B), \quad \forall A, B \in R.$

Indeed,

$$\begin{aligned} d(A+B) &= d\left(\begin{bmatrix} 0 & a_1 \\ 0 & b_1 \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ 0 & b_2 \end{bmatrix}\right) \\ &= d\left(\begin{bmatrix} 0 & a_1 + a_2 \\ 0 & b_1 + b_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & a_1 + a_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix} \\ &= d\left(\begin{bmatrix} 0 & a_1 \\ 0 & b_1 \end{bmatrix}\right) + d\left(\begin{bmatrix} 0 & a_2 \\ 0 & b_2 \end{bmatrix}\right) \\ &= d(A) + d(B). \end{aligned}$$

Thus, d is an additive mapping.

Now, we will prove that

$$d(AB) = d(A)B + Ad(B), \quad \forall A, B \in R.$$

Indeed

$$\begin{aligned} d(AB) &= d\left(\begin{bmatrix} 0 & a_1 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} 0 & a_2 \\ 0 & b_2 \end{bmatrix}\right) \\ &= d\left(\begin{bmatrix} 0 & a_1 b_2 \\ 0 & b_1 b_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & a_1 b_2 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{2.1}$$

and,

$$\begin{aligned} d(A)B + Ad(B) &= d\left(\begin{bmatrix} 0 & a_1 \\ 0 & b_1 \end{bmatrix}\right) \begin{bmatrix} 0 & a_2 \\ 0 & b_2 \end{bmatrix} + \begin{bmatrix} 0 & a_1 \\ 0 & b_1 \end{bmatrix} d\left(\begin{bmatrix} 0 & a_2 \\ 0 & b_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & a_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a_2 \\ 0 & b_2 \end{bmatrix} + \begin{bmatrix} 0 & a_1 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_1 b_2 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{2.2}$$

From 2.1 and 2.2 we get

$$d(AB) = d(A)B + Ad(B), \quad \forall A, B \in R.$$

Therefore, d is a derivation of R.

Remark 2.1.1.
Obviously, every derivation is a Jordan derivation. But the converse need not true in general.

Now, we can prove that the converse need not true, in the following example

Example 2.1.2.

Let $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid \quad c^2 = 0 \text{ and } a, b, c, d \in \mathbb{C} \right\} \subseteq M_2(\mathbb{C})$ *be a ring of* 2×2 *matrices with respect to usual addition and multiplication in matrices. Let the mapping* $d : R \to R$ *defined by*

$$d\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}, \quad \text{for every } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R.$$

Then, d is a Jordan derivation of R.

Assume that, $C = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in R$ *and* $D = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in R$*, where* $a_1, b_1, c_1, d_1, a_2, b_2, c_2,$ $d_2 \in \mathbb{C}$.

Then, we shall prove that $\quad d(C + D) = d(C) + d(D), \quad \forall C, D \in R.$

Indeed,

$$\begin{aligned} d(C + D) &= d\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \\ &= d\left(\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & c_1 + c_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & c_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ 0 & 0 \end{bmatrix} \\ &= d\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + d\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \\ &= d(C) + d(D). \end{aligned}$$

Thus, d is an additive mapping.

Now, let $X=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R$, *we will prove that* $d(X^2) = d(X)X + Xd(X), \quad \forall X \in R$. *Indeed,*

$$\begin{aligned} d(X^2) &= d\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\ &= d\left(\begin{bmatrix} a^2+bc & ab+bd \\ ca+dc & cd+d^2 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & ca+dc \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & (a+d)c \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{2.3}$$

and,

$$\begin{aligned} d(X)X + Xd(X) &= d\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} d\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & c \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} c^2 & cd \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & ac \\ 0 & c^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & cd \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & ac \\ 0 & 0 \end{bmatrix} \qquad (Since,\ c^2 = 0) \\ &= \begin{bmatrix} 0 & cd+ac \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & (a+d)c \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{2.4}$$

From 2.3 and 2.4 we get

$$d(X^2) = d(X)X + Xd(X), \quad \forall X \in R.$$

Hence, d is a Jordan derivation of R.
But, we will show that d is not a derivation i.e.,

$$d(AB) \neq d(A)B + Ad(B), \quad \textit{for some } A, B \in R$$

such that

$A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \in R$ *and* $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in R$*, where* $a \in \mathbb{C}$.
Since,

$$\begin{aligned} d(X^2) &= d\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \\ &= d\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{2.5}$$

and,

$$\begin{aligned} d(A)B + Ad(B) &= d\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} d\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{2.6}$$

From 2.5 and 2.6 we get

$$d(AB) \neq d(A)B + Ad(B), \quad \textit{for some } A, B \in R.$$

Therefore, d is not a derivation of R.

Definition 2.1.3.
Let R be a ring. For some fixed $r \in R$, the mapping $I_r : R \to R$ given by

$$I_r(x) = [r, x], \text{ for all } x \in R$$

is called an inner derivation.

Now, we can show that every inner derivation is a derivation by the following lemma

Lemma 2.1.1.
Every inner derivation is derivation.

Proof. Let $I_r : R \to R$ be an inner derivation, for some fixed $r \in R$, then

$$I_r(x) = [r, x], \forall x \in R. \tag{2.7}$$

Firstly, we will show that

$$I_r(x + y) = I_r(x) + I_r(y), \quad \forall x, y \in R.$$

Since,

$$\begin{aligned} I_r(x + y) &= [r, x + y] \\ &= [r, x] + [r, y] \qquad \text{(By lemma 1.3.1)} \\ &= I_r(x) + I_r(y) \end{aligned}$$

Thus, I_r is an additive mapping.

Now, replacing x by xy, in 2.7 where $y \in R$, we get

$$\begin{aligned} I_r(xy) &= [r, xy] \\ &= [r, x]y + x[r, y] \qquad \text{(By lemma 1.3.1)} \\ &= I_r(x)y + xI_r(y), \quad \forall x, y \in R. \end{aligned}$$

That is I_r is a derivation.

□

Remark 2.1.2.
The converse of lemma 2.1.1 is not necessary true as the following example

Example 2.1.3.

Let $R = \left\{ \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \mid \quad a, b \in \mathbb{Z} \right\} \subseteq M_2(\mathbb{Z})$ *be a ring of* 2×2 *matrices with respect to usual addition and multiplication in matrices. Let the mapping* $d : R \to R$ *defined by*

$$d\left(\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \quad \textit{for every } \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \in R.$$

Then, d is a derivation of R.(see example 2.1.1) But, now we can see that d is not inner derivation.

Let $A = \begin{bmatrix} 0 & 2 \\ 0 & d \end{bmatrix} \in R$ and $B = \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix} \in R$, where $b, d \in \mathbb{Z}$.

Indeed,

$$
\begin{aligned}
d(AB) &= d\left(\begin{bmatrix} 0 & 2 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix}\right) \\
&= d\left(\begin{bmatrix} 0 & 2 \\ 0 & d \end{bmatrix}\right) \\
&= \quad \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}
\end{aligned} \tag{2.8}
$$

and,

$$
\begin{aligned}
I_A(B) = [A, B] = AB - BA & \qquad \text{(By definitions 1.3.1 and 2.1.3)} \\
= \begin{bmatrix} 0 & 2 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & d \end{bmatrix} & \\
= \begin{bmatrix} 0 & 2 \\ 0 & d \end{bmatrix} - \begin{bmatrix} 0 & bd \\ 0 & d \end{bmatrix} & \\
= \begin{bmatrix} 0 & 2 - bd \\ 0 & 0 \end{bmatrix}. &
\end{aligned} \tag{2.9}
$$

From 2.8 and 2.9 we get

$$
d(AB) \neq I_A(B), \quad \text{for some } A, B \in R.
$$

Therefore, d is not an inner derivation of R.

