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Research Project Report (MAT699)

Topological Degrees and Applications

By

Nawal Jaber Ali Alhelali

Supervisor:

Dr Ismail DJEBALI

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Contents

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Introduction

The concept of degree dates back to the end of the eighteenth century. Kronecker[1] introduced the degree for C^1 functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ in 1869. Poincaré[2], Böhler [3], and Hadamard[4] developed the degree for continuous in the beginning of the twentieth century. L.E. Brouwer[5] extended the degree for continuous maps between manifolds of same dimension and presented some applications.

The case of infinite dimension was considered by Leray and Schauder in 1934 [6]. They constructed the degree starting from Brouwer's degree for the class of compact perturbations of identity and their approximation by maps with finite-dimensional range.

In this project, two topological degrees are investigated. First, Brouwer's topological degree for continuous functions defined on open bounded subsets of Euclidean space is presented in Chapter 2. The construction considers the regular and singular cases, separately. The main properties of the degree are proved in detail. The classical Brouwer fixed point theorem as well as some equivalent forms are derived from the general theory of this degree. The

[1] Krocker, L. (1869) Uber systeme von funktionen mehrer variabel n, Monatsberichte. Acad. Wiss. Berlin, pp. 159-193, 688-698

[2] Poincaré, H. (1892,1899) Méthodes nouvelles de la mécanique céleste (3 volumes). Gauthiers-Villars, Paris.

[3] Böhl, P. (1904) Uber die bewegung eines machaniscsches systems in der nähe einer Gleichgewichtslage. J. Reine Angew. Math. 127, 176, 179

[4] Hadamard, J. (1910) Sur quelques applications de l'indice de Klonecker; dans "introduction à la théorie des fonctions d'une variable", par J. Tannery, Vol. II, Hermann, Paris, pp. 875-915

[5] Brouwer, L.E.J. (1912) Uber abbildung von Mannigfaltigkeitein. Math. Ann 71; pp. 97-115.

[6] Leray, Jean; Schauder, Jules. Topologie et équations fonctionnelles. (French) Ann. Sci. École Norm. Sup. (3) 51 (1934), 45–78

retraction theory is also introduced and the non-retraction of the unit ball is proved in the finite dimension case.

From Brouwer's degree, Schauder's topological degree in infinite dimension is introduced in Chapter 3 by an approximation method for the class of compact perturbations of the identity mapping. The construction relies on an approximation result of compact mappings due to Schauder. Then the main properties follow from those proved in the finite-dimensional case. The Schauder fixed point theorem and some variants including nonlinear alternatives and boundary condition results are then derived.

Chapter 4 is devoted to some applications to the solvability of some initial and boundary value problems associated with differential equations. We show how Leray-Schauder topological degree helps in providing some existence theorems.

In Chapter 1, we have collected several auxiliary results from Topology, Functional Analysis, and Vector Calculus that we have used throughout the project.

Chapter 1

Preliminaries

In this chapter we will present some of the definitions, theorems and symbols that will be used throughout this research. For more details, we refer to [5, 7].

1.1 Topology

In this section, we present some of the important concepts associated to topology such that the compactness and connectedness which will be needed in this research project.

1.1.1 Topological Space

First, we introduce some basic definitions and theorems related to topology.

Definition 1.1.1. *A topology on a set X is a collection τ of subsets of X having the following properties:*
(1) $\emptyset$ and X are in τ.
(2) The union of the elements of any sub-collection of τ is in τ.
(3) The intersection of any finite sub-collection of τ is in τ.
A set for which topology τ has been specified is called a topological space and is denoted by (X, τ).

Examples 1.1.1. If X is any set, the collection of all the subsets of X is a topology on X called *the discrete topology*. The collection that consist of X and $\emptyset$ only is also topology and called the *indiscrete topology*.

Definition 1.1.2. *$U \subset X$ is an open subset of X if $U \in \tau$.*

Definition 1.1.3. *$C \subset X$ is a closed subset of X if $X \backslash C$ is an open.*

In any topological space, we can see that $\emptyset$ and X both are open and closed sets. We give an equivalent characterization of open sets.

Proposition 1.1.1. *Let (X, τ) be a topological space. A nonempty subset $A \subset X$ is open if and only if for all $x \in X$, there exists $U \in \tau$ such that $x \in U \subset A$.*

Proof. Assume that A is an open set then for all $x \in A$, $x \in A \subset A$. Conversely, we have $A = \bigcup_{x \in A} U_x$ which is open since τ is topology. □

Theorem 1.1.1. *Let (X, τ) be a topological space. Then,*
(1) The intersection of closed subsets of X is a closed set.
(2) The finite union of closed subsets of X is a closed set.

Proof. (1) Let $(U_\alpha)_\alpha$ be a collection of closed subsets in X. In order to show that $\bigcap_\alpha U_\alpha$ is closed, we show that its complement is open. Indeed, note that $X \backslash (\bigcap_\alpha U_\alpha) = \bigcup_\alpha (X \backslash U_\alpha)$. Since U_α is closed, $X \backslash U_\alpha$ is open and so $\bigcup_\alpha (X \backslash U_\alpha)$ is open. It follows that $\bigcap_\alpha U_\alpha$ is closed.
(2) Let $\bigcup_{i=1}^n U_i$ be a finite collection of closed subsets in X. We follow the same method mentioned in the proof of part (1), i.e, we show $X \backslash \bigcup_{i=1}^n U_i$ is open. Indeed, since $X \backslash \bigcup_{i=1}^n U_i = \bigcap_{i=1}^n X \backslash U_i$ which is open as $X \backslash U_i$ is open. Hence, $\bigcup_{i=1}^n U_i$ is closed. □

Definition 1.1.4. *Let V is a subset of X containing x. We say that V is a neighborhood of X, if there exists an open set U such that $x \in U \subset V$. The set of all neighborhoods of x will be denoted by $\mathcal{N}_x$.*

Definition 1.1.5. *Let A subset of X. The closure of A is the intersection of all closed sets containing A and will be denoted by $\overline{A}$. An equivalent definition to the closure can be given as, $x \in \overline{A}$ if and only if for every $U \in \mathcal{N}_x$ containing x, we have $U \cap A \neq \emptyset$.*

Examples 1.1.2. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, a, b, \{a, b\}\}$. Set $A = \{b\}$. Then, the closed sets are $X, \emptyset, \{b, c\}, \{a, c\}$ *and* $\{c\}$. Hence, $\overline{A} = \{b, c\}$.

Definition 1.1.6. *Let f be a real-valued function defined on topological space X. The support of f, denoted* supp f, *is defined by*

$$\operatorname{supp} f = \overline{\{x \in X : \ f(x) \neq 0\}}.$$

Definition 1.1.7. *Let $A \subset X$. A point $x \in A$ is called interior point of A if there is an open subset U such that $x \in U \subset A$. The set of all interior points of A is denoted $int(A)$.*

Definition 1.1.8. *Let A subset of a topological space X and $x \in X$. We say x is a limit point of A if for every $U \in \mathcal{N}_x$, we have $U \cap A \setminus \{x\} \neq \emptyset$.The set of all limit points of A is denoted A'.*

Definition 1.1.9. *Let A be a subset of topological space X and $x \in X$. We say x is an isolated point of A if there exists $U \in \mathcal{N}_x$, such that $U \cap A = \{x\}$.*

Examples 1.1.3. Let $X = \mathbb{R}$ and $A = (0, 1] \cup \{2\}$. Then, $A' = [0, 1]$ as every neighborhood of 0 intersects A in other point than 0 itself so, $0 \in A'$.The same holds for every point in $[0, 1]$. For the element 2, there is a neighborhood which intersects A only in 2, i.e, $2 \notin A'$, but this shows that 2 is an isolated point. The set of isolated points is $\{2\}$.

Theorem 1.1.2. *Let A subset of topological space X. A closed if and only if $A = \overline{A}$.*

Proof. If $A = \overline{A}$, then clearly A is a closed set. Conversely, assume that A is closed. From the definition of $\overline{A}$, we already know that $A \subset \overline{A}$. Conversely, if $x \in \overline{A}$, then every closed set containing A contains A in particular. Hence, $x \in A$ and $A = \overline{A}$. □

There is another way to describe the closure of a set, that involves the limit point which will be considered now.

Theorem 1.1.3. *Let A subset of topological space X. Then $\overline{A} = A \cup A'$.*

Proof. By definition, A and A' are subsets of $\overline{A}$. Hence $A \cup A' \subset \overline{A}$. Conversely, let $x \in \overline{A}$ and $x \notin A$. Then, every neighborhood of x intersects A in at least one point different from x, proving that $x \in A'$. □

Definition 1.1.10. *Let (X, τ) be a topological space. If A is a subset of X, the collection $\tau_A = \{A \cap U : U \in \tau\}$ is a topology on A called the subspace topology, and A is called a topological subspace of X or just a subspace.*

Next we discuss the continuity of function defined in topological spaces.

Definition 1.1.11. *If X and Y be topological spaces. A function $f : X \longrightarrow Y$ is said to be continuous if any open subset V of Y, $f^{-1}(U)$ is an open subset of X.*

Examples 1.1.4. Let $X = \{1, 2\}$ with topology $\tau_X = \{\emptyset, \{1\}, X\}$ and $Y = \{1, 2, 3\}$ with topology $\tau_Y = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, Y\}$. Consider $f : X \longrightarrow Y$ where $f(1) = 2$ and $f(2) = 1$, then $f^{-1}(\{2\}) = \{1\}$ which is open in X and $f^{-1}(Y) = X$ also open in X thus, f is continuous.

Theorem 1.1.4. *Let X and Y be topological spaces. A function $f : X \longrightarrow Y$ is continuous if and only if for every closed subset $C \subset Y$, $f^{-1}(C)$ is closed in X.*

Proof. Suppose that C is closed set in Y, i.e, $Y \backslash C = U$ is open, since f is continuous we have $f^{-1}(U)$ open. Note that $f^{-1}(C) = f^{-1}(Y \backslash U) = X \backslash f^{-1}(U)$, which implies that $X \backslash f^{-1}(U) = f^{-1}(C)$ is closed. □

Definition 1.1.12. *Let X and Y are topological spaces. A function $f : X \longrightarrow Y$ is said to be sequentially continuous at a point x_0 if for all $(x_n)_n \subset X$ such that $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$. f is sequentially continuous if f is sequentially continuous at each $x \in X$.*

Theorem 1.1.5. *If f is a continuous function, then f is sequentially continuous. The converse is true if X is a metric space.*

Lemma 1.1.1 (Pasting Lemma). Let $X = A \cup B$ where A and B are closed. Let $f : A \longrightarrow Y$ and $g : B \longrightarrow Y$ are continuous functions and $f(x) = g(x)$ for all $x \in A \cap B$. If

$$h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B, \end{cases}$$

Then h is continuous.

Definition 1.1.13. *(1) Let X and Y be topological spaces. A function $f : X \longrightarrow Y$ is said to be homeomorphism if f is bijective map and f and f^{-1} are continuous.*
(2) If there exists a homeomorphism function between two topological spaces, then we say that the spaces are homeomorphic.

Definition 1.1.14. *A topological space X is called a Hausdorff space if for each $x, y \in X$ with $x \neq y$, there exists an open set U containing x and an open set V containing y such that $U \cap V = \emptyset$.*

Definition 1.1.15. *Let X be a set. A metric on X is a function $d : X \times X \longrightarrow \mathbb{R}$ satisfying the following properties:*

1. $d(x,y) \geq 0$, $\forall\, x,y \in X$ *and*, $d(x,y) = 0 \Leftrightarrow x = y$.

2. $d(x,y) = d(y,x)$.

3. $d(x,y) \leq d(x,y) + d(y,z)$, $\forall\, x,y,z \in X$. *(Triangular Inequality)*

If d is a metric on a set X, then the pair (X,d) is called a *metric space* and the number $d(x,y)$ is called *the distance between x and y*. Given $\varepsilon > 0$, the set $B(x,\varepsilon) = \{y \in X :\ d(x,y) < \varepsilon\}$ is called *the open ball centered at x with radius ε*.

Definition 1.1.16. *Let (X,d) be a metric space. A set $A \subset X$ is said to be bounded if there is $r > 0$ and $x \in X$ such that $A \subset B(x,r)$.*

Theorem 1.1.6. *Let (X,d) be a metric space and $A \subset X$. Then $x \in \overline{A}$ if and only if there is a sequence x_n in A such that x_n converges to x.*

Corollary 1.1.1. *Let (X,d) be a metric space and $A \subset X$. A is closed if and only if for any sequence $x_n \in A$ such that x_n converges to x, we have $x \in A$.*

Proof. Suppose A is closed and $x_n \in A$ with $x_n \longrightarrow x$, as $n \to \infty$. By the previous theorem, we have $x \in \overline{A} = A$. Conversely, let $x \in \overline{A}$. By the previous theorem there is x_n such that $x_n \longrightarrow x$, as $n \to \infty$. Hence by assumption $x \in A$, i.e, $\overline{A} \subset A$. Therefore $\overline{A} = A$, and so A is closed. □

Definition 1.1.17. *Let X be a metric space with metric d. Let A subset of X. We define the distance of a point $x \in X$ and the set A by*

$$d(x,A) = \inf\{d(x,y)\ :\ y \in A\}.$$

Definition 1.1.18. *Let X be a metric space with metric d. We say that a subset A of X is bounded if there is some number M such that $d(x,y) \leq M$, for all $x,y \in A$. If A is bounded and nonempty, then we define the diameter of A by*

$$diam A = \sup\{d(x,y) :\ x,y \in A\}.$$

Theorem 1.1.7. *Let (X,d) be a metric space and $A \subset X$ be nonempty subset. $x \in \overline{A}$ if and only if $d(x,A) = 0$.*

Proof.

$$\begin{aligned} x \in \overline{A} &\Leftrightarrow \forall\, r > 0 \; B(x,r) \cap A \neq \emptyset \\ &\Leftrightarrow \forall\, r > 0 \text{ there exists } y \in A \text{ such that } d(x,y) < r \\ &\Leftrightarrow \inf_{y \in A} d(x,y) = 0 \\ &\Leftrightarrow d(x,A) = 0. \end{aligned}$$

□

Corollary 1.1.2. *Let A be a closed subset in a metric space X, and $x \notin A$. Then, $d(x,A) > 0$.*

Proof. A is closed if and only if $A = \overline{A}$. Since $x \notin A$, then by the previous theorem we have

$$d(x,A) \neq 0 \Leftrightarrow d(x,A) > 0$$

. □

Theorem 1.1.8. *Every metric space is a Hausdorff space.*

Proof. Suppose $x, y \in X$ where X is a metric space and $x \neq y$, we take $r_1, r_2 > 0$ such that, $0 < r_1 + r_2 < d(x,y)$ then $B(x,r_1) \cap B(y,r_2) = \emptyset$. Otherwise, there exists some $z \in B(x,r_1) \cap B(y,r_2)$. Then

$$0 \leq d(x,y) \leq d(x,z) + d(z,y) < r_1 + r_2 < d(x,y),$$

which is a contradiction. Hence the space X is Hausdorff. □

Examples 1.1.5. Since $\mathbb{R}^n$ is a metric space, $\mathbb{R}^n$ is Hausdorff space.

Theorem 1.1.9. *If X is a Hausdorff space, then the one-element set $\{x\}$ is closed.*

Proof. Let $Y = X \setminus \{x\}$ and $y \in Y$ be such that $x \neq y$. Since X is Hausdorff, there exist open sets $U_y \ni x$ and $V_y \ni y$ open such that $U_y \cap V_y = \emptyset$. The set $V = \bigcup_{y \in Y} V_y$ is open. Indeed, if $y \in V$ then there exists y_0 such that $y \in V_{y_0}$, and so $y \neq x$ i.e, $y \in V \subset Y$. Hence, Y is open i.e, $\{x\}$ is closed. □

Theorem 1.1.10. *Every finite set in Hausdorff space X is closed.*

Proof. Consider $X \supset C = \{x_1, x_2, \ldots, x_N\} = \bigcup_{i=1}^{N} \{x_i\}$ which is the union of a finite number of closed sets. Using Theorem 1.1.9, it is closed as a finite union of closed sets (see Theorem 1.1.1). □

Corollary 1.1.3. *If $x \in \mathbb{R}^n$ and f is continuous, then $f^{-1}(\{x\})$ is a closed set.*

Proof. $\mathbb{R}^n$ is Hausdorff and so $\{x\}$ is closed. Since f is continuous then $f^{-1}(\{x\})$ is closed according to Theorem 1.1.4. □

Two important concepts in topology are now considered.

1.1.2 Connected Space

Definition 1.1.19. *Let X be a topological space. A separation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. X is said to be connected if there is no separation of X.*

Examples 1.1.6. If $X = \{a, b\}$, then X is not connected for the discrete topology, since there is a separation of X, $X = \{a\} \cup \{b\}$ and $\{a\} \cap \{b\} = \emptyset$, where $\{a\}$ and $\{b\}$ are open. X is connected for the indiscrete topology since the only open sets are X and $\emptyset$.

Lemma 1.1.2. If C and D form a separation of X and if Y is connected subspace of X, then Y lies entirely in C or in D.

Proof. The sets $C \cap Y$ and $D \cap Y$ are open in Y since C and D are open in X. Also $(C \cap Y) \cap (D \cap Y) = \emptyset$ since $C \cap D = \emptyset$. Note that, $Y = Y \cap X = Y \cap (C \cup D) = (C \cap Y) \cup (D \cap Y)$, which form a separation of Y. But Y is connected, hence $C \cap Y = \emptyset$ or $D \cap Y = \emptyset$. If $C \cap Y = \emptyset$, then $D \cap Y = Y$ and $Y \subset D$. If $D \cap Y = \emptyset$, then $C \cap Y = Y$ and $Y \subset C$. □

Lemma 1.1.3. Let Y subspace of X and (A, B) a partition of Y. Then, (A, B) is a separation of Y if and only if $A' \cap B = \emptyset$ and $A \cap B' = \emptyset$.

Theorem 1.1.11. *Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is connected.*

Proof. Let C and D form a separation of B. Since $A \subset B$, using Lemma 1.1.2, $A \subset C$ or $A \subset D$ because A is connected. Suppose that $A \subset C$, then $\overline{A} \subset \overline{C}$. Furthermore, $\overline{C} \cap D = (C' \cup C) \cap D = (C' \cap D) \cup (C \cap D) = \emptyset$ as C and D form a separation. Since $A \subset B \subset \overline{A} \subset \overline{C}$, then $B \subset \overline{C}$ and $B \cap D = \emptyset$. This is a contradiction with the fact $D \subset B$. Hence B is connected. □

Corollary 1.1.4. *If A is connected subspace of X, then $\overline{A}$ is connected.*

Proof. Since $A \subset \overline{A} \subset \overline{A}$, $\overline{A}$ is connected according to the previous theorem. □

Definition 1.1.20. *Define a relation $\sim$ on X defined by $x \sim y$ if there is a connected subspace of X containing both x and y.This relation is an equivalence relation and the equivalence classes are called the components or the connected components of X. For $x \in X$, we denote by $C(x)$ the equivalence class of x.*

Theorem 1.1.12. *The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.*

Remark 1.1.1. For $x \in X$, $C(x)$ is the maximal connected subspace of X containing x.

Theorem 1.1.13. *Each connected component is connected and closed.*

Theorem 1.1.14. *If $x, y \in X$, then $C(x)$ and $C(y)$ are either equal or disjoint.*

Proof. Suppose that $C(x) \cap C(y) \neq \emptyset$. Let $z \in C(x) \cap C(y)$. $z \in C(x)$ implies $C(x) \subset C(z)$ since $C(z)$ is the maximal connected subspace that contains z and $C(x)$ is connected. For the same reason $x \in C(z)$ implies $C(z) \subset C(x)$ and so $C(x) = C(z)$. Similarly, $C(z) = C(y)$ hence $C(x) = C(y)$. □

Proposition 1.1.2. *If X has a finite number of the components, then every component is open.*

Proof. Let C_i, $i = 1, 2, \ldots, n$. be all the components of X, then $X = \bigcup_{i=1}^{n} C_i$, hence $X \backslash C_j = \bigcup_{i=1, i \neq j}^{n} C_i$, $j = 1, 2, \ldots, n$ is closed since the finite union of closed set is a closed set. Thus, $X \backslash C_j$ is closed i.e, C_j is open. □

Definition 1.1.21. *Let X be a topological space and $x, y \in X$. A path in X from x to y is a continuous map $f : [a, b] \to X$, such that $f(a) = x$ and $f(b) = y$, where $[a, b]$ is a closed interval in $\mathbb{R}$. X is said to be path connected if for every $x, y \in X$, there a path joining x to y.*

Theorem 1.1.15. *If X is a path connected, then X is connected.*

Definition 1.1.22. *Define another equivalence relation $\sim$ on X by $x \sim y$ if there is a path in X from x to y. The equivalence classes are called the path components of X.*

Recall that a collection $\mathcal{A}$ of subsets of a space X is a *covering* of X, if the union of the elements of $\mathcal{A}$ is equal to X. For example, the components covers X. It is called *open covering* if the elements of $\mathcal{A}$ are open subsets of X. So now we introduce the compactness.

1.1.3 Compact Space

Definition 1.1.23. *A topological space X is said to be compact if every open covering of X has a finite sub-covering.*

Examples 1.1.7. $\mathbb{R}$ is not compact. Indeed, consider the open cover $\mathbb{R} = \bigcup_{n\in\mathbb{N}}(-n,n)$. Suppose there is a finite subcover i.e, there is $N \in \mathbb{N}$ such that $\mathbb{R} = \bigcup_{n=1}^{N}(-n,n)$. This is impossible since $\mathbb{R}$ unbounded.

Theorem 1.1.16. *Let X be a compact space, and $A \subset X$ a closed subspace. Then, A is compact.*

Proof. Suppose that $A \subset \bigcup_{\alpha\in I} U_\alpha$, where U_α is an open in X for each $\alpha \in I$. Let $B = X\backslash A$, then $X = B \cup \bigcup_{\alpha\in I} U_\alpha$ which is an open cover. Since X is compact, there is some $N \in \mathbb{N}$ such that $X = B \cup \bigcup_{i=1}^{N} U_{\alpha_i}$. Thus, $A \subset B \cup \bigcup_{i=1}^{N} U_{\alpha_i} = X$, since $A \in X\backslash B$. Then $A \subset \bigcup_{i=1}^{N} U_{\alpha_i} = X$ and A is compact. □

Theorem 1.1.17. *Let X be a compact space and f be continuous function. Then $f(X)$ is a compact space.*

Proof. Suppose $f : X \longrightarrow Y$ is a continuous map. Let $f(X) \subset \bigcup_{\alpha\in I} U_\alpha$, where U_α is an open subset of Y for each $\alpha \in I$. Then, $X \subset f^{-1}(\bigcup_{\alpha\in I} U_\alpha) = \bigcup_{\alpha\in I} f^{-1}(U_\alpha)$. Since f is continuous, then $f^{-1}(U_\alpha)$ is an open subset in X for each $\alpha \in I$. Since, X is compact, then $X = \bigcup_{i=1}^{N} f^{-1}(U_{\alpha_i})$, for some $N \in \mathbb{N}$. Hence, $f(X) = f(\bigcup_{i=1}^{N} f^{-1}(U_{\alpha_i})) = \bigcup_{i=1}^{N} f(f^{-1}(U_{\alpha_i})) \subset \bigcup_{i=1}^{N} U_{\alpha_i}$. Therefore, $f(X)$ is compact. □

Definition 1.1.24. *Let X be a topological space. The space X is said to be sequentially compact if every sequence of points of X has a convergent subsequence.*

Theorem 1.1.18. *Let A be a subset of metric space. Then, A is compact if and only if A is sequentially compact.*

Theorem 1.1.19. *Let A be compact subset of metric space X, then A is closed and bounded.*

Proof. Suppose $x \in \overline{A}$. Then there is a sequence $x_n \in A$ such that $x_n \longrightarrow x$, as $n \to \infty$. Since A is compact, then A is sequentially compact. Thus, x_n has a convergent subsequence x_{n_k} which converges to x. Then $x \in A$, i.e, A is closed. Now, we show that A is bounded. Fix $x \in X$. Then $X = \bigcup_{n\in\mathbb{N}} B(x,n)$, which is an open cover. Since, A is compact and $A \subset \bigcup_{n\in\mathbb{N}} B(x,n)$, then there is $N \in \mathbb{N}$ such that $A \subset \bigcup_{i=1}^{N} B(x,i) = B(x,N)$. Hence, A is bounded. □

Definition 1.1.25. *Let $A \subset X$, then A is relatively compact if the closure of A is compact.*

Theorem 1.1.20. *If C is compact then, $A \subset C$ is relatively compact.*

Proof. $A \subset C$ which implies that $\overline{A} \subset \overline{C} = C$. Since $\overline{A}$ is a closed subset of C which is compact, then $\overline{A}$ is compact and so A is relatively compact. □

Theorem 1.1.21. *If A is relatively compact, then $\overline{A}$ is sequentially compact.*

Proof. Since A is relatively compact then $\overline{A}$ is compact, and so sequentially compact. □