2.2 Some Properties of Derivation

In this section, we will prove Posner's First Theorem that are easily conjectured, namely (2.2.1). In a prime ring of characteristics not 2, if the composition of two derivations is a derivation, then one of them is zero. For a derivation $d : R \to R$, and let U be a non-zero right ideal of R. we consider the following Theorem (2.2.2). If d acts as a homomorphism or an anti-homomorphism on U, then $d = 0$ on R. Furthermore, an additive mapping d is a homomorphism or anti-homomorphism respectively $d(xy) = d(x)d(y)$ *or* $d(xy) = d(y)d(x),$ *for all* $x, y \in U$.

The following lemma which is necessary for developing the proof of Posner's First Theorem

Lemma 2.2.1.
Let d be a derivation of a prime ring R and r be an element of R. If $rd(x) = 0$ for all $x \in R$ then either $r = 0$ or d is zero.

Proof. Let R be a prime ring , and let d be a derivation of R. By assumpition, we have

$$rd(x) = 0, \quad \forall x \in R. \tag{2.10}$$

Replacing x by xy in 2.10, $y \in R$ and then, from definition 2.1.1 of derivation we get

$$0 = rd(xy) = r[d(x)y + xd(y)] = rd(x)y + rxd(y)$$

That's mean

$$rd(x)y + rxd(y) = 0, \quad \forall x, y \in R. \tag{2.11}$$

Using 2.10 then, 2.11 becomes

$$rxd(y) = 0, \quad \forall x \in R$$

indeed,

$$rRd(y) = \{0\}$$

Hence, by the definition 1.2.1 of a prime ring, we have either $r = 0$ or $d(y) = 0$. If d is not zero, that is, there exist some y in R such that $d(y) \neq 0$ therefore, $r = 0$. □

Now we introduce the Posner's First Theorem

Theorem 2.2.1. (***Posner's First Theorem***)
Let R be a prime ring with characteristic not 2 and d,d' derivations of R such that the composition dd' is also a derivation then, one at least of d,d' is zero.

Proof. Let R be a prime ring with $char R \neq 2$. If dd' is a derivation on R. Then, by the definition 2.1.1 of a derivation dd', we have

$$(dd')(xy) = (dd')(x)y + x(dd')(y), \quad \forall x, y \in R. \tag{2.12}$$

On the other side, d and d' are derivations on R. Then, by the definition 2.1.1 of a derivations d and d', we have

$$\begin{aligned}(dd')(xy) &= d(d'(xy)) \\ &= d(d'(x)y + xd'(y)) \\ &= d(d'(x)y) + d(xd'(y)) \\ &= d(d'(x))y + d'(x)d(y) + d(x)d'(y) + xd(d'(y)) \\ &= (dd')(x)y + d'(x)d(y) + d(x)d'(y) + x(dd')(y).\end{aligned} \tag{2.13}$$

Now, from 2.12 and 2.13, we get

$$d'(x)d(y) + d(x)d'(y) = 0, \quad \forall x, y \in R. \tag{2.14}$$

Replacing x by $xd(z)$ in 2.14, we get

$$d'(xd(z))d(y) + d(xd(z))d'(y) = 0, \quad \forall x, y, z \in R. \tag{2.15}$$

Using the definition 2.1.1 of a derivations d and d' then, 2.15 becomes

$$[d'(x)d(z) + xd'(d(z))]d(y) + [d(x)d(z) + xd(d(z))]d'(y) = 0, \quad \forall x, y, z \in R.$$

That means

$$d'(x)d(z)d(y) + xd'(d(z))d(y) + d(x)d(z)d'(y) + xd(d(z))d'(y) = 0, \quad \forall x, y, z \in R. \tag{2.16}$$

In 2.14, replace x by $d(z)$ then, we get

$$d'(d(z))d(y) + d(d(z))d'(y) = 0, \quad \forall y, z \in R.$$

Multiplying the last equation by x then, becomes

$$x[d'(d(z))d(y) + d(d(z))d'(y)] = 0, \quad \forall x, y, z \in R. \tag{2.17}$$

Comparing between 2.16 and 2.17, we have

$$d'(x)d(z)d(y) + d(x)d(z)d'(y) = 0, \quad \forall x, y, z \in R. \tag{2.18}$$

Replacing x by z in 2.14 we get

$$d(z)d'(y) = -d'(z)d(y), \quad \forall y, z \in R.$$

Hence, 2.18 becomes

$$d'(x)d(z)d(y) - d(x)d'(z)d(y) = 0, \quad \forall x, y, z \in R.$$

That's mean

$$(d'(x)d(z) - d(x)d'(z))d(y) = 0, \quad \forall x, y, z \in R. \tag{2.19}$$

Now, by using lemma 2.2.1 on 2.19, we get

$$d'(x)d(z) - d(x)d'(z) = 0, \quad \forall x, z \in R. \tag{2.20}$$

unless d is zero, $\forall y \in R$.
Then, by replacing between y and z in 2.14, tell us that

$$d'(x)d(z) + d(x)d'(z) = 0, \quad \forall x, z \in R. \tag{2.21}$$

Adding 2.20 and 2.21, we get

$$2d'(x)d(z) = 0, \quad \forall x, z \in R.$$

Since, $char R \neq 2$ then

$$d'(x)d(z) = 0, \quad \forall x, z \in R. \tag{2.22}$$

Again, by using lemma 2.2.1 on 2.22, we get

$$d \text{ is zero} \quad \text{ or } \quad d'(x) = 0, \quad \forall x \in R.$$

Therefore, d is zero or d' is zero. □

Theorem 2.2.2.
Let R be a prime ring and U a non-zero right ideal of R. If d is a derivation of R which acts as a homomarphism or an anti-homomorphism on U, then $d = 0$ on R.

Proof. Since d acts a as homomarphism on U, then we have

$$xd(y) + d(x)y = d(xy) = d(x)d(y), \quad \forall x, y \in U. \tag{2.23}$$

Subsituting x^2 for x in 2.23, we get

$$x^2 d(y) + d(x^2)y = d(x^2)d(y)$$

or

$$x^2d(y) + xd(x)y + d(x)xy = xd(x)d(y) + d(x)xd(y) \quad \text{(By definition 2.1.2)}$$

or

$$x(xd(y) + d(x)y) + d(x)xy = xd(x)d(y) + d(x)xd(y)$$

Recalling 2.23, we conclude that

$$xd(x)d(y) + d(x)xy - xd(x)d(y) - d(x)xd(y) = 0$$

or

$$d(x)x(y - d(y)) = 0, \quad for\ all\ x, y \in U. \quad (2.24)$$

Replacing y by yr, $r \in R$, we have

$$d(x)x(yr - d(yr)) = 0$$

implies that,

$$d(x)x(y - d(y))r - d(x)xyd(r) = 0. \quad \text{(By definition 2.1.1)}$$

Which together with 2.24, gives

$$d(x)xyd(r) = 0.$$

Now, replacing y by $ys, s \in R$, yields

$$d(x)xysd(r) = 0$$

or equivalently,

$$d(x)xyRd(r) = \{0\}, \quad \forall x, y \in U \text{ and } r \in R. \tag{2.25}$$