Theorem 1.1.22. *If (X,d) is a metric space, then the following statements are equivalent:*

(i) *A is relatively compact.*

(ii) *For each sequences $x_n \in A$ there is a sub-sequence of x_n which converges in X.*

Proof. Let A is relatively compact. Then $\bar{A}$ is sequentially compact, and so if $x_n \in A \subset \bar{A}$, then there is exists a sub-sequence x_{n_k} converges to some $x \in \bar{A} \subset X$. Conversely, we show that A is relatively compact, i.e, $\bar{A}$ is compact. It is sufficient to show $\bar{A}$ is sequentially compact. Indeed, let $x_n \in \bar{A}$, by the one of the characterization of the closure that we have introduced in Theorem 1.1.6, there is $y_n \in A$ for all $n \in \mathbb{N}$ such that,

$$d(x_n, y_n) < \frac{1}{n}.$$

By hypothesis there is a sub-sequence $y_{n_k} \in A$ converges to some limit $x_0 \in X$. Since $\bar{A}$ is closed $x_0 \in \bar{A}$. Finally, by triangular inequality we have

$$d(x_{n_k}, x_0) \leq d(x_{n_k}, y_{n_k}) + d(y_{n_k}, x_0) \longrightarrow 0,$$

as $k \to \infty$. □

Theorem 1.1.23. *Let X be a Hausdorff space and $Y \subset X$ is a compact subspace, then Y is closed.*

Proposition 1.1.3. *If every component in a compact space is open, then the number of components is finite.*

Proof. Let $(C(x))_x$ be a collection of components of X, then $X = \bigcup_{x\in X} C(x)$ with $C(x)$ is open, hence $\bigcup_{x\in X} C(x)$ is an open cover of X, as X is compact then, there exists $N \in \mathbb{N}$ such that $X = \bigcup_{i=1}^{N} C(x_i)$. This implies that X has N components. □

Definition 1.1.26. *A space X is said to be locally compact at x, if there is some compact set Y of X that contains a neighborhood of x. If X is locally compact at each of its points, X is said to be locally compact.*

Examples 1.1.8. $\mathbb{R}$ is locally compact, since for $x \in \mathbb{R}$, there is $[x-1, x+1] \in \mathcal{N}_x$, because $x \in (x-1, x+1) \subset [x-1, x+1]$. Hence, $x \in [x-1, x+1] \subset [x-1, x+1]$ which is compact.

Examples 1.1.9. $\mathbb{R}^n$ is locally compact, similar argument to $\mathbb{R}$, we take $x \in \mathbb{R}^n$, then $x \in [x_1 - 1, x_1 + 1] \times [x_2 - 1, x_2 + 1] \times \dots [x_n - 1, x_n + 1] \in \mathcal{N}_x$ which is compact.

Theorem 1.1.24 (Characterization of locally compact spaces). *X is locally compact if and only if the components of every open subset of X is path component.*

Let $U \subset \mathbb{R}^n$ be an open set and $C \subset U$, then C is component if and only if C is path component (in this case, the components are open sets).

1.2 Functional Analysis

Definition 1.2.1. *Let X be a vector space. A norm on X is a map $\cdot||$: $X \longrightarrow [0, \infty)$ satisfying the following properties:*

(1) **(Positivity)** $\|x\| \geq 0$, *and* $\|x\| = 0 \;\Leftrightarrow\; x = 0$.
(2) **(Homogeneity)** $\|\lambda x\| = |\lambda|\,\|x\|$, *where* λ *is any scalar and for any* $x \in X$.
(3) **(Triangular inequality)** $\|x + y\| \leq \|x\| + \|y\|$, *for all* $x, y \in X$.
We call the space $(X, \|\cdot\|)$ *a normed space.*

Proposition 1.2.1. *Let* X *be normed space. Then,*

$$\Big|\|x\| - \|y\|\Big| \leq \|x - y\|.$$

for all $x, y \in X$.

Proof.

$$\begin{aligned}\|x\| =&\|x - y + y\| \leq \|x - y\| + \|y\|\\ \Rightarrow&\|x\| - \|y\| \leq \|x - y\|.\end{aligned}$$

In addition,

$$\begin{aligned}\|y\| =&\|y - x + x\| \leq \|x - y\| + \|x\|\\ \Rightarrow&\|y\| - \|x\| \leq \|x - y\|.\end{aligned}$$

Combining the inequalities, we obtain

$$\Big|\|x\| - \|y\|\Big| \leq \|x - y\|.$$

□

Proposition 1.2.2. *Let* X *be a normed space then* $B(x, r) = \{x\} + rB(0, 1)$.

Proof. Let $y \in B(x, r)$; then

$$\|y - x\| < r \Rightarrow \frac{1}{r}\|y - x\| < 1.$$

Set $z = \frac{1}{r}(y - x)$, then $y = x + rz$ which implies that $y \in \{x\} + rB(0, 1)$. Conversely, let $y \in \{x\} + rB(0, 1)$, then $y = x + rz$ with $\|z\| < 1$. Since,

$$\|y - x\| = \|rz\| = r\|z\| < r,$$

then $y \in B(x, r)$. □

Corollary 1.2.1. *Let* X *be normed space. If* $y \in B(x, r)$, *then there exists* $z \in X$ *with* $\|z\| < 1$ *such that* $y = x - rz$.

Proof. By Proposition 1.2.2, $y \in \{x\} + rB(0,1)$ i.e, $y = x + rz'$ for some $z' \in X$ with $\|z'\| < 1$. Set $z = -z'$, then $\|z\| < 1$ and $y = x - rz$. □

Lemma 1.2.1 (Riesz's Theorem). Let E be a real normed space and $M \subset E$ be a proper closed subspace. Then, for any $\varepsilon \in (0,1)$, there exists $x_0 \in E$ such that $\|x_0\| = 1$ and $d(x_0, M) = \inf_{y \in M} \|x_0 - y\| > \varepsilon$.

An important consequence of Lemma 1.2.1 is given by the following lemma.

Lemma 1.2.2. Let E be a real normed space. Then, the unit closed ball $\overline{B(0,1)} = \{x : \|x\| \leq 1\}$ is compact if and only if dimension of E is finite.

Definition 1.2.2. *Let X be topological space and let $A \subset X$ be nonempty. The subset A is called a retract of X if there is a continuous function $r : X \longrightarrow A$ such that $r(x) = x$ for all $x \in A$. The map r is called a retraction.*

Examples 1.2.1. Let X be normed space and $A = \overline{B(x_0, R)}$ an arbitrary closed ball then A is a retract of X with a retraction given by

$$r(x) = \begin{cases} x, & x \in \overline{B(x_0, R)} \\ x_0 + \frac{R(x - x_0)}{\|x - x_0\|}, & x \notin B(x_0, R) \end{cases}$$

$x_0 + \frac{R(x-x_0)}{\|x-x_0\|}$ is continuous because $\|x - x_0\| \geq R > 0$, then r is continuous by the pasting lemma 1.1.1. Moreover, as $r(x) = x_0 + \frac{R(x-x_0)}{\|x-x_0\|}$, then $\|r(x) - x_0\| = \frac{R\|x-x_0\|}{\|x-x_0\|} = R$, i.e, $r(x) \in S(x_0, R) \subset \overline{B(x_0, R)}$. r is called the radical retraction. Therefore, every closed ball of normed space is retract of this space.

Theorem 1.2.1 (Dugundji's Retraction Theorem). *Every closed convex subset C of a normed space X is retract.*

Lemma 1.2.3 (Approximation Lemma). Let $K \subset \mathbb{R}^n$ be compact and $f : K \longrightarrow \mathbb{R}^n$ continuous then, there exists $\varepsilon_0 > 0$, such that for every $0 < \varepsilon < \varepsilon_0$, there exist $f_\varepsilon \in C^\infty(\mathbb{R}^n)$ and

$$\|f(x) - f_\varepsilon(x)\| < \varepsilon$$

Definition 1.2.3. *Let X be a vector space over $\mathbb{R}$. An inner product is scalar valued function $\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{R}$ such that for all $x, y, z \in X$ and for all $\alpha \in \mathbb{R}$, we have*

1. $\langle x, x\rangle \geq 0$,

2. $\langle x, x\rangle = 0 \Leftrightarrow x = 0$,

3. $\langle x, y\rangle = \langle y, x\rangle$,

4. $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$,

5. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

The space $(X, \langle\cdot,\cdot\rangle)$ *is called an inner product space over* $\mathbb{R}$.

Theorem 1.2.2 (Cauchy-Schwarz Inequality)**.** *Let* $(X, \langle\cdot,\cdot\rangle)$ *be an inner product. Then for all* $x, y \in X$

$$|\langle x, y\rangle| \leq \sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle}$$

Theorem 1.2.3. *Let* $(X, \langle\cdot,\cdot\rangle)$ *over* $\mathbb{R}$. *For each* $x \in X$, *define*

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Then $\|\cdot\|$ *defines a norm on* X.

Remark 1.2.1. Using the norm, Cauchy-Schwarz Inequality becomes

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

1.3 Differential Calculus

Definition 1.3.1. *Let* $f \in C(\overline{\Omega}, \mathbb{R}^n) \cap C^1(\Omega, \mathbb{R}^n)$. *If the component of* $f(x)$ *are* $f_i(x)$, *we denote* $f'(x)$ *the derivative matrix*

$$f'(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \cdots & \frac{\partial f_2}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \frac{\partial f_n}{\partial x_2}(x) & \cdots & \frac{\partial f_n}{\partial x_n}(x) \end{pmatrix},$$

and denote $J_f(x)$ *the determinant of* $f'(x)$ *which called the Jacobian.*

Theorem 1.3.1 (**The inverse function theorem**). *Suppose* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is continuously differentiable on some open set containing* x *and suppose* $J_f(x) \neq 0$. *Then there is some open set* U *containing* x *and an open set* V *containing* $f(x)$ *such that* $f : U \longrightarrow V$ *has continuous inverse* $f^{-1} : V \longrightarrow U$ *which is differentiable for all* $y \in V$.

Theorem 1.3.2 (**Change of variable in** $\mathbb{R}^n$). *Suppose* $\Omega \subset \mathbb{R}^n$ *is a bounded set and let* $\eta : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a one to one linear map and* $\eta \in C^1(\mathbb{R}^n)$. *If* $f : \eta(\Omega) \longrightarrow \mathbb{R}$ *is an integrable function then,*

$$\int_{\eta(\Omega)} f(y)dy = \int_{\Omega} f(\eta(x))|J_f(x)|dx.$$

Theorem 1.3.3 (**Continuity of integral with respect to parameter**). *Let* $\Omega \subset \mathbb{R}^n$ *open, and* $F : [0, T] \longrightarrow \mathbb{R}^n$. *defined by,*

$$F(t) = \int_{\Omega} f(x,t)dx.$$

where f *is continuous in* t *and* x. *Then* F *is continuous with respect to* t.

Lemma 1.3.1 (**Sard's Lemma**). Let $\Omega \subset \mathbb{R}^n$ be open and $f \in C^1(\Omega)$. If

$$S_f(\Omega) = \{x \in \Omega : J_f(x) = 0\}.$$

Then $\mu_n(f(S_f(\Omega))) = 0$, where μ_n is the Lebesgue measure in $\mathbb{R}^n$.

Chapter 2

Brouwer's Topological Degree (the finite-dimensional case)

In this chapter, our main focus will be in the case of finite dimensional spaces. We introduce Brouwer's degree for continuous functions and explain the construction of this degree. Also we present the main properties associated to this important tool in Analysis. As an important consequence, we prove the Brouwer's fixed point theorem together with some of its applications. For the results of this chapter, we refer to [1, 4, 6].

2.1 Regular Case

Let Ω be a bounded open subset of $\mathbb{R}^n$ and $f \in C(\overline{\Omega}, \mathbb{R}^n) \cap C^1(\Omega, \mathbb{R}^n)$.To define the Brouwer's degree, we first recall the Jacobian $J_f(x)$ of f at x (see Definition 1.3.1).

Definition 2.1.1. *Let $x \in \Omega \subset \mathbb{R}^n$, x is a regular point if $J_f(x) \neq 0$. Otherwise, x is said to be a singular point i.e, $J_f(x) = 0$ and we denote the set of singular points by*

$$S_f(\Omega) = \{x \in \Omega : J_f(x) = 0\}.$$

Definition 2.1.2. *We say that p is a regular value, if for all $x \in f^{-1}(p), x$ is a regular point in other word, if $f^{-1}(p) \cap S_f(\Omega) = \emptyset$. If p is not a regular value then, p is called a singular value.*

Proposition 2.1.1. *If p is a regular value and $f^{-1}(p) \cap \partial\Omega = \emptyset$, then, $f^{-1}(p)$ is finite.*

Proof. Since p is a regular value then, for all $x \in f^{-1}(p), J_f(x) \neq 0$. Using the Inverse Function Theorem 1.3.1, for all $x \in f^{-1}(p)$, there is $U_x \in \mathcal{N}_x$ such that, $f|_{U_x}$ is homeomorphism. Thus, x must be an isolated point of $f^{-1}(p)$ otherwise, $f|_{U_x}$ is not bijective which is a contradiction. Hence, $f^{-1}(p)$ consists only of isolated points. As, $f^{-1}(p) \subset \Omega \subset \overline{\Omega}$, and $\overline{\Omega}$ is a closed and bounded subset of $\mathbb{R}^n$ then, $\overline{\Omega}$ is compact. Also, $f^{-1}(p)$ is closed and so compact as the closed subset of a compact set compact. From the Inverse Function Theorem 1.3.1, $f^{-1}(p) \subset \bigcup_{x \in f^{-1}(p)} U_x$, which is an open cover of $f^{-1}(p)$ thus, there is $N \in \mathbb{N}$ such that $f^{-1}(p) \subset \bigcup_{i=1}^{N} U_{x_i}$, since x_i are isolated point for each $i = 1, 2, \ldots, n$, then $U_{x_i} \cap f^{-1}(p) = \{x_i\}, \forall\, x_i \in f^{-1}(p)$. Therefore, the number of elements of $f^{-1}(p)$ is finite □

The following definition makes sense because of the previous proposition. We start constructing Brouwer's topological degree.

Definition 2.1.3 (**Degree in the regular case**). *Let $\Omega \subset \mathbb{R}^n$ open and bounded subset and $f \in C^1(\Omega) \cap C(\overline{\Omega})$. If $p \notin [f(\partial\Omega) \cup f(S_f(\Omega))]$, then we define*

$$\begin{cases} \deg(f, \Omega, p) = \sum_{x \in f^{-1}(p)} \operatorname{sgn} J_f(x) & \text{if } f^{-1}(p) \neq \emptyset \\ \deg(f, \Omega, p) = 0 & \text{if } f^{-1}(p) = \emptyset, \end{cases}$$

where sgn *denotes the sign function.*

Thus $\deg(f, \Omega, p) \in \mathbb{Z}$. The next result is an integral representation of the degree defined above and will be considered as an equivalent definition.

Proposition 2.1.2. *Suppose Ω, f and p as in Definition 2.1.3. Let $(\varphi_\varepsilon)_{\varepsilon>0} \subset C(\mathbb{R}^n, \mathbb{R})$ be a family of positive functions with $\operatorname{supp} \varphi_\varepsilon \subset B_\varepsilon(0) = B(0, \varepsilon)$ and $\int_{\mathbb{R}^n} \varphi_\varepsilon(x)\, dx = 1$. Then, there exists $\varepsilon_0(p, f) > 0$ such that*

$$\deg(f, \Omega, p) = \int_\Omega \varphi_\varepsilon(f(x) - p) J_f(x) dx,$$

for all $0 < \varepsilon < \varepsilon_0$.

Proof. Suppose that $f^{-1}(p) = \{x_1, x_2, \ldots, x_N\}$. If $f^{-1}(p) = \emptyset$, then there is nothing to prove. Let $\psi_\varepsilon(x) = \varphi_\varepsilon(f(x) - p)$, where $x \in \Omega \subset \overline{\Omega}$. Thus $\operatorname{supp} \psi_\varepsilon \subset \overline{\Omega}$ and so $\operatorname{supp} \psi_\varepsilon$ is compact. Assume $C_1, C_2, \ldots, C_N$ are the components of $x_1, x_2, \ldots, x_N$ respectively in $\operatorname{supp} \psi_\varepsilon$. Hence,

$$I = \int_\Omega \varphi_\varepsilon(f(x) - p) J_f(x) dx = \int_{\operatorname{supp} \psi_\varepsilon} \psi_\varepsilon(x) J_f(x) dx.$$

Since $\operatorname{supp}\psi_\varepsilon$ covered by $C_1, C_2, \dots, C_N$, then

$$I = \int_{\operatorname{supp}\psi_\varepsilon} \psi_\varepsilon(x) J_f(x) dx = \sum_{i=1}^{N} \int_{C_i} \psi_\varepsilon(x)|J_f(x)|\operatorname{sgn} J_f(x) dx.$$

Note that $J_f(x)$ is constant on each $C_i, 1 \leq i \leq N$.

$$I = \sum_{i=1}^{N} \int_{C_i} \psi_\varepsilon(x)|J_f(x)|\operatorname{sgn} J_f(x) dx = \sum_{i=1}^{N} \operatorname{sgn} J_f(x) \int_{C_i} \psi_\varepsilon(x)|J_f(x)|dx.$$

$$= \sum_{i=1}^{N} \operatorname{sgn} J_f(x) \int_{C_i} \varphi_\varepsilon(f(x) - p)|J_f(x)|dx$$

Since $J_{f-p(x)} = J_f(x)$ and applying the change of variable on $\mathbb{R}^n$ (see Theorem 1.3.2), we get

$$I = \sum_{i=1}^{N} \operatorname{sgn} J_f(x) \int_{C_i} \varphi_\varepsilon(f(x) - p)|J_f(x)|dx = \sum_{i=1}^{N} \operatorname{sgn} J_f(x) \int_{B_\varepsilon(0)} \varphi_\varepsilon(y) dy$$

As $\operatorname{supp}\varphi_\varepsilon \subset B_\varepsilon(0)$ and given that $\int_{\mathbb{R}^n} \varphi_\varepsilon(x)\, dx = 1$, we conclude

$$\int_{\Omega} \varphi_\varepsilon(f(x) - p) J_f(x) dx = \sum_{i=1}^{N} \operatorname{sgn} J_f(x) = \deg(f, \Omega, p)$$

□

Examples 2.1.1. Let the function $\varphi_\varepsilon(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ be given by

$$\varphi_\varepsilon(x) := \begin{cases} c \exp\left(-\frac{1}{1-\frac{\|x\|^2}{\varepsilon^2}}\right) & \|x\| < \varepsilon \\ 0 & \|x\| \geq \varepsilon, \end{cases}$$

where c is constant such that $\int_{\mathbb{R}^n} \varphi_\varepsilon = 1$. We show that φ_ε satisfies the conditions stated in Proposition 2.1.2. Indeed, consider the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$g(t) = \begin{cases} \exp\left(-\frac{1}{t}\right) & t > 0 \\ 0 & t \leqslant 0 \end{cases}$$

We can write $\varphi_\varepsilon(x) = g\left(1 - \frac{\|x\|^2}{\varepsilon^2}\right)$. We prove that, $g \in C^\infty(\mathbb{R})$. Indeed,

$$g'(t) = \frac{1}{t^2}\exp\left(-\frac{1}{t}\right) = P_2\left(\frac{1}{t}\right)\exp\left(-\frac{1}{t}\right) \to 0 \ \ as \ \ t \to 0^+$$

$$g''(t) = \left(\frac{1}{t^4} - \frac{2}{t^3}\right)\exp\left(-\frac{1}{t}\right) = P_4\left(\frac{1}{t}\right)\exp\left(-\frac{1}{t}\right) \to 0 \ \ as \ \ t \to 0^+$$

where $P_2 = (\frac{1}{t})^2$ and $P_4 = (\frac{1}{t})^4 - 2(\frac{1}{t})^3$. Thus by induction we conclude that

$$g^{(n)}(t) = P_{2n}\left(\frac{1}{t}\right)\exp\left(-\frac{1}{t}\right) \to 0, \quad \text{as} \ \ t \to 0^+,$$

where P_{2n} is a polynomial of degree $2n$. Since $g^{(n)}(0^-) = 0$, and $g^{(n)}(0^+) = 0$, we have $g^{(n)}(0) = 0$. We deduce that $g \in C^\infty(\mathbb{R})$.The function $x \mapsto \|x\|^2$ is of class $C^\infty(\mathbb{R})$ and so the composition φ_ε is of class $C^\infty(\mathbb{R})$. Let $K = \{x \in \mathbb{R}^n : \varphi_\varepsilon(x) \neq 0\}$. If $x \in K$ we have, $\varphi_\varepsilon(x) \neq 0$, i.e, $g(1 - \frac{\|x\|^2}{\varepsilon^2}) \neq 0$. From the definition of g we have, $1 - \frac{\|x\|^2}{\varepsilon^2} > 0$, which implies $\|x\| < \varepsilon$, i.e, $K \subset B_\varepsilon(0)$. Therefore, $\operatorname{supp}\varphi_\varepsilon(x) = \overline{K} \subset \overline{B_\varepsilon(0)}$.

The previous example guarantees the existences of φ_ε mentioned in Proposition 2.1.2. The proof of the following lemma can be found, e.g., in [6].

Lemma 2.1.1. Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset, $f \in C(\overline{\Omega}) \cap C^1(\Omega)$, and let $v \in C_c^1(\mathbb{R}^n)$ the space of all real valued continuous functions on $\mathbb{R}^n$ with compact support and $\operatorname{supp} v \cap f(\partial\Omega) = \emptyset$. Then there exists $u \in C_c^1(\mathbb{R}^n)$ such that

$$\operatorname{div} u(x) = J_f(x)\operatorname{div} v(f(x)),$$

where $\operatorname{div} v = \nabla \cdot v$.

Proposition 2.1.3. *Suppose that $\Omega \subset \mathbb{R}^n$ be open, bounded set, $f \in C(\overline{\Omega}) \cap C^1(\Omega)$ and p_1 and p_2 are regular values which lie in a ball of $\mathbb{R}^n \backslash f(\partial\Omega)$. Then,*

$$\deg(f, \Omega, p_1) = \deg(f, \Omega, p_2).$$

Proof. We show that $\deg(f, \Omega, p_1) - \deg(f, \Omega, p_2) = 0$. Indeed, take $0 < \varepsilon < \min(\varepsilon_0^1, \varepsilon_0^2) = \varepsilon_0$, where $\varepsilon_0^1 = \varepsilon_0^1(p_1, f)$ and $\varepsilon_0^2 = \varepsilon_0^2(p_2, f)$. According to Proposition 2.1.2 we have,

$$\deg(f, \Omega, p_1) - \deg(f, \Omega, p_2) = \int_\Omega \left[\varphi_\varepsilon(f(x) - p_1) - \varphi_\varepsilon(f(x) - p_2)\right] J_f(x)dx.$$

We can choose ε small enough so that $\varphi_\varepsilon(f(x)-p_i)=0$, for all $x \in \partial\Omega$ and $i=1,2$. Let $v(y)=\varphi_\varepsilon(y-p_1)-\varphi_\varepsilon(y-p_2)$. Then, $\operatorname{supp} v \cap f(\partial\Omega)=\emptyset$. By Lemma 2.1.1, there exists $u \in C^1(\mathbb{R}^n)$ such that $\operatorname{supp} u \subset \Omega$ and

$$[\varphi_\varepsilon(f(x)-p_1)-\varphi_\varepsilon(f(x)-p_2)]\, J_f(x)=\operatorname{div} u.$$

By integration, we get

$$\deg(f,\Omega,p_1)-\deg(f,\Omega,p_2)=\int_\Omega \operatorname{div} u(x)dx=\int_{\partial\Omega} u\cdot\eta dx=0,$$

where the second equality is the Divergence Theorem with η the outer normal to Ω. □

2.2 Singular Case

Let p be a singular value. By Sard's Lemma 1.3.1, there exists a regular value p_ε such that $\|p-p_\varepsilon\|<\varepsilon$. Let p be a singular value, $p \notin f(\partial\Omega)\subset\mathbb{R}^n$. Then for all ε there exists a regular value p_ε such that $p_\varepsilon \in B(p,\varepsilon)$. Suppose by contradiction that, there exists $\varepsilon_0>0$, such that for all regular values p', we have $p' \notin B(p,\varepsilon_0)$ which means that $B(p,\varepsilon_0)\subset f(S_f(\Omega))$. This implies that

$$\begin{aligned}&\mu_n(B(p,\varepsilon_0))\le\mu_n(f(S_f(\Omega)))\\ &\Leftrightarrow \operatorname{diam}(B(p,\varepsilon_0))\le\mu_n(f(S_f(\Omega)))\\ &\Leftrightarrow 2\varepsilon_0\le 0,\end{aligned}$$

which is a contradiction.

Definition 2.2.1 (Degree in the singular case). *Let $\Omega\subset\mathbb{R}^n$ be an open bounded subset, $f\in C(\overline{\Omega})\cap C^1(\Omega)$, and $p\notin f(\partial\Omega)$. Then define*

$$\deg(f,\Omega,p)=\deg(f,\Omega,p'),$$

where p' is a regular value such that $\|p-p'\|<\operatorname{dist}(p,f(\partial\Omega))$.