Assume that $d(R) \neq \{0\}$. Then from 2.25, we have

$$d(x)xy = 0. \quad \forall x, y \in U. \tag{2.26}$$

Thus from 2.24, we have

$$d(x)xd(y) = 0, \quad \forall x, y \in U. \tag{2.27}$$

Note that

$$d(x(xy)) = d(x)d(xy).$$

That is

$$xd(xy) + d(x)xy = d(x)d(xy), \quad \forall x, y \in U$$

or

$$x^2d(y) + xd(x)y + d(x)xy = d(x)xd(y) + d(x)d(x)y.$$

In view of 2.26, and 2.27, this reduces to

$$x^2d(y) = (d(x)d(x) - xd(x))y, \quad \forall x, y \in U. \tag{2.28}$$

Since $d(x^2) = (d(x))^2 \ \forall x \in U$, we have

$$(d(x))^2 - xd(x) = d(x)x \qquad (By\ definition\ 2.1.2)$$

hence, 2.26 and 2.28 yield

$$x^2 d(y) = 0, \quad \forall x, y \in U. \tag{2.29}$$

Let r be an element of R and replacing y by yr in 2.29, gives

$$x^2 y d(R) = \{0\} \qquad (By\ definition\ 2.1.1)$$

from which we conclude that

$$x^2 y R d(R) = \{0\}$$

and hence, $x^2 y = 0$, for all $x, y \in U$. In particular $x^3 = 0$, for all $x \in U$. Since R is prime ring, then it has no nil right ideals, so we have a contradiction. Thus $d(R) = (0)$ $i.e.,$ $d = 0$.

Now, since d acts as an anti-homomarphism on U, then we have

$$xd(y) + d(x)y = d(xy) = d(y)d(x), \quad \forall x, y \in U. \tag{2.30}$$

In 2.30, replacing y by xy, we get

$$xd(xy) + d(x)xy = d(xy)d(x)$$

or equivalently,

$$xd(xy) + d(x)xy = xd(y)d(x) + d(x)yd(x). \quad \text{(By definition 2.1.1)} \tag{2.31}$$

Indeed, the first terms on the two sides of 2.31 are equal, we conclude that

$$d(x)xy = d(x)yd(x), \quad for\ all\ x, y \in U. \tag{2.32}$$

Replacing y by yr, $r \in R$ in 2.32 gives

$$d(x)xyr = d(x)yrd(x)$$

on other hand, right-multiplying 2.32 by r gives

$$d(x)xyr = d(x)yd(x)r.$$

Thus, we have

$$d(x)yrd(x) = d(x)yd(x)r$$

or

$$d(x)y[r, d(x)] = 0, \quad \forall x, y \in U \textit{ and } r \in R. \tag{2.33}$$

Replacing y by ys, $s \in R$ in 2.33 gives

$$d(x)yR[r, d(x)] = \{0\}, \quad \forall x, y \in U \textit{ and } r \in R.$$

Since R is prime, either

$$d(x)y = 0 \textit{ or } d(x) \in Z.$$

Thus either

$$d(U)U = \{0\} \textit{ or } d(U) \subseteq Z.$$

For $d(U) \subseteq Z$, d acts as a homomorphism on U and there is nothing to do. Now assume that $d(U)U = \{0\}$, since

$$xd(y) + d(x)y = d(y)d(x), \quad \forall x, y \in U$$

we have

$$xd(y) = d(y)d(x), \quad \forall x, y \in U \tag{2.34}$$

Replacing x by xr, we get

$$xrd(y) = d(y)(xd(r) + d(x)r)$$

hence,

$$xrd(y) = d(y)d(x)r, \quad for\ all\ x, y \in U, and\ r \in R.$$

From 2.34, we see that

$$xd(y)r = d(y)d(x)r,$$

hence $x[r, d(y)] = 0$. Replacing x by xs yields

$$xR[r, d(y)] = \{0\}, \quad for\ all\ x, y \in U, and\ r \in R.$$

Since R is prime ring, therefore $d(U) \subseteq Z$. so d to be a homomorphism of U. It follows $d = 0$. This completes the proof of the Theorem 2.2.2. □

2.3 Commutatively of Prime Ring

In this section R is a prime ring with char$R \neq 2$. We prove the result (2.3.1) since d is a non-zero derivation on R. If $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative.

Theorem 2.3.1.
Let R be a prime ring with char$R \neq 2$ and d be a non-zero derivation on R. If $[d(x), d(y)] = 0$, for all $x, y \in R$, then R is commutative.

Proof. Let R be a prime ring with char$R \neq 2$ and d be a non-zero reverse derivation of R.

$$\text{Let } [d(x), d(y)] = 0, \quad \forall x, y \in R. \tag{2.35}$$

Replacing y by xy in 2.35, we get

$$\begin{aligned} 0 &= [d(x), d(xy)] = [d(x), d(x)y + xd(y)] = [d(x), d(x)y] + [d(x), xd(y)] \\ &= d(x)[d(x), y] + [d(x), d(x)]y + [d(x), x]d(y) + x[d(x), d(y)] \\ &= d(x)[d(x), y] + [d(x), x]d(y), \quad \forall x, y \in R. \end{aligned} \tag{2.36}$$

Replacing y by yr in 2.36 for $r \in R$, we obtain

$$\begin{aligned} 0 &= d(x)[d(x), yr] + [d(x), x]d(yr) \\ &= d(x)y[d(x), r] + d(x)[d(x), y])r + [d(x), x]d(y)r + [d(x), x]yd(r). \end{aligned}$$

By using 2.36, the above relation reduces to

$$[d(x), x]yd(r) = 0, \quad \forall x, y \in R. \tag{2.37}$$

Replacing r by $d(z)$ in 2.37, we get

$$[d(x), x]yd^2(z) = 0, \quad \forall x, y, z \in R.$$

Since R is prime and $d \neq 0$, then

$$[d(x), x] = 0, \quad \forall x \in R. \tag{2.38}$$

Replacing x by $x + y$ in 2.38, we get

$$\begin{aligned} 0 &= [d(x + y), x + y] \\ &= [d(x), x] + [d(x), y] + [d(y), x] + [d(y), y] \\ &= [d(x), y] + [d(y), x], \quad \forall x, y \in R. \end{aligned} \tag{2.39}$$

Replacing y by yx in 2.39, we have

$$[d(x), yx] + [d(yx), x] = 0.$$

Then, we get

$$y[d(x), x] + [d(x), y]x + [d(y)x, x] + [yd(x), x] = 0.$$

On simplification, we get $[y, x]d(x) = 0$, for all $x, y \in R$.
From Lemma 2.2.1 and Since $d \neq 0$, then $[y, x] = 0$, for all $x, y \in R$. Hence R is commutative.

$\square$

Chapter 3

$(\theta, \phi)-$Derivation on Prime Ring

The primary purpose of this chapter is to investigate about $(\theta, \phi)-$ derivation d which is a ring homomorphism or anti-homomorphism on R. Bell and Kappe ([5]) proved that if d is a derivation of R which is either an endomorphism or anti-endomorphism in semi-prime ring R then $d = 0$, and if d acts as a homomorphism or anti-homomorphism on a non-zero right ideal U of prime ring R, then $d = 0$ on R. It is our aim in this chapter to extend the above mentioned results to a more general situation.

In this chapter, R represent an associative ring with unity 1, center Z, where char$R \neq 2$ and U is a non-zero left ideal of R. Recall that a ring R is prime if $aRb = 0$ implies that $a = 0$ or $b = 0$. Let R be a ring and θ, ϕ be two mappings of R. We write $[x, y]$, $[x, y]_{\theta,\phi}$ for $xy - yx$ and $x\theta(y) - \phi(y)x$, respectively and make extensive use of basic lie product identities $[xy, z]_{\theta,\phi} = x[y, z]_{\theta,\phi} + [x, \phi(z)]y = x[y, \theta(z)] + [x, z]_{\theta,\phi}y$.