Remark 2.2.1. (1) Note that in the previous definition, $\partial\Omega$ is closed and bounded in $\mathbb{R}^n$, hence compact which implies that $f(\partial\Omega)$ is compact as f is continuous. Hence, $f(\partial\Omega)$ is closed set and so, $\mathbb{R}^n\backslash f(\partial\Omega)$ is open, then $\operatorname{dist}(p,f(\partial\Omega))>0$ by Corollary 1.1.2.

(2) Definition 2.2.1 does not depend on the choice of p'. To see that suppose p' and p'' are two regular values such that

$$\|p - p'\| < \text{dist}(p, f(\partial\Omega)) = d,$$

and

$$\|p - p''\| < \text{dist}(p, f(\partial\Omega)) = d.$$

Then $p', p'' \in B(p, d) \subset \mathbb{R}^n \backslash f(\partial\Omega)$. We check the last statement. Let $y \in B(p, d)$ and $x \in \partial\Omega$, we have

$$\begin{aligned}
\|y - f(x)\| &= \|(f(x) - p) - (y - p)\| \\
&\geq \Big| \|f(x) - p\| - \|y - p\| \Big| \\
&= \|f(x) - p\| - \|y - p\| \\
&\geq d - \|y - p\| > 0, \qquad (\text{since} \quad y \in B(p, d)).
\end{aligned}$$

That is, $y \in \mathbb{R}^n \backslash f(\partial\Omega)$, and by Proposition 2.1.3, we have

$$\deg(f, \Omega, p') = \deg(f, \Omega, p'').$$

2.3 Properties of the Degree

In this section, we give some of the main properties of the degree when $f \in C^1(\Omega) \cap C(\overline{\Omega})$ and $p \notin f(\partial\Omega)$. We start with the concept of homotopy.

Definition 2.3.1. *Let X, Y be topological spaces and $f, g : X \longrightarrow Y$ continuous functions. A homotopy from f to g is a continuous function $H : X \times [0, 1] \longrightarrow Y$ satisfying*

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x), \text{ for all } \ x \in X.$$

We write $H(x, t) = H_t(x)$, $\forall\, x \in X$, $\forall\, t \in [0, 1]$. We say that f and g are homotopic.

Examples 2.3.1. If $f, g : \overline{\Omega} \longrightarrow \mathbb{R}^n$ are two continuous functions, one may define

$$H(x, t) = tf(x) + (1 - t)g(x),$$

where $t \in [0, 1]$, then $H(x, 1) = g(x)$ and $H(x, 0) = f(x)$, i.e, H is homotopy. We refer to H as a convex combination of f and g.

Proposition 2.3.1. *If $f \in C(\overline{\Omega})$, and $p \notin f(\partial\Omega)$, then the there is $g \in C^1(\Omega) \cap C(\overline{\Omega})$ such that, $\|f(x) - g(x)\| < \mathrm{dist}(p, f(\partial\Omega))$ and $p \notin g(S_g(\Omega))$.*

Proof. Let $h \in C^1(\Omega) \cap C(\overline{\Omega})$ such that $\|f(x) - h(x)\| < \frac{1}{2}\mathrm{dist}(p, f(\partial\Omega))$, h exists by approximation lemma 1.2.3. Now by Sard's lemma 1.3.1 there exists $q \notin h(S_h(\Omega))$, regular value such that $\|p - q\| < \frac{1}{2}\mathrm{dist}(p, f(\partial\Omega))$. We set $g(x) = h(x) + p - q$, then $g \in C^1(\Omega) \cap C(\overline{\Omega})$ and

$$\begin{aligned}\|f(x) - g(x)\| &= \|f(x) - h(x) - p + q\| \\ &\leq \|f(x) - h(x)\| + \|p - q\| \\ &< \mathrm{dist}(p, f(\partial\Omega)).\end{aligned}$$

Moreover,

$$g(x) = p \Leftrightarrow h(x) + p - q = p \Leftrightarrow h(x) = q, \qquad \forall\, x \in \overline{\Omega}.$$

Since $J_g(x) = J_h(x)$, then $q \notin h(S_h(\Omega))$ implies $p \notin g(S_g(\Omega))$. To see why, using contradiction suppose that $y \in g^{-1}(p) \cap S_g(\Omega)$ then $g(y) = p$ and $J_g(y) = 0$ which equivalent to $h(y) = q$ and $J_h(p) = 0$, which contradicts the fact $q \notin h(S_h(\Omega))$. Finally we check that $p \notin g(\partial\Omega)$. Indeed

$$\begin{aligned}\|p - g(x)\| &= \|p - f(x) + f(x) - g(x)\| \\ &\geq \Big| \|f(x) - p\| - \|f(x) - g(x)\| \Big| \\ &= \|f(x) - p\| - \|f(x) - g(x)\| \\ &> \|f(x) - p\| - \mathrm{dist}(p, f(\partial\Omega)) \geq 0.\end{aligned}$$

□

Now, we collect the main properties of Brouwer's topological degree.

Theorem 2.3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded subset and $f \in C^1(\Omega) \cap C(\overline{\Omega})$. If $p \notin f(\partial\Omega)$, we have the following properties:*

(1) If $p \notin \partial\Omega$ then, $deg(\mathrm{Id}, \Omega, p) = \begin{cases} 1, & p \in \Omega \\ 0, & p \notin \overline{\Omega}, \end{cases}$ where Id is the identity map.

(2) If $p \notin -(\partial\Omega)$ then, $deg(-\mathrm{Id}, \Omega, p) = \begin{cases} (-1)^n, & p \in \Omega \\ 0, & p \notin \overline{\Omega}, \end{cases}$ where Id is the identity map.

(3) **(Continuity with respect to p)** *If $p_1 \notin f(\partial\Omega)$ and $d_1 = \mathrm{dist}(p_1, f(\partial\Omega))$,*

then $B(p_1, d_1) \subset \mathbb{R}^n \backslash f(\partial\Omega)$. *Let* $p_2 \in \mathbb{R}^n$ *be such that* $\|p_1 - p_2\| < d_1$. *We have,*
(a) $p_2 \notin f(\partial\Omega)$.
(b) $\deg(f, \Omega, p_1) = \deg(f, \Omega, p_2)$.
(4) **(Invariance by homotopy of the degree)** *Let* $H(x,t) : \overline{\Omega} \times [0,1] \longrightarrow \mathbb{R}^n$ *be a continuously differentiable function and* $p_t : [0,1] \longrightarrow \mathbb{R}^n$ *continuous function such that* $p_t \notin H(x,t)$, *for all* $t \in [0,1]$, *and for each* $x \in \partial\Omega$. *Then* $\deg(H(\cdot,t), \Omega, p_t)$ *does not depend on the parameter* t.
(5) Let $p \notin f(\partial\Omega)$ *and* $f, g : \overline{\Omega} \longrightarrow \mathbb{R}^n$ *are in* $C^1(\Omega) \cap C(\bar{\Omega})$, *such that for each* $x \in \partial\Omega$,

$$\|f(x) - g(x)\| < \|f(x) - p\|$$

then,

$$p \notin g(\partial\Omega) \quad \textit{and} \quad \deg(f, \Omega, p) = \deg(g, \Omega, p).$$

(6) **(Existence property)** *If* $\deg(f, \Omega, p) \neq 0$, *then there exists* $x \in \Omega$ *such that* $f(x) = p$.
(7) **(Domain decomposition)** *Let* $(\Omega_i)_{i \in I} \subset \Omega$ *be a family of disjoint open subsets of* Ω *such that either*
(a) $\bigcup_{i \in I} \Omega_i = \Omega$ *and* $p \notin f(\partial\Omega)$ *or*
(b) $\bigcup_{i \in I} \Omega_i \subset \Omega$ *and* $p \notin f(\overline{\Omega} \backslash \bigcup_{i \in I} \Omega_i)$. *Then*

$$\deg(f, \Omega, p) = \sum_{i \in I} \deg(f, \Omega_i, p),$$

where only a finite number of terms are nonzero in the sum.
(8) **(Excision property)** *Let* $K \subset \Omega$ *closed subset and* $p \notin f(K) \cup f(\partial\Omega)$. *Then,*

$$\deg(f, \Omega, p) = \deg(f, \Omega \backslash K, p)$$

(9) Let $x_0 \in \Omega$ *be an isolated solution of the equation* $f(x_0) = p$, *then there exists* $r_0 > 0$ *such that the degree* $\deg(f, B(x_0, r), p)$ *is constant for all* $0 < r \leq r_0$.
(10) **(Multiplicity property)** *Let* $U \subset \mathbb{R}^n$ *and* $V \subset \mathbb{R}^m$, $f \in C^1(U) \cap C(\bar{U})$ *and* $g \in C^1(V) \cap C(\bar{V})$, *where* U *and* V *are open bounded subsets of* $\mathbb{R}^n$ *and* $\mathbb{R}^m$, *respectively. Let* $p \notin f(\partial U)$ *and* $q \notin g(\partial V)$. *Then,*

$$\deg(f \times g, U \times V, (p,q)) = \deg(f, U, p) \cdot \deg(g, V, q)$$

where $(f \times g)$ *defined by*

$$(f \times g)(x_1, x_2) = (f(x_1), g(x_2)), \forall (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m.$$

(11) **(Shifting property)** *If $p \notin f(\partial\Omega)$ then*

$$\deg(f, \Omega, p) = \deg(f - q, \Omega, p - q)$$

Proof. (1) (a) $p \notin \mathrm{Id}(\partial\Omega) = \partial\Omega$ implies that either $p \in \Omega$ or $p \notin \overline{\Omega}$. Assume that $p \in \Omega$. Then, $\mathrm{Id}^{-1}(p) = \{p\}$, now calculate the $J_{\mathrm{Id}}(p)$ to decide whether p is regular value or not. Let $x = (x_1, x_2, \ldots, x_n) \in \overline{\Omega}$, the jacobian matrix of the identity map is given by

$$\mathrm{Id}'(x) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I_n$$

Then $J_{\mathrm{Id}}(p) = det(\mathrm{Id}'(x)) = 1 \neq 0$, thus by Definition 2.1.3

$$\deg(\mathrm{Id}, \Omega, p) = \sum_{x \in \mathrm{Id}^{-1}(p)} \mathrm{sgn} J_{\mathrm{Id}}(x) = \mathrm{sgn} J_{\mathrm{Id}}(p) = \mathrm{sgn}(1) = 1.$$

(b) If $p \notin \overline{\Omega} = \mathrm{Id}(\overline{\Omega})$, then $\deg(\mathrm{Id}, \Omega, p) = 0$. Indeed, since $\mathrm{Id}(x) = p \Leftrightarrow x = p$ which impossible because $x \in \overline{\Omega}$ and $p \notin \overline{\Omega}$ i.e, $\mathrm{Id}^{-1}(p) = \emptyset$. Making use of Proposition 2.1.2 and definition 2.2.1, we get

$$deg(\mathrm{Id}, \Omega, p) = \int_{\Omega} \varphi_{\varepsilon}(x - p) dx,$$

where $\mathrm{supp}\, \varphi_{\varepsilon} \subset B(0, \varepsilon)$, for small ε, choose ε such that $B(0, \varepsilon) \subset \mathbb{R}^n \backslash (\mathrm{Id}(\partial\Omega)) = \mathbb{R}^n \backslash \partial\Omega$. As $\partial\Omega$ is closed, then $\mathbb{R}^n \backslash \partial\Omega$ is open. Since, $\overline{\Omega}$ is compact we have, $\inf_{x \in \overline{\Omega}} \|x - p\| = min_{x \in \overline{\Omega}} \|x - p\| > 0$, thus we choose $0 < \varepsilon < \inf_{x \in \overline{\Omega}} \|x - p\|$, this implies that $x - p \notin B(0, \varepsilon)$, $\forall\, x \in \Omega$, and so, $x - p \notin supp\, \varphi_{\varepsilon}$. Hence,

$$deg(\mathrm{Id}, \Omega, p) = \int_{\Omega} \varphi_{\varepsilon}(x - p) dx = 0.$$

(2) We follow the same procedure as in part (1) but, note that for the jacobian matrix we get the following

$$(-\mathrm{Id})'(x) = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix}$$

Therefore, $J_{-\mathrm{Id}}(x) = (-1)^n$, and we conclude the desired result.
(3) Note first that, since $\partial\Omega$ is closed and bounded thus, compact and f is continuous, $f(\partial\Omega)$ is compact, hence closed. Then

$$p \notin f(\partial\Omega) = \overline{f(\partial\Omega)} \;\Rightarrow\; d_1 > 0.$$