Recall that, an additive mapping $d : R \to R$ is called a derivation (resp. Jordan derivation) of a ring R if $d(ab) = d(a)b + ad(b)$, (resp. $d(a^2) = d(a)a + ad(a)$) for all $a, b \in R$. A derivation I_a is an inner if there exist an $a \in R$ such that $I_a(x) = [a, x]$ holds for all $x \in R$. And d is called $(\theta, \phi)-$derivation if $d(xy) = d(x)\theta(y) + \phi(x)d(y)$, holds for all $x, y \in R$.

3.1 Definitions and Properties

Definition 3.1.1.
Let R be a ring and let $\theta, \phi : R \to R$ be two mappings. Define $(\theta, \phi)-$Lie product $[\ ,\ ]_{\theta,\phi}$ on R as follows

$$[x, y]_{\theta,\phi} = x\theta(y) - \phi(y)x, \quad \text{for all } x, y \in R.$$

Lemma 3.1.1.
Let R be a ring and let $\theta, \phi : R \to R$ be two mappings. Then for all $x, y, z \in R$, we have

1. $[x + y, z]_{\theta,\phi} = [x, z]_{\theta,\phi} + [y, z]_{\theta,\phi}$.
2. $[xy, z]_{\theta,\phi} = x[y, z]_{\theta,\phi} + [x, \phi(z)]y = x[y, \theta(z)] + [x, z]_{\theta,\phi}(y)$.

Proof.

$$\begin{aligned} 1.\ [x + y, z]_{\theta,\phi} &= (x + y)\theta(z) - \phi(z)(x + y) \\ &= x\theta(z) + y\theta(z) - \phi(z)x - \phi(z)y \\ &= [x, z]_{\theta,\phi} + [y, z]_{\theta,\phi}. \qquad \text{(By definition 3.1.1)} \end{aligned}$$

To prove that 2. $[xy, z]_{\theta,\phi} = x[y, z]_{\theta,\phi} + [x, \phi(z)]y$

$$\begin{aligned} [xy, z]_{\theta,\phi} &= (xy)\theta(z) - \phi(z)(xy) \\ &= (xy)\theta(z) - x\phi(z)y + x\phi(z)y - \phi(z)(xy) \\ &= x(y\theta(z) - \phi(z)y) + (x\phi(z) - \phi(z)x)y \\ &= x[y, z]_{\theta,\phi} + [x, \phi(z)]y. \qquad \text{(By definitions 3.1.1 and 1.3.1)} \end{aligned}$$

Now, to prove that $[xy, z]_{\theta,\phi} = x[y, \theta(z)] + [x, z]_{\theta,\phi}(y)$

$$\begin{aligned}[xy, z]_{\theta,\phi} &= (xy)\theta(z) - \phi(z)(xy) \\ &= (xy)\theta(z) - x\theta(z)y + x\theta(z)y - \phi(z)(xy) \\ &= x(y\theta(z) - \theta(z)y) + (x\theta(z) - \phi(z)x)y \\ &= x[y, \theta(z)] + [x, z]_{\theta,\phi}(y). \qquad \text{(By definitions 1.3.1 and 3.1.1)}\end{aligned}$$

□

When θ, ϕ are two homomorphisms on R, we can get the following lemma

Lemma 3.1.2.

Let R be a ring and let $\theta, \phi : R \to R$ be two homomorphisms. Then for all $x, y, z \in R$, we have

1. $[x, y + z]_{\theta,\phi} = [x, y]_{\theta,\phi} + [x, z]_{\theta,\phi}$,
2. $[x, yz]_{\theta,\phi} = [x, y]_{\theta,\phi}\theta(z) + \phi(y)[x, z]_{\theta,\phi}$.

Proof.

1. $$\begin{aligned}[x, y + z]_{\theta,\phi} &= x\theta(y + z) - \phi(y + z)x \\ &= x(\theta(y) + \theta(z)) - (\phi(y) + \phi(z))x \quad \text{(Since, } \theta, \phi \text{ are homomorphisms)} \\ &= x\theta(y) + x\theta(z) - \phi(y)x - \phi(z)x \\ &= [x, y]_{\theta,\phi} + [x, z]_{\theta,\phi}. \qquad \text{(By definition 3.1.1)}\end{aligned}$$

2. $$\begin{aligned} [x, yz]_{\theta,\phi} &= x\theta(yz) - \phi(yz)x \\ &= x\theta(y)\theta(z) - \phi(y)\phi(z)x \qquad \text{(Since, } \theta, \phi \text{ are homomorphisms)} \\ &= x\theta(y)\theta(z) + \phi(y)x\theta(z) - \phi(y)x\theta(z) - \phi(y)\phi(z)x \\ &= (x\theta(y) + \phi(y)x)\theta(z) + \phi(y)(x\theta(z) - \phi(z)x) \\ &= [x, y]_{\theta,\phi}\theta(z) + \phi(y)[x, z]_{\theta,\phi}. \qquad \text{(By definition 3.1.1)} \end{aligned}$$

□

Definition 3.1.2.

Let R be a ring. An additive mapping $d : R \to R$ is called (θ, ϕ)–derivation where $\theta, \phi : R \to R$ are two mappings of R, if $d(xy) = d(x)\theta(y) + \phi(x)d(y)$, holds for all $x, y \in R$. And, we say that d is a Jordan $(\theta, \phi)-$derivation if $d(x^2) = d(x)\theta(x) + \phi(x)d(x)$, holds for all $x \in R$.

It is clear that every derivation is (θ, ϕ)–derivation, but the converse is not true as the following example, shows

Example 3.1.1.

Let

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid \quad a, b, c, d \in \mathbb{Z} \right\} \subseteq M_2(\mathbb{Z})$$

be a ring of 2×2 matrices with respect to usual addition and multiplication in matrices. Let the mapping $d : R \to R$, defined by

$$d\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, \quad \textit{for all} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R.$$

Also, let $\theta, \phi : R \to R$ are two mappings of R such that

$$\theta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}, \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \quad \textit{for all} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R.$$

Then, d is $(\theta, \phi)-$derivation of R.

Suppose that, $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in R$ *and* $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in R$,

where $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in \mathbb{Z}$.
Now, we prove that

$$d(A+B) = d(A) + d(B), \qquad for\ all\ A, B \in R.$$

Indeed,

$$\begin{aligned} d(A+B) &= d\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \\ &= d\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix} \\ &= d\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + d\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \\ &= d(A) + d(B). \end{aligned}$$

Thus, d is an additive mapping.

Moreover, let us show that d satisfies

$$d(AB) = d(A)\theta(B) + \phi(A)d(B), \qquad for\ all\ A, B \in R.$$

Since,

$$\begin{aligned} d(AB) &= d\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \\ &= d\left(\begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{3.1}$$

And,

$$\begin{aligned} d(A)\theta(B) + \phi(A)d(B) &= d\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) \theta\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) + \phi\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) d\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ c_2 & d_2 \end{bmatrix} + \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} b_1c_2 & b_1d_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_1a_2 & a_1b_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{3.2}$$

From 3.1 and 3.2 we get

$$d(AB) = d(A)\theta(B) + \phi(A)d(B), \qquad for\ all\ x, y \in R.$$

Therefore, d is $(\theta, \phi)-$derivation of R.
But, now we can see that d is not derivation.