To prove (a), let $y \in B(p_1, d_1)$ and $x \in \partial\Omega$ we have

$$\begin{array}{rcll} \|y - f(x)\| & = & \|(f(x) - p_1) - (y - p_1)\| & \\ & \geq & \Big| \|f(x) - p_1\| - \|y - p_1\| \Big| & \\ & = & \|f(x) - p_1\| - \|y - p_1\| & \\ & \geq & d_1 - \|y - p_1\| > 0, & \text{(since} \quad y \in B(p_1, d_1)). \end{array}$$

Hence $y \in \mathbb{R}^n \backslash f(\partial\Omega)$.
(a) Since $\|p_1 - p_2\| < d_1,\ p_2 \in B(p_1, d_1)$. By (a), we have $p_2 \in B(p_1, d_1) \subset \mathbb{R}^n \backslash f(\partial\Omega)$, i.e, $p_2 \notin f(\partial\Omega)$.
(b) Let $0 < \varepsilon < d_1 - \|p_1 - p_2\|$. By Sard's Lemma 1.3.1, there exists regular values, $p_1' = p_1'(\varepsilon)$ and $p_2' = p_2'(\varepsilon)$ such that

$$\|p_1 - p_1'\| < \varepsilon \qquad \text{and} \qquad \|p_2 - p_2'\| < \varepsilon.$$

Since $\varepsilon < d_1$,

$$p_1' \in B(p_1, \varepsilon) \subset B(p_1, d_1)$$

and, using the triangular inequality

$$\begin{array}{rcl} \|p_2' - p_1\| & = & \|p_2' - p_2 + p_2 - p_1\| \\ & \leq & \|p_2' - p_2\| + \|p_2 - p_1\| \\ & < & \varepsilon + \|p_2 - p_1\| < d_1 - \|p_2 - p_1\| + \|p_2 - p_1\| = d_1. \end{array}$$

Hence, $p_2' \in B(p_1, \varepsilon)$, i.e, $p_1', p_2' \in B(p_1, d_1)$. Since p_1' and p_2' are regular values and lies in a ball subset of $\mathbb{R}^n \backslash f(\partial\Omega)$ by Proposition 2.1.3 we have

$$\deg(f, \Omega, p_1') = \deg(f, \Omega, p_2'). \tag{2.1}$$

By Definition 2.2.1, we have

$$\deg(f, \Omega, p_1) = \deg(f, \Omega, p_1'), \tag{2.2}$$

and

$$\deg(f, \Omega, p_2) = \deg(f, \Omega, p_2'). \tag{2.3}$$

Combining (2.1), (2.2) and (2.3) we get

$$\deg(f,\Omega,p_1)=\deg(f,\Omega,p_2),$$

as claimed.
(4) Since $p_t\notin H_t(\partial\Omega)$ for all $t\in[0,1]$, then by Sard's Lemma 1.3.1, there exists p'_t regular value, such that $\|p_t-p'_t\|<\operatorname{dist}(p_t,H_t(\partial\Omega))$. Hence, by Definition 2.2.1 we have

$$\begin{aligned}\deg(H(\cdot,t),\Omega,p_t)&=\deg(H(\cdot,t),\Omega,p'_t)\\&=\int_\Omega \varphi_\varepsilon(H_t(x)-p'_t)J_{H_t}(x)dx,\end{aligned}$$

where $supp\,\varphi_\varepsilon\subset B(0,\varepsilon)$ with $\varepsilon>0$. Then, we define the map

$$\begin{aligned}d_t:&[0,1]\to\mathbb{Z}\\&t\mapsto\deg(H(\cdot,t),\Omega,p_t)\end{aligned}$$

which is continuous by the continuity of the integral with respect to parameter t by Theorem 1.3.3. Since φ_ε, H_t and p'_t are continuous, then the composition is continuous. Then d_t is constant for all $t\in[0,1]$, otherwise d_t is not continuous, a contradiction. We deduce that $\deg(H(\cdot,t),\Omega,p_t)$ is constant on $[0,1]$, i.e, $\deg(H(\cdot,t),\Omega,p_t)$ is independent of the parameter t.
(5) Suppose that $p\in g(\partial\Omega)$, then there exists $x\in\partial\Omega$, such that $p=g(x)$. By the hypothesis we have

$$\|f(x)-p\|=\|f(x)-g(x)\|<\|f(x)-p\|,$$

which is a contradiction. For the second part of the property, let $H(x,t)=tg(x)+(1-t)f(x)$, where $x\in\overline{\Omega}$ and $t\in[0,1]$. Then H is continuously differentiable. Moreover,

$$\begin{aligned}\|p_t-H(x,t)\| &= \|p_t-tg(x)-(1-t)f(x)\|\\&= \|(p_t-f(x))-t(g(x)-f(x))\|\\&\geq \Big|\|p_t-f(x)\|-t\|g(x)-f(x)\|\Big|\\&= \|p_t-f(x)\|-t\|g(x)-f(x)\|\\&\geq \|p_t-f(x)\|-\|g(x)-f(x)\|>0\quad(\text{as }\ t\leq 1).\end{aligned}$$

Then, $p_t\notin H(x,t)$ for all $t\in[0,1]$. By Property (4), the degree $\deg(H(\cdot,t),\Omega,p)$ is constant on $[0,1]$. In particular,

$$\begin{aligned}&\deg(H(\cdot,0),\Omega,p)=\deg(H(\cdot,1),\Omega,p)\\&\Leftrightarrow\deg(f,\Omega,p)=\deg(g,\Omega,p).\end{aligned}$$

(6) Assume by contradiction that $p \notin f(\Omega)$. Since $p \notin f(\partial\Omega)$, then $p \notin f(\overline{\Omega})$. Since, $f(\overline{\Omega})$ is compact, then $\text{dist}(p, f(\overline{\Omega})) > 0$. By proposition 2.3.1 we may choose $g \in C^1(\Omega) \cap C(\overline{\Omega})$ such that

$$\|f(x) - g(x)\| < \text{dist}(p, f(\overline{\Omega})) \leq \|f(x) - p\|, \quad \forall\, x \in \overline{\Omega},$$

and $p \notin g(S_g(\Omega)) \cup g(\partial\Omega)$, then $p \notin g(\overline{\Omega})$. Otherwise, there exist $x_0 \in \overline{\Omega}$ such that $g(x_0) = p$, in particular,

$$\|f(x_0) - p\| = \|f(x_0) - g(x_0)\| < \|f(x_0) - p\|,$$

which is a contradiction. Thus $p \notin g(\overline{\Omega})$, i.e, $g^{-1}(p) = \emptyset$ then, $\deg(g, \Omega, p) = 0$. By Property (5) we have

$$0 = \deg(g, \Omega, p) = \deg(f, \Omega, p),$$

which is a contradiction.
(7) (a) Since $\partial\Omega_i \subset \partial\Omega$, then $p \notin f(\partial\Omega)$ implies $p \notin f(\partial\Omega_i)$ for all $i \in I$ and so $\deg(f, \Omega_i, p)$ is well defined. By Proposition 2.3.1 we choose $g \in C^1(\Omega) \cap C(\overline{\Omega})$ such that $\|f(x) - g(x)\| < \text{dist}(p, f(\partial\Omega)) \leq \|f(x) - p\|$ for all $x \in \partial\Omega$ and $p \notin g(S_g(\Omega))$. Then

$$\|f(x) - g(x)\| < \text{dist}(p, f(\partial\Omega)) \leq \text{dist}(p, f(\partial\Omega_i) \leq \|f(x) - p\|,$$

for all $x \in \partial\Omega_i$ by Property (5) we have

$$\deg(f, \Omega_i, p) = \deg(g, \Omega_i, p), \quad \text{and} \quad \deg(f, \Omega, p) = \deg(g, \Omega, p).$$

Since $p \notin g(S_g(\Omega))$, then $g^{-1}(p)$ is finite, i.e, $g^{-1}(p) = \{x_1, x_2, \ldots, x_N\}$. $\bigcup_{i \in I} \Omega_i = \Omega$, for $x \in \Omega$ there exist $i \in I$ such that $x \in \Omega_i$. Thus, $\deg(g, \Omega_i, p) \neq 0$, for $i = 1, 2, \ldots, K$, $K \leq N$, and $\deg(g, \Omega_i, p) = 0$, $i \geq K+1$.

Then,

$$\begin{aligned}\deg(f,\Omega,p) = \deg(g,\Omega,p) &= \sum_{x\in g^{-1}(p)} \operatorname{sgn}(J_g(x)) \\ &= \sum_{i=1}^{N} \operatorname{sgn}(J_g(x_i)) \\ &= \sum_{i=1}^{K} \deg(g,\Omega_i,p), \quad \text{for } K \le N \\ &= \sum_{i=1}^{\infty} \deg(g,\Omega_i,p) \\ &= \sum_{i=1}^{\infty} \deg(f,\Omega_i,p)\end{aligned}$$

(b) First we show the statement true for $i \in \{1,2\}$, i.e,
If Ω_1 and Ω_2 are disjoint open subsets of Ω, such that $\Omega_1 \cup \Omega_2 \subset \Omega$ and $p \notin f(\overline{\Omega}\backslash(\Omega_1 \cup \Omega_2))$. Then,

$$\deg(f,\Omega,p) = \deg(f,\Omega_1,p) + \deg(f,\Omega_2,p) \tag{2.4}$$

Indeed, since $\Omega_1 \cup \Omega_2$ is open, then $\partial(\Omega_1 \cup \Omega_2)) \subset \overline{\Omega}\backslash(\Omega_1 \cup \Omega_2)$, and so $f(\partial(\Omega_1 \cup \Omega_2)) \subset f(\overline{\Omega}\backslash(\Omega_1 \cup \Omega_2))$, which implies that $p \notin f(\partial(\Omega_1 \cup \Omega_2))$ as $p \notin f(\overline{\Omega}\backslash(\Omega_1 \cup \Omega_2))$. Thus, by part (a)

$$\deg(f,\Omega_1 \cup \Omega_2,p) = \deg(f,\Omega_1,p) + \deg(f,\Omega_2,p).$$

First we consider the regular case, i.e, $p \notin S_f(\Omega)$. Then

$$\deg(f,\Omega,p) = \sum_{x\in f^{-1}(p)\cap\Omega} \operatorname{sgn} J_f(x) = \sum_{x\in f^{-1}(p)\cap(\Omega_1\cup\Omega_2)} \operatorname{sgn} J_f(x),$$

because, $x \in f^{-1}(p) \Rightarrow f(x) = p$ where $x \in \Omega$ is equivalent to $f(x) = p$ for $x \in \Omega_1 \cup \Omega_2$ as $f(x) \neq p$ for all $x \in \overline{\Omega}\backslash(\Omega_1 \cup \Omega_2)$. For the singular case we consider p as a singular value. By Preposition 2.3.1 there is $f' \in C(\overline{\Omega}) \cap C^1(\Omega)$ such that $p \notin S_{f'}(\Omega)$ and $\|f(x) - f'(x)\| < \operatorname{dist}(p,\partial\Omega) \le \|f(x) - p\|$ for all $x \in \partial\Omega$. By part (a) we have $\deg(f,\Omega_i,p)$ is well defined and by Property

(5) $\deg(f,\Omega_i,p)=\deg(f',\Omega_i,p)$ for $i=1,2$. Hence, again by Property (5) and the result in the regular case, we obtain

$$\begin{aligned}\deg(f,\Omega,p)&=\deg(f',\Omega,p)\\&=\deg(f',\Omega_1\cup\Omega_2,p)\\&=\deg(f,\Omega_1,p)+\deg(f,\Omega_2,p)\end{aligned}$$

We show the statement true for $i\in I$. i.e, if $\bigcup_{i\in I}\Omega_i\subset\Omega$ and $p\notin f(\overline{\Omega}\setminus\bigcup_{i\in I}\Omega_i)$, then

$$\deg(f,\Omega,p)=\sum_{i\in I}\deg(f,\Omega_i,p),$$

where only a finite number of terms is nonzero in the sum. Indeed, let $U=\bigcup_{i\in I}\Omega_i$ and $V=\Omega\setminus\overline{U}$ open subset of Ω. Since $\overline{\Omega}\setminus(U\cup V)\subset\overline{\Omega}\setminus U$, then $f(\overline{\Omega}\setminus(U\cup V))\subset f(\overline{\Omega}\setminus U)$, as $p\notin f(\overline{\Omega}\setminus U)$ then $p\notin f(\overline{\Omega}\setminus(U\cup V))$. By the result of case of two sets, we have as in (2.4),

$$\deg(f,\Omega,p)=\deg(f,U,p)+\deg(f,V,p). \tag{2.5}$$

Since $\overline{\Omega\setminus\overline{U}}\subset\overline{\Omega}\setminus U$, then

$$f(\overline{V})=f(\overline{\Omega\setminus\overline{U}})\subset f(\overline{\Omega}\setminus U)$$

which implies $p\notin f(\overline{V})$ as $p\notin f(\overline{\Omega}\setminus U)$. By the existence property (6), we get

$$\deg(f,V,p)=0. \tag{2.6}$$

Thus, by (2.5), (2.6) and part (a) we obtain

$$\begin{aligned}\deg(f,\Omega,p)&=\deg(f,U,p)\\&=\deg(f,\bigcup_{i\in I}\Omega_i,p)\\&=\sum_{i\in I}\deg(f,\Omega_i,p).\end{aligned}$$

(8) Since K is closed subset of Ω, then $\Omega\setminus K$ is open because Ω is open. In addition,

$$\begin{aligned}p\notin f(\partial\Omega)\cup f(K)&\Leftrightarrow p\notin f(\partial\Omega\cup K)\\&\Leftrightarrow p\notin f((\Omega\cup\partial\Omega)\setminus(\Omega\setminus K))\\&\Leftrightarrow p\notin f(\overline{\Omega}\setminus(\Omega\setminus K))\end{aligned}$$

Since, $\Omega\backslash K \subset \Omega$, then by the property (7) part (*b*) we have

$$\deg(f, \Omega, p) = \deg(f, \Omega\backslash K, p).$$

(9) Since x_0 is as isolated solution of equation $f(x) = p$, then there exist $r_0 > 0$ such that $f(x_0) = p$ and for all $x \in \overline{B}_{r_0}(x_0)\backslash\{x_0\}$, $f(x) \neq p$. For $0 < r < r_0$ let $\Omega = B_{r_0}(x_0)$, and

$$K = B_{r_0}(x_0)\backslash B_r(x_0) \subset \Omega.$$

Then K is closed in Ω because $K = \mathbb{R}^n\backslash B_r(x_0) \cap B_{r_0}(x_0)$, and $\mathbb{R}^n\backslash B_r(x_0)$ is closed in $\mathbb{R}^n$. Also $\Omega\backslash K = B_r(x_0)$ and $p \notin f(B_{r_0}(x_0)\backslash B_r(x_0)) \cup f(S_{r_0}(x_0))$, where $S_{r_0}(x_0) = \partial B_{r_0}(x_0)$. Indeed, because x_0 is the unique solution of $f(x) = p$ in $\overline{B}_{r_0}(x_0)$. Then by the excision property (Property (8))

$$\deg(f, B_{r_0}(x_0), p) = \deg(f, B_r(x_0), p).$$

(10) First we will consider the regular case, i.e, $p \notin S_f(U)$ and $q \notin S_g(V)$. By Preposition 2.1.2 we have

$$\deg(f, U, p) = \int_U \varphi_\varepsilon(f(x) - p) J_f(x) dx$$

and

$$\deg(g, V, p) = \int_V \psi_\varepsilon(g(x) - q) J_g(x) dx,$$

where suppφ_ε and suppψ_ε are subsets of $B(0, \varepsilon)$, also $\int_{\mathbb{R}^n} \varphi_\varepsilon(x)dx = \int_{\mathbb{R}^m} \psi_\varepsilon(x)dx = 1$, $\varphi_\varepsilon \in C(\mathbb{R}^n, \mathbb{R})$ and $\psi_\varepsilon \in C(\mathbb{R}^m, \mathbb{R})$. We define

$$\begin{aligned} \varphi_\varepsilon \times \psi_\varepsilon : &\mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}. \\ &(x, y) \longmapsto \varphi_\varepsilon(x)\psi_\varepsilon(y). \end{aligned}$$

Then $\varphi_\varepsilon \times \psi_\varepsilon \in C(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R})$. Let $(x, y) \in \text{supp}(\varphi_\varepsilon \times \psi_\varepsilon)$

$$\begin{aligned} &\Leftrightarrow \varphi_\varepsilon \times \psi_\varepsilon(x, y) \neq 0 \\ &\Leftrightarrow \varphi_\varepsilon(x) \neq 0 \ \text{ and } \ \psi_\varepsilon(y) \neq 0. \end{aligned}$$

$\Leftrightarrow x \in \text{supp}\varphi_\varepsilon$ and $y \in \text{supp}\psi_\varepsilon$. Hence, $\text{supp}\varphi_\varepsilon \times \psi_\varepsilon = \text{supp}\varphi_\varepsilon \times \text{supp}\psi_\varepsilon \subset B(0, \varepsilon) \times B(0, \varepsilon)$. By Fubini's Theorem, we have

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} \varphi_\varepsilon \times \psi_\varepsilon(x, y) dx dy = \int_{\mathbb{R}^n} \varphi_\varepsilon(x) dx \cdot \int_{\mathbb{R}^m} \psi_\varepsilon(y) dy = 1.$$

Again, by Proposition 2.1.2, we have by Fubini's Theorem

$$\begin{aligned}
\deg(f\times g, U\times V,(p,q)) = \\
&\int_{\mathbb{R}^n\times\mathbb{R}^m} \varphi_\varepsilon\times\psi_\varepsilon(f\times g(x,y)-(p,q))J_{f\times g}(x,y)dxdy. \\
&= \int_U\int_V \varphi_\varepsilon\times\psi_\varepsilon(f(x)-p,g(y)-q)J_{f\times g}(x,y)dxdy \\
&= \int_U\int_V \varphi_\varepsilon(f(x)-p)\psi_\varepsilon(g(y)-q)J_f(x)J_g(y)dxdy \\
&= \int_U \varphi_\varepsilon(f(x)-p)J_f(x)dx\cdot\int_V \psi_\varepsilon(g(y)-q)J_g(y)dy) \\
&= \deg(f,U,p)\cdot\deg(g,V,q).
\end{aligned}$$

Next, we extend the result to the singular case, by Definition 2.2.1 we have

$$\deg(f,\Omega,p)=\deg(f,\Omega,p'),$$

where $\|p-p'\|<\text{dist}(p,f(\partial\Omega))$ and p' is a regular value of f. Also,

$$\deg(g,\Omega,q)=\deg(g,\Omega,q'),$$

where $\|q-q'\|<\text{dist}(q,g(\partial\Omega))$ and q' is a regular value of g. Thus, by the first part of this proof and by Property (5), we have

$$\begin{aligned}
\deg(f\times g, U\times V,(p,q)) &= \deg(f\times g, U\times V,(p',q')) \\
&= \deg(f,U,p')\cdot\deg(g.V,q') \\
&= \deg(f,U,p')\cdot\deg(g.V,q').
\end{aligned}$$

(11) By Definition, 2.2.1 we have

$$\deg(f,\Omega,p)=\deg(f,\Omega,p'),$$

where $\|p-p'\|<\text{dist}(p,f(\partial\Omega))$ and p' is a regular value of f. Since p' is a regular value of f then $p'-q$ is a regular value of $f-q$. Indeed, let $x\in(f-q)^{-1}(p'-q)$, then

$$\begin{aligned}
&(f-q)(x)=p'-q \\
&\Leftrightarrow f(x)=p' \\
&\Leftrightarrow x\in f^{-1}(p') \\
&\Rightarrow J_f(x)\neq 0, \quad \text{as } p \text{ is regular value of } f \\
&\Rightarrow J_{f-q}(x)\neq 0, \quad \text{as } \ J_f=J_{f-q}.
\end{aligned}$$

Since $p \notin f(\partial\Omega)$, then $p-q \notin (f-q)(x)$, otherwise there exists $x_0 \in \partial\Omega$ such that $(f-q)(x_0) = p-q$, i.e, $f(x_0) = p$ contradiction. Hence $\deg(f-q, \Omega, p'-q)$ is well defined and,

$$\begin{aligned}
\deg(f, \Omega, p) &= \deg(f, \Omega, p') \\
&= \int_\Omega \varphi_\varepsilon(f(x) - p')J_f(x)dx, \quad (\text{where}\, \text{supp}\varphi_\varepsilon \subset \Omega) \\
&= \int_\Omega \varphi_\varepsilon(f(x) - q - (p' - q))J_{f-q}(x)dx \\
&= \deg(f - q, \Omega, p' - q).
\end{aligned}$$

That is,

$$\deg(f, \Omega, p) = \deg(f - q, \Omega, p' - q). \tag{2.7}$$

In addition, p and p' lie in a ball of $\mathbb{R}^n \backslash f(\partial\Omega)$ because $p' \in B(p, \text{dist}(p, f(\partial\Omega))) \subset \mathbb{R}^n \backslash f(\partial\Omega)$ by Property (3), since

$$\|p - q - (f - q)(x)\| = \|p - f(x)\| \quad \forall\, x \in \overline{\Omega}$$

$\text{dist}(p, f(\partial\Omega)) = \text{dist}(p - q, f - q(\partial\Omega))$, which implies that

$$\|(p' - q) - (p - q)\| = \|p' - q\| < \text{dist}(p, f(\partial\Omega))$$

Then by Definition 2.2.1 and (2.7), we have

$$\begin{aligned}
\deg(f, \Omega, p) &= \deg(f - q, \Omega, p' - q) \\
&= \deg(f - q, \Omega, p - q)
\end{aligned}$$

□

Corollary 2.3.1. *Let $f, g \in C^1(\Omega) \cap C(\bar{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is open, bounded subset and*

$$\|f(x) - g(x)\| < \text{dist}(p, f(\partial\Omega)) \qquad \forall\, x \in \Omega$$

then

$$\deg(f, \Omega, p) = \deg(g, \Omega, p).$$

Proof. Since

$$\|f(x) - g(x)\| < \text{dist}(p, f(\partial\Omega)) = \inf_{x \in \partial\Omega} \|f(x) - p\| = \min_{x \in \partial\Omega} \|f(x) - p\|,$$

for each $x \in \partial\Omega$, as $\partial\Omega$ is compact. Then by Property (5), we have

$$\deg(f, \Omega, p) = \deg(g, \Omega, p).$$

□

Corollary 2.3.2 (Invariance property on the boundary). *Let $f, g \in C^1(\Omega) \cap C(\bar{\Omega})$ where $\Omega \subset \mathbb{R}^n$ open be such that $f(x) = g(x)$ for each $x \in \partial\Omega$. Then*

$$\deg(f, \Omega, p) = \deg(g, \Omega, p).$$

Proof. Since

$$\|f(x) - g(x)\| = 0 < \|f(x) - p\|.$$

Then by Property (5), we have

$$\deg(f, \Omega, p) = \deg(g, \Omega, p).$$

□

2.4 Extension of Brouwer's Degree to Continuous Functions

Definition 2.4.1. *Let $f \in C(\overline{\Omega})$ and $p \notin f(\partial\Omega)$. We set*

$$\deg(f, \Omega, p) = \deg(f_\varepsilon, \Omega, p),$$

where $f_\varepsilon \in C^1(\Omega) \cap C(\overline{\Omega})$ is an approximation function of f is given in Lemma 1.2.3, and $\varepsilon < \operatorname{dist}(p, f(\partial\Omega))$.

Remark 2.4.1. (1) Note that this definition is well defined, for $p \notin f(\partial\Omega)$, as $\varepsilon < \operatorname{dist}(p, f(\partial\Omega))$. By the approximation lemma 1.2.3, there exists f_ε such that $\|f(x) - f_\varepsilon(x)\| < \varepsilon$ for all $x \in \partial\Omega$. Then,

$$\begin{aligned}
\|f_\varepsilon(x) - p\| &= \|f_\varepsilon(x) - f(x) + f(x) - p\| \\
&\geq \Big| \|f_\varepsilon(x) - f(x)\| - \|f(x) - p\| \Big| \\
&= \|f(x) - p\| - \|f_\varepsilon(x) - f(x)\| \\
&> \|f(x) - p\| - \varepsilon \geq 0.
\end{aligned}$$

That is $p \notin f_\varepsilon(\partial\Omega)$.
(2) This definition does not depend on the choice of the function f_ε. Since, $p \notin f(\partial\Omega)$, let $0 < \varepsilon < \inf_{x\in\partial\Omega} \frac{\|f(x)-p\|}{2}$, the infimum exist and

$$\inf_{x\in\partial\Omega} \frac{\|f(x) - p\|}{2} = \min_{x\in\partial\Omega} \frac{\|f(x) - p\|}{2}$$

because $\partial\Omega$ is compact. Suppose that f_ε and g_ε are approximation functions of f, then $\|f(x) - f_\varepsilon(x)\| < \varepsilon$ and $\|f(x) - g_\varepsilon(x)\| < \varepsilon$, we show that $\deg(f_\varepsilon, \Omega, p) = \deg(g_\varepsilon, \Omega, p)$. Indeed, consider $H_t(x) = tf_\varepsilon(x) + (1-t)g_\varepsilon(x)$, where $t \in [0,1]$. Then, H is continuously differentiable and,

$$\begin{aligned}\|H_t(x) - f(x)\| &= \|tf_\varepsilon(x) + (1-t)g_\varepsilon(x) - f(x)\| \\ &= \|t(f_\varepsilon(x) - f(x)) + (1-t)(g_\varepsilon(x) - f(x))\| \\ &\leq t\|f_\varepsilon - f(x)\| + (1-t)\|g_\varepsilon(x) - f(x)\| \\ &< t\varepsilon + (1-t)\varepsilon = \varepsilon.\end{aligned}$$

Then, for every $x \in \partial\Omega$, we have

$$\begin{aligned}\|H_t(x) - p\| &\geq \Big| \|H_t(x) - f(x)\| - \|f(x) - p\| \Big| \\ &= \|f(x) - p\| - \|H_t(x) - f(x)\| \\ &> 2\varepsilon - \varepsilon = \varepsilon > 0.\end{aligned}$$

Hence, $\|H_t(x) - p\| > 0$ which equivalent saying that, $p \notin H_t(\partial\Omega)$ and using Theorem 2.3.1 Property (4), we get the desired result.