Indeed,

$$\begin{aligned} d(A)B + Ad(B) &= d\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right)\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} d\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_1a_2 & a_1b_2 \\ c_1a_2 & c_1b_2 \end{bmatrix} \\ &= \begin{bmatrix} 2a_1a_2 + b_1c_2 & 2a_1b_2 + b_1d_2 \\ c_1a_2 & c_1b_2 \end{bmatrix}. \end{aligned} \tag{3.3}$$

Hence, by 3.1 and 3.3 we get

$$d(AB) \neq d(A)B + Ad(B) \quad for\ some\ A, B \in R.$$

Thus, d is not a derivation of R.

Definition 3.1.3.
Let R be a ring and suppose that $\theta, \phi : R \to R$ are two mappings of R. For some fixed $r \in R$, the mapping $I_r : R \to R$ given by

$$I_r(x) = [r, x]_{\theta,\phi},\ \textit{for all}\ x \in R$$

is said to be (θ,ϕ)–inner derivation.

Now, we can prove that every (θ,ϕ)–inner derivation is (θ,ϕ)–derivation by the following lemma

Lemma 3.1.3.
Every (θ,ϕ)–inner derivation is a (θ,ϕ)–derivation, where $\theta, \phi : R \to R$ are two homomorphism.

Proof. Let $I_r : R \to R$ be an (θ,ϕ)–inner derivation, for some fixed $r \in R$,

then
$$I_r(x) = [r, x]_{\theta,\phi}, \forall x \in R. \tag{3.4}$$

Firstly, we will show that

$$I_r(x + y) = I_r(x) + I_r(y), \quad \forall x, y \in R.$$

Since,

$$\begin{aligned} I_r(x + y) &= [r, x + y]_{\theta,\phi} \\ &= [r, x]_{\theta,\phi} + [r, y]_{\theta,\phi} \qquad \text{(By lemma 3.1.2)} \\ &= I_r(x) + I_r(y), \quad \forall x, y \in R. \end{aligned}$$

Thus, I_r is an additive mapping.

Now, replacing x by xy, in 3.4 where $y \in R$, we get

$$\begin{aligned} I_r(xy) &= [r, xy]_{\theta,\phi} \\ &= [r, x]_{\theta,\phi}\theta(y) + \phi(x)[r, y]_{\theta,\phi} \qquad \text{(By lemma 3.1.2)} \\ &= I_r(x)\theta(y) + \phi(x)I_r(y), \quad \forall x, y \in R. \end{aligned}$$

Therefore, I_r is a $(\theta, \phi)-$derivation. □

3.2 Some Results of $(\theta, \phi)-$derivation.

Throughout the present section R is a prime ring with characteristic not two. For $(\theta, \phi)-$derivation $d : R \to R$, we prove the following results (3.2.1). If d is a homomorphism on R then $d = 0$. (3.2.2). If d is a anti-homomorphism on R then $d = 0$. On the other hand we said that an additive map d :

$R \to R$ is a homomorphism or anti-homomorphism respectively $d(xy) = d(x)d(y)$ *or* $d(xy) = d(y)d(x),\ for\ all\ x, y \in R.$

Theorem 3.2.1.

Let R be a prime ring and θ, ϕ are automorphisms on R. If d is $(\theta, \phi)-$ derivation of R which is a homomorphism on R, then d=0.

Proof. Assume R is a prime ring with char $R \neq 2$. Since, d acts as a homomorphism on R, we have

$$d(xy) = d(x)d(y). \tag{3.5}$$

And, d acts as $(\theta, \phi)-$derivation of R, we have

$$d(xy) = d(x)\theta(y) + \phi(x)d(y). \tag{3.6}$$

Now, by 3.5 and 3.6, we get

$$d(x)\theta(y) + \phi(x)d(y) = d(x)d(y). \tag{3.7}$$

Substituting xr for x in 3.7 where $r \in R$, we get

$$d(xr)\theta(y) + \phi(xr)d(y) = d(xr)d(y).$$

Since d is an homomorphism on R and ϕ is an automorphism of R, becomes

$$d(x)d(r)\theta(y) + \phi(x)\phi(r)d(y) = d(x)d(r)d(y).$$

Expanding the last equation one obtains,

$$\begin{aligned} d(x)d(r)\theta(y) + \phi(x)\phi(r)d(y) &= d(x)d(ry) \\ &= d(x)(d(r)\theta(y) + \phi(r)d(y)) \\ &= d(x)d(r)\theta(y) + d(x)\phi(r)d(y) \end{aligned}$$

or equivalently,

$$\begin{aligned} 0 &= d(x)\phi(r)d(y) - \phi(x)\phi(r)d(y) \\ &= (d(x) - \phi(x))\phi(r)d(y). \end{aligned}$$

Indeed, ϕ is an automorphism of R, we get

$$(d(x) - \phi(x))Rd(y) = \{0\}, \quad for\ all\ x, y \in R.$$

Since, R is a prime ring, we conclude that

$$d(x) = \phi(x),\ for\ all\ x \in R \quad or \quad d = 0. \tag{3.8}$$

Suppose $d(x) = \phi(x)$ for all $x \in R$. Replacing x by xy, $y \in R$ in this equation, we have

$$d(xy) = \phi(xy) = \phi(x)\phi(y).$$

On the left hand side, recalling d is a $(\theta, \phi)-$derivation and 3.8, it follows

$$d(x)\theta(y) + \phi(x)d(y) = \phi(x)d(y)$$

then, we have

$$d(x)\theta(y) = 0 \quad for\ all\ x, y \in R.$$

Since, R is a prime ring and lemma 2.2.1, we see that $d = 0$ on R. □

Theorem 3.2.2.
Let R be a prime ring and θ, ϕ are automorphisms on R. If d is $(\theta, \phi)-$ derivation of R which is an anti-homomorphism on R, then d=0.

Proof. Suppose R is a prime ring with char$R \neq 2$. Since, d acts as an anti-homomorphism on R, we have

$$d(xy) = d(y)d(x). \tag{3.9}$$

And, d acts as a $(\theta, \phi)-$derivation of R, we have

$$d(xy) = d(x)\theta(y) + \phi(x)d(y). \tag{3.10}$$

Now, by 3.9 and 3.10, we get

$$d(x)\theta(y) + \phi(x)d(y) = d(y)d(x). \tag{3.11}$$

Replacing y by xy in 3.11, becomes

$$d(x)\theta(xy) + \phi(x)d(xy) = d(xy)d(x).$$

Recall that θ is automorphisms and d is $(\theta, \phi)-$derivation of R which is an anti-homomorphism on R, we have

$$d(x)\theta(x)\theta(y) + \phi(x)d(y)d(x) = d(x)\theta(y)d(x) + \phi(x)d(y)d(x)$$

Indeed, the second terms on the both sides are equal, we conclude that

$$d(x)\theta(x)\theta(y) - d(x)\theta(y)d(x) = 0, \quad for\ all\ x, y \in R. \tag{3.12}$$

Substitutrng yr for y in 3.12 where $r \in R$, we get

$$\begin{aligned} 0 &= d(x)\theta(x)\theta(yr) - d(x)\theta(yr)d(x) \\ &= d(x)\theta(x)\theta(y)\theta(r) - d(x)\theta(y)\theta(r)d(x). \end{aligned}$$

Using 3.12, it gives

$$\begin{aligned} 0 &= d(x)\theta(y)d(x)\theta(r) - d(x)\theta(y)\theta(r)d(x) \\ &= d(x)\theta(y)(d(x)\theta(r) - \theta(r)d(x)) \\ &= d(x)\theta(y)[d(x), \theta(r)] \qquad \text{(By definition 1.3.1)} \end{aligned}$$

Since θ, ϕ are automorphisms of R, we obtain

$$d(x)R[d(x), \theta(r)] = \{0\}, \qquad for\ all\ x, r \in R$$

Since R is a prime ring,

$$d(x) = 0 \quad or \quad [d(x), \theta(r)] = 0$$

Hence,

$$d(x) = 0 \quad or \quad d(x) \in Z, \quad for\ all\ x \in R.$$

If $d(x) = 0$ then $d(x) \in Z$. So, we can take $d(R) \subseteq Z$ which forces $d(x)d(y) = d(y)d(x)\ \forall x, y \in R$. And, so d to be an homomorphism of R. It follows $d = 0$ from Theorem 3.2.1. This completes the proof of the Theorem 3.2.2.

□

3.3 Commutatively of Prime Ring

In this section R is a prime ring with char$R \neq 2$ and U is a non-zero left ideal of R. For $(\theta, \phi)-$derivation $d : R \to R$, we shall prove the following Theorem (3.3.1). If $d(xy) = d(yx)$, for all $x, y \in R$ then R is commutative.

Before proceeding the proof of the main theorem we first state a few known results which will be used in subsequent discussion.

Lemma 3.3.1.

Let R be a ring without any non-zero nilpotent ideal. Then any element of R which commutes with all elements of [R,R] must lie in the center of R.

Proof. Let a be a fixed element of R, which is commute with all elements of $[R, R]$. Assume $x, y \in R$, then a commutes with $[x, y] = xy - yx$ and a commutes with $[x, xy] = x(xy) - (xy)x = x(xy - yx)$. Hence, we get

$$\begin{aligned} ax(xy - yx) &= x(xy - yx)a \\ &= xa(xy - yx) \qquad \text{(Since } a \text{ commutes with } [x, y]) \end{aligned}$$

or equivalently,

$$(xa - ax)(xy - yx) = 0, \quad \forall x, y \in R. \tag{3.13}$$

Replacing y by ya in $(xy - yx)$, we get

$$\begin{aligned} x(ya) - (ya)x &= x(ya) - yxa + yxa - (ya)x \\ &= (xy - yx)a + y(xa - ax). \end{aligned} \tag{3.14}$$

From 3.13, if we replace y by ya, and using 3.14, we have

$$\begin{aligned} 0 &= (xa - ax)(x(ya) - (ya)x) \\ &= (xa - ax)((xy - yx)a + y(xa - ax)) \\ &= (xa - ax)(xy - yx)a + (xa - ax)y(xa - ax) \\ &= (xa - ax)y(xa - ax) \end{aligned}$$

This results in

$$(xa - ax)R(xa - ax) = \{0\}.$$

But then $(xa - ax)R$ is a nilpotent right ideal so is $\{0\}$. Since R has no nilpotent ideals we get from this $xa - ax = 0$, that is, a must be in the center of R. □

Lemma 3.3.2.
Let R be a prime ring, and suppose that $a \in R$ centralizes a non-zero right ideal of R. Then $a \in Z$

Proof. Suppose that, a centralizes the non-zero right ideal J of R. If $x \in R$, $r \in J$ then $rx \in J$ hence, $a(rx) = (rx)a$. But $ar = ra$, we thus get that $r(ax - ax) = 0$, which is to say, $J(ax - xa) = \{0\}$, for all $x \in R$. Since R is prime and $J \neq \{0\}$, we conclude that $ax = xa$, for all $x \in R$, hence $a \in Z$. □

Lemma 3.3.3. *[3]*
Let d be a non-zero $(\theta,\phi)-$derivation, where θ,ϕ are two homomorphisms on R, U an ideal of a ring R and $a \in R$. if $[d(U),a]_{\theta,\phi} = \{0\}$, then $a \in Z$.

Now, we will introduce and prove the main theorem in this section

Theorem 3.3.1.
Let R be a prime ring of char$R \neq 2$. If d is a non-zero $(\theta,\phi)-$derivation of R and d(xy)= d(yx), for all x,y $\in R$, then R is a commutative ring.

Proof. Let R be a prime ring with char$R \neq 2$, and d is acts as $d(xy) = d(yx)$, $\forall x,y \in R$. Suppose $c \in R$ such that $d(c) = 0$, for example $c = [x,y], \forall z \in R$ we have

$$\begin{aligned} d(zc) &= d(z)\theta(c) + \phi(z)d(c) = d(z)\theta(c), \quad \text{(By definition 3.1.2)} \\ d(cz) &= d(c)\theta(z) + \phi(c)d(z) = \phi(c)d(z). \quad \text{(By definition 3.1.2)} \end{aligned} \tag{3.15}$$

By hypothesis, $d(cz) = d(zc), \forall c,z \in R$, and using 3.15, we get

$$d(z)\theta(c) = \phi(c)d(z),$$

that is,

$$d(z)\theta(c) - \phi(c)d(z) = 0.$$

Thus, we have

$$[d(z), c]_{\theta,\phi} = 0, \quad \textit{for all } z \in R. \quad \text{(By definition 3.1.1)} \tag{3.16}$$

This reduces $c \in Z$ for all $c \in R$ such that $d(c) = 0$ by Lemma 3.3.3. In view of 3.16, we obtain $[x, y] \in Z$ for all $x, y \in$ R because of $d([x, y]) = 0$. Therefore, R is commutative by Lemma 3.3.1.

□

Chapter 4

Reverse Derivation of Prime Ring

Bresar and Vukman [7] have introduced the notion of a reverse derivation in rings. Samman and Alyamani [19] have investigated some properties of reverse derivations in prime or semiprime rings. In this chapter we present some properties of reverse derivations in prime rings. Throught this chapter R will denote an associative ring with unity 1 and Z its center.

4.1 Definitions and Examples

In this siction, we will present some definitions and examples that will be used throughout this chapter.

Definition 4.1.1.
Let R be a ring, an additive mapping $d : R \to R$ is a reverse derivation if $d(xy) = d(y)x + yd(x)$, for all x, $y \in R$.

Remark 4.1.1.
If R is commutative then both derivation and reverse derivation are same.

We provide some examples to show that it is not the case in general.

Example 4.1.1.

We consider $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid \quad a, b \in \mathbb{Z} \right\} \subseteq M_2(\mathbb{Z})$ *is a ring of* 2×2 *matrices with respect to usual addition and multiplication in matrices. We define the mapping* $d : R \to R$ *by*

$$d\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \quad \text{for every } \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in R.$$

Let A, B *be any elements of* R*, where* $A = \begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}$*, where* $a_1, b_1, a_2, b_2 \in \mathbb{Z}$*.*

Firstly, we will prove that

$$d(A + B) = d(A) + d(B), \quad \forall A, B \in R.$$

Indeed,

$$\begin{aligned} d(A + B) &= d\left(\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}\right) \\ &= d\left(\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & a_1 + a_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix} \\ &= d\left(\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}\right) + d\left(\begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}\right) \\ &= d(A) + d(B). \end{aligned}$$

Thus, d is an additive mapping.

Now, we will prove that

$$d(AB) = d(A)B + Ad(B), \quad \forall A, B \in R.$$

Since,

$$\begin{aligned} d(AB) &= d\left(\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}\right) \\ &= d\left(\begin{bmatrix} a_1a_2 & a_1b_2 \\ 0 & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & a_1a_2 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{4.1}$$

and,

$$\begin{aligned} d(A)B + Ad(B) &= d\left(\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} d\left(\begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & a_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_1a_2 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{4.2}$$

Thus, from 4.1 and 4.2 we conclude that

$$d(AB) = d(A)B + Ad(B), \quad \forall A, B \in R.$$

Hence, d is a derivation of R.

Moreover, let us show that d satisfies

$$d(AB) = d(B)A + Bd(A), \quad \forall A, B \in R.$$

Indeed,

$$\begin{aligned} d(B)A + Bd(A) &= d\left(\begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix} d\left(\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & a_1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_2a_1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_1a_2 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{4.3}$$

Finally, from 4.1 and 4.3 we have

$$d(AB) = d(B)A + Bd(A), \quad \forall A, B \in R.$$

Therefore, d is also a reverse derivation of R.

Example 4.1.2.

We consider the ring as in the above example. If we define $d : R \to R$ by

$$d\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}, \quad \textit{for all} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in R.$$

Then, clearly d is an additive mapping i.e.,

$$d(A + B) = d(A) + d(B), \quad \forall A, B \in R.$$

Since,

$$\begin{aligned} d(A + B) &= d\left(\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}\right) \\ &= d\left(\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & 0 \end{bmatrix}\right) \end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} 0 & b_1 + b_2 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b_2 \\ 0 & 0 \end{bmatrix} \\
&= d\left(\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}\right) + d\left(\begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}\right) = d(A) + d(B).
\end{aligned}
$$

Now, we need to prove that

$$d(AB) = d(A)B + Ad(B), \quad \forall A, B \in R.$$

Indeed,

$$
\begin{aligned}
d(AB) &= d\left(\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}\right) \\
&= d\left(\begin{bmatrix} a_1a_2 & a_1b_2 \\ 0 & 0 \end{bmatrix}\right) \\
&= \begin{bmatrix} 0 & a_1b_2 \\ 0 & 0 \end{bmatrix}
\end{aligned} \tag{4.4}
$$

and,

$$
\begin{aligned}
d(A)B + Ad(B) &= d\left(\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} d\left(\begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}\right) \\
&= \begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & b_2 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & a_1b_2 \\ 0 & 0 \end{bmatrix}.
\end{aligned} \tag{4.5}
$$

From 4.4 and 4.5 we get

$$d(AB) = d(A)B + Ad(B), \quad \forall A, B \in R.$$

Hence, d is a derivation of R.
But, let us show that d is not reverse derivation of R, i.e.,

$$d(AB) \neq d(B)A + Bd(A), \qquad for\ some\ A, B \in R.$$

Indeed,

$$\begin{aligned} d(B)A + Bd(A) &= d\left(\begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix} d\left(\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & b_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & b_1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_2 b_1 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{4.6}$$

Finally, from 4.4 and 4.6 we have

$$d(AB) \neq d(B)A + Bd(A), \qquad for\ some\ A, B \in R.$$

Therefore, d is not a reverse derivation of R.

4.2 Properties of Reverse Derivations

Throughout this section R will denote a ring with unity 1 and Z its center. We know that an additive mapping $d : R \to R$ is a reverse derivation if $d(xy) = d(y)x + yd(x)$ for all $x, y \in R$. And we said that d is a homomorphism or anti-homomorphism respectively if $d(xy) = d(x)d(y)$ *or* $d(xy) = d(y)d(x)$, for all $x, y \in R$.

In this section, we prove the following results (4.2.1) if a reverse derivation d acts as homomorphism or an anti-homomorphism on a non-zero right ideal U of a prime ring R, then $d = 0$.

Now we consider the following results

Theorem 4.2.1.
Let R be a prime ring and U a non-zero right ideal of R. Suppose $d : R \to R$ is a reverse derivation of R

1. *If d acts as a homomorphism on U, then $d = 0$ on R.*

2. *If d acts as an anti-homomorphism on U, then $d = 0$ on R.*

Proof. (1) If d acts as a homomorphism on U, then we have

$$d(y)d(x) = d(yx) = d(x)y + xd(y), \quad for\ all\ x, y \in U. \tag{4.7}$$

We replace $y = yx$ in equation 4.7, then

$$d(yx)d(x) = d(x)yx + xd(yx), \quad for\ all\ x, y \in U. \tag{4.8}$$

By multiplying 4.7 with $d(x)$ on right side and using d is a homomorphism on U, we get

$$d(yx)d(x) = d(x)yd(x) + xd(y)d(x)$$

$$d(yx)d(x) = d(x)yd(x) + xd(yx). \tag{4.9}$$

By combining equations 4.8 and 4.9, we get

$$d(x)yx = d(x)yd(x), \quad for\ all\ x, y \in U \tag{4.10}$$

i.e., $x = d(x)$. So, $(d(x)–x)d(x) = 0$. Thus

$$d(x^2) = xd(x)$$

Since d is a reverse derivation, we have $d(x)x = 0$, $for\ all\ x \in U$.

By linearizing x, we obtain

$$\begin{aligned} 0 &= d(x+y)(x+y) \\ &= d(x)x + d(y)x + d(x)y + d(y)y \\ &= d(x)y + d(y)x, \quad for\ all\ x, y \in U. \end{aligned} \tag{4.11}$$

We replace y by xy in equation 4.11, we have

$$\begin{aligned} 0 &= d(x)xy + d(xy)x \\ &= d(x)xy + d(y)xx + yd(x)x \quad \text{(By definition 4.1.1)} \\ &= d(y)xx, \quad for\ all\ x, y \in U. \end{aligned} \tag{4.12}$$

If we right multiply by x in equation 4.11, we get

$$d(x)yx + d(y)xx = 0, \quad for\ all\ x, y \in U.$$

From the above equations, we obtain

$$d(x)yx = 0, \quad for\ all\ x, y \in U.$$

By substituting y by ys in this equation, we get $d(x)ysx = 0$, for all $x, y \in U$ and $s \in R$. Thus for each $x \in U$, the primeness of R implies that either $d(x)y = 0$ or $x = 0$. But $x \neq 0$, implies that

$$d(x)y = 0, \quad for\ all\ x, y \in U. \tag{4.13}$$

If we replace x by xr in equation 4.13, we get

$$d(xr)y = 0, \quad for\ all\ x, y \in U\ and\ r \in R.$$

Then, $d(r)xy + rd(x)y = 0$. So by 4.13, we get

$$d(r)xy = 0, \quad for\ all\ x, y \in U\ and\ r \in R. \tag{4.14}$$

Again we replace x by xs in equation 4.14. We have

$$d(r)xsy = 0, \quad for\ all\ x, y \in U\ and\ r, s \in R$$
$$i.e.,\ d(r)xRy = \{0\}, \quad for\ all\ x, y \in U\ and\ r \in R.$$

Since R is prime, it follows that

$$d(r)x = 0, \quad for\ all\ x, y \in U\ and\ r \in R. \tag{4.15}$$

In equation 4.15, we substitute r by rs, becomes

$$\begin{aligned} 0 &= d(rs)x \\ &= (s)rx + sd(r)x \\ &= d(s)rx, \quad \forall x \in U\ and\ r, s \in R \end{aligned} \tag{4.16}$$

$$i.e., \quad d(s)Rx = 0, \quad \forall x \in U\ and\ s \in R.$$

Since R is prime, either $d(s) = 0$ or $x = 0$. But $x \neq 0$, then $d(s) = 0$, for all $s \in R$, then $d = 0$ on R.

(2) Suppose d acts as an anti-homomorphism on U. By our hypothesis, we have

$$d(y)d(x) = d(xy) = d(y)x + yd(x), \quad \forall x, y \in U. \tag{4.17}$$

By substituting y by xy in $d(y)d(x)$,then

$$\begin{aligned} d(xy)d(x) &= d(x(xy)) \\ &= d((xx)y), \quad \forall x, y \in U. \end{aligned}$$

Thus

$$d(xy)d(x) = d(y)xx + yd(xx), \quad \forall x, y \in U. \tag{4.18}$$

On the other hand, by using definition 4.1.1

$$d(xy)d(x) = d(y)xd(x) + yd(x)d(x), \quad \forall x, y \in U. \tag{4.19}$$

By combining equations 4.18 and 4.19. Then

$$\begin{aligned} d(y)xd(x) &= d(y)xx \quad \forall x, y \in U \qquad (4.20) \\ i.e.,\ d(x) &= x, \quad \forall x \in U. \\ So\ (d(x) – x) &= 0, \quad \forall x \in U. \end{aligned}$$

We right multiply last equation with $d(x)$. Then

$$\begin{aligned} (d(x) – x)d(x) &= 0, \quad \forall x \in U. \\ \text{Thus} \quad d(x^2) &= xd(x), \quad \forall x \in U. \end{aligned}$$

Since d is a reverse derivation, we have $d(x)x = 0$, $\forall x \in U$.
By linearazing x, we obtain

$$\begin{aligned} 0 &= d(x + y)(x + y) \\ &= d(x)x + d(y)x + d(x)y + d(y)y \\ &= d(x)y + d(y)x, \quad \forall\ x, y \in U. \end{aligned} \tag{4.21}$$

We replace y by xy in equation 4.21, we have

$$\begin{aligned} 0 &= d(x)xy + d(xy)x \\ &= d(x)xy + d(y)xx + yd(x)x \quad \text{(By definition 4.1.1)} \\ &= d(y)xx, \quad \forall\ x, y \in U. \end{aligned}$$

Hence, we have obtained equation 4.12. The remaining proof is same as in proof of (1). □

4.3 Commutatively of Prime Ring

In this section R is a prime ring with char$R \neq 2$ and U is a non-zero right ideal of R. We prove the following results (4.3.2). Let d be a non-zero reverse derivation of R. If $[d(x), x] = 0$ for all $x \in U$, then R is commutative. (4.3.3). Let d be a non-zero reverse derivation of R. If $[d(x), d(y)] = 0$ for all $x, y \in U$, then R is commutative.

The following Theorem which is necessary for developing the proof of Theorem 4.3.2

Theorem 4.3.1. *[20]*
Let I be a non-zero right ideal of a prime ring R. If I is commutative, then R is commutative.

Theorem 4.3.2.
Let R be a prime ring with char$R \neq 2$, U a non-zero right ideal of R and d be a non-zero reverse derivation of R. If $[d(x), x] = 0$ for all $x \in U$, then R is commutative.

Proof. Let R be a prime ring with char$R \neq 2$, U a non-zero right ideal of R and d be a non-zero reverse derivation of R. We have

$$[d(x), x] = 0, \quad \forall x \in U. \tag{4.22}$$

By linearizing x in equation 4.22, we obtain

$$\begin{aligned} 0 &= [d(x+y), x+y] \\ &= [d(x), x] + [d(x), y] + [d(y), x] + [d(y), y] \\ &= [d(x), y] + [d(y), x], \quad \forall x, y \in U. \end{aligned}$$

Thus,

$$[d(x), y] - [x, d(y)] = 0, \quad \forall x, y \in U. \tag{4.23}$$

By substituting y with yx in equation 4.23, we get

$$\begin{aligned} 0 &= [d(x), yx] - [x, d(yx)] \\ &= [d(x), y]x + y[d(x), x] - [x, d(x)y] - [x, xd(y)] \\ &= [d(x), y]x - [x, d(x)]y - d(x)[x, y] - [x, x]d(y) - x[x, d(y)] \quad \text{(By lemma 1.3.1)} \end{aligned}$$

since, $[d(x), y] = [x, d(y)]$ then we get,

$$d(x)[x, y] = 0, \quad \forall x, y \in U. \tag{4.24}$$

We replace y by yz in equation 4.24, we have

$$\begin{aligned} 0 &= d(x)[x, yz] \\ &= d(x)[x, y]z + d(x)y[x, z] \\ &= d(x)y[x, z], \quad \forall x, y, z \in U. \end{aligned}$$

Again by substituting y by yr in last equation, we have

$$d(x)yr[x, z] = 0, \quad \forall\ x, y, z \in U and\ r \in R$$

or equivalently,

$$d(x)yR[x,z] = \{0\}, \quad \forall\, x, y, z \in U.$$

Since R is prime, either $d(x)y = 0$ or $[x, z] = 0$. If $d(x)y = 0$, then $d(U)U = \{0\}$. But $d(U)U \neq \{0\}$, since $d \neq 0$, $U \neq \{0\}$ and R is prime. Thus $[x, z] = 0$, for all $x, z \in U$. So U is commutative.
Hence by Theorem 4.3.1, R is commutative. □

Theorem 4.3.3.
Let R be a prime ring with char$R \neq 2$, U be a non-zero right ideal of R and d be a non-zero reverse derivation of R. If $[d(x), d(y)] = 0$, for all $x, y \in U$, then R is commutative.

Proof. Let R be a prime ring with char$R \neq 2$, U a non-zero right ideal of R and d be a non-zero reverse derivation of R. We have

$$[d(x), d(y)] = 0, \quad \forall x, y \in U. \tag{4.25}$$

By taking y by yx in equation 4.25, we have

$$\begin{aligned} 0 &= [d(x), d(yx)] \\ &= [d(x), d(x)y + xd(y)] \\ &= d(x)[d(x), y] + [d(x), d(x)]y + x[d(x), d(y)] + [d(x), x]d(y), \quad \forall x, y \in U. \end{aligned}$$

We get

$$d(x)[d(x), y] + [d(x), x]d(y) = 0, \quad \forall x, y \in U. \tag{4.26}$$

By substituting $d(y)$ with $d(z)y$ in equation 4.26, we have

$$d(x)[d(x), y] + [d(x), x]d(z)y = 0, \quad \forall x, y, z \in U. \tag{4.27}$$

Again we take y by yr, $r \in R$ in equation 4.27. Then we have

$$\begin{aligned} 0 &= d(x)[d(x), yr] + [d(x), x]d(z)yr \\ &= d(x)y[d(x), r] + d(x)[d(x), y]r + [d(x), x]d(z)yr, \quad \forall x, y, z \in U and\ r \in R. \end{aligned} \tag{4.28}$$

From equations 4.27 and 4.28, we get

$$\begin{aligned} d(x)y[d(x), r] &= 0, \quad \forall x, y, z \in U \ and\ r \in R. \\ d(x)U[d(x), r] &= \{0\}. \end{aligned}$$

Again we replace y by ys, $s \in R$ in last equation, we get

$$d(x)UR[d(x), r] = \{0\}.$$

Since R is prime we have either $d(x)U = \{0\}$ or $[d(x), r] = 0$. Since $d \neq 0$, $U \neq \{0\}$ and R is prime it follows that $d(x)U \neq \{0\}$. So $[d(x), r] = 0$. Then $d(x) \in Z$, center of R. Hence $[d(x), x] = 0$, for all $x \in U$. From Theorem 4.3.2, R is commutative. □

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