Remark 2.4.2. Thanks to this extension we can reduce the condition on f to be continuous Theorem 2.3.1.

We give now an additional property of Brouwer's degree.

Theorem 2.4.1. (Invariance on connected components) *Let f be continuous, p_1 and p_2 are in the same component C of $\mathbb{R}^n \backslash f(\partial\Omega)$. Then*

$$\deg(f, \Omega, p_1) = deg(f, \Omega, p_2).$$

Proof. Let C be component of $\mathbb{R}^n \backslash f(\partial\Omega)$ containing p_1 and p_2. Since, $\mathbb{R}^n \backslash f(\partial\Omega)$ is open, then C is path component by Theorem 1.1.24. Thus, there exists a continuous path

$$\gamma : [0,1] \longrightarrow C,$$

such that $\gamma(0) = p_1$ and $\gamma(1) = p_2$. Since $C \subset \mathbb{R}^n \backslash f(\partial\Omega)$, then $f(\partial\Omega) \subset \mathbb{R}^n \backslash C$ and as $p_1, p_2 \in \gamma[0,1] \subset C$. Then

$$p_1, p_2 \notin f(\partial\Omega).$$

Since γ is continuous (after reducing the condition to be continuous), we apply Property (4) to get, $\deg(f, \Omega, \gamma(t))$ is constant on $[0, 1]$. In particular,

$$\deg(f, \Omega, \gamma(0)) = \deg(f, \Omega, \gamma(1))$$
$$\Leftrightarrow \deg(f, \Omega, p_1) = \deg(f, \Omega, p_2).$$

□

Remark 2.4.3. Since the ball is connected, then Theorem 2.4.1 is a generalization of Property (3) in Theorem 2.3.1.

2.5 Brouwer's Fixed Point Theorem and Applications

Theorem 2.5.1 (Brouwer's Fixed Point Theorem). *Let $f : \overline{B_R(0)} \subset \mathbb{R}^n \longrightarrow \overline{B_R(0)}$ be a continuous map. Then f has a fixed point in $\overline{B_R(0)}$.*

Proof. If there exists a fixed point on $\partial B_R(0)$ the proof is done. Otherwise, $f(x) \neq x$ for all $x \in \partial B_R(0)$. Set $H(x, t) = x - tf(x) = (\mathrm{Id} - tf)(x)$ for all $t \in [0, 1]$ and $x \in \overline{B_R(0)}$, then $H(x, 0) = x$, $H(x, 1) = x - f(x)$ and H_t is continuous as f is continuous. We show that $0 \notin H_t(\partial B_R(0))$. Suppose by contradiction that, there exists $x_0 \in \partial B_R(0)$ such that $H(t_0, x_0) = 0$ for some $t_0 \in [0, 1]$; then $x_0 = t_0 f(x_0)$. Since, $x_0 \in \partial B_R(0) = S_R(0)$ i.e, $R = \|x_0\| = \|t_0 f(x_0)\| = t_0 \|f(x_0)\|$. Then $t_0 > 0$ as $R \neq 0$. Also as $f(x_0) \neq x_0$, then $t_0 \neq 1$ i.e, $0 < t_0 < 1$ which is implies that $R = t_0 \|f(x_0)\| < \|f(x_0)\| \leq R$, as $f(x_0) \in \overline{B_R(0)}$, which is a contradiction. Hence, $0 \notin H_t(\partial B_R(0))$ and by Property (4), $\deg(H_t, B_R(0), 0)$ is well-defined and is constant for all $t \in [0, 1]$. In particular,

$$\deg(H(x, 1), B_R(0), 0) = \deg(H(x, 0), B_R(0), 0)$$
$$\Leftrightarrow \deg(\mathrm{Id} - f, B_R(0), 0) = \deg(\mathrm{Id}, B_R(0), 0) = 1$$

Thus $\deg(\mathrm{Id} - f, B_R(0), 0) \neq 0$ and by the existence property (6), there exists $x \in B_R(0)$ such that $f(x) = x$. Therefore, in all cases f has a fixed point in $\overline{B_R(0)}$. □

Remark 2.5.1. In dimension $n = 1$, Theorem 2.5.1 follows from the Intermediate Value Theorem. Consider $g(x) = f(x) - x$, since $f : [a, b] \longrightarrow [a, b]$

is continuous, then g is continuous, $g(a) \geq 0$ and $g(b) \leq 0$. If $g(a) = 0$ or $g(b) = 0$ the proof is done. Otherwise, $g(a) > 0$ and $g(b) < 0$, by the The Intermediate Value Theorem, there exist $a < x < b$ such that $g(x) = 0$ i.e, $f(x) = x$.

Theorem 2.5.2. *Let $C \in \mathbb{R}^n$ be nonempty bounded closed convex subset and $f : C \longrightarrow C$ be a continuous map. Then f has a fixed point in C.*

Proof. Since C is bounded, let $B(0, R)$ be such that $C \subset B(0, R)$, also as C is closed convex set then by Dugundji's Retraction Theorem 1.2.1, there is a retraction $h : X \longrightarrow C$, we take the restriction on $\overline{B_R(0)}$, which is $h|_{\overline{B_R(0)}} = r : \overline{B_R(0)} \longrightarrow C$. Thus, $f \circ r : \overline{B_R(0)} \longrightarrow C \subset \overline{B_R(0)}$ is a continuous map. Then by Theorem 2.5.1 there is $x_0 \in \overline{B_R(0)}$ such that $f(r(x_0)) = x_0$. If $x_0 \notin C$ then $(f \circ r)(x_0) = x_0 \notin C$ contradiction. Thus, $x_0 \in C$ and so $f(x_0) = f(r(x_0)) = x_0$ i.e, f has a fixed point in C. □

Theorem 2.5.3. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous map and $0 \in \Omega \subset \mathbb{R}^n$ where Ω is open bounded subset. If the inner product $\langle f(x), x \rangle > 0$ for all $x \in \partial\Omega$, then*

$$\deg(f, \Omega, 0) = 1$$

Proof. Set $H(x, t) = tx + (1-t)f(x)$ continuous and $H(x, 0) = f(x)$, $H(x, 1) = x = \mathrm{Id}(x)$. We show that $0 \notin H_t(\partial\Omega)$. Suppose by contradiction there is $x_0 \in \partial\Omega$ such that $x_0 \in H_t(\partial\Omega)$, i.e, $tx_0 + (1-t)f(x_0) = 0$, then $t > 0$. Otherwise, $f(x_0) = 0$ and this contradicts the fact $\langle f(x), x \rangle > 0$ for all $x \in \partial\Omega$. Similarly if $t = 1$ we have $x_0 = 0$ which is again a contradiction and so

$$\begin{aligned}
&\langle x_0 t + (1-t)f(x_0), x_0 \rangle = 0 \\
&\Rightarrow t \langle x_0, x_0 \rangle + (1-t) \langle f(x_0), x_0 \rangle = 0 \\
&\Rightarrow (1-t) \langle f(x_0), x_0 \rangle = -t\|x_0\|^2 < 0.
\end{aligned}$$

Since $(1-t) > 0$, then $\langle f(x_0), x_0 \rangle < 0$, which is a contradiction. Then by Property (4), we have $\deg(H_t, \Omega, 0)$ constant for all $t \in [0, 1]$. In particular

$$\deg(H_0, \Omega, 0) = \deg(H_1, \Omega, 0) \Rightarrow \deg(f, \Omega, 0) = \deg(\mathrm{Id}, \Omega, 0) = 1.$$

□

Corollary 2.5.1. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous map. If*

$$\lim_{\|x\| \to \infty} \frac{\langle f(x), x \rangle}{\|x\|} = \infty,$$

then $f(\mathbb{R}^n) = \mathbb{R}^n$.

Proof. By the hypothesis and using the definition of the limit, there is $R > 0$ such that $\|x\| \geq R$ implies $\frac{\langle f(x),x\rangle}{\|x\|} > 0$ and by Theorem 2.5.3 we get $\deg(f, B_R(0), 0) = 1$, then by Property (6) there is $x \in B_R(0) \subset \mathbb{R}$ such that $f(x) = 0$. If $p \in \mathbb{R}^n$ such that $\|p\| > 0$, then again by the hypothesis there exists $R > 0$ such that for all $\|x\| \geq R$,

$$\begin{aligned} \frac{\langle f(x),x\rangle}{\|x\|} > \|p\| \quad &\Rightarrow \quad \frac{\langle f(x),x\rangle}{\|x\|} - \|p\| > 0 \\ &\Rightarrow \quad 0 < \frac{\langle f(x),x\rangle - \|x\|\|p\|}{\|x\|} \\ &\qquad \leq \frac{\langle f(x),x\rangle - \langle x,p\rangle}{\|x\|} \\ &\qquad = \frac{\langle f(x)-p,x\rangle}{\|x\|}, \end{aligned}$$

where we have used Cauchy-Schwarz Inequality (Theorem 1.2.2). Thus, $\langle f(x) - p, x\rangle > 0$ for all $x \in S_R(0)$. By Theorem 2.5.3 we have

$$\deg(f - p, B_R(0), 0) = 1$$

Thus by the existence property (6), there exists $x_0 \in B_R(0) \subset \mathbb{R}^n$ such that $f(x_0) - p = 0$ i.e, $f(x_0) = p$ and since p was arbitrary then $f(\mathbb{R}^n) = \mathbb{R}^n$. □

Theorem 2.5.4 (Non-retraction of the sphere). *There is no retraction of the closed ball of $\mathbb{R}^n$ onto its boundary.*

Proof. Suppose by contradiction that there exists a retraction $r : \overline{B_R(0)} \longrightarrow S_R(0)$. Then, $-r : \overline{B_R(0)} \longrightarrow S_R(0) \subset \overline{B_R(0)}$ is a continuous map. By Brouwer's fixed point theorem 2.5.1 there is exists $x_0 \in \overline{B_R(0)}$ such that $-r(x_0) = x_0$. Then $x_0 \in S_R(0)$, since r is a retraction, $r(x_0) = x_0$. Hence, $x_0 = -x_0$ then $x_0 = 0$ which is a contradicts the fact $\|x_0\| = R > 0$. □

Theorem 2.5.5. *Theorem 2.5.4 implies Brouwer's fixed point theorem.*

Proof. Let $f : \overline{B_R(0)} \longrightarrow \overline{B_R(0)}$ be a continuous map. By contradiction, suppose that f has no fixed point, i.e, $f(x) \neq x$ for all $x \in \overline{B_R(0)}$. Define the map $r : \overline{B_R(0)} \longrightarrow \overline{B_R(0)}$ by

$$r(x) = [f(x), x) \cap S_R(0),$$

where $[f(x), x)$ is the ray (a half-line) originated from $f(x)$. More precisely

$$r(x) = tx + (1 - t)f(x),$$

where $t > 0$ is such that $\|r(x)\| = R$. We can find the value $t = t(x)$. We have,

$$\begin{aligned}
\|r(x)\| = R &\Leftrightarrow \langle r(x), r(x) \rangle = R^2 \\
&\Leftrightarrow \langle f(x) + t(x - f(x)), f(x) + t(x - f(x)) \rangle = R^2 \\
&\Leftrightarrow \|f(x)\|^2 + t^2 \|x - f(x)\|^2 + 2t \langle f(x), x - f(x) \rangle = R^2
\end{aligned}$$

$$\Leftrightarrow t^2\|x-f(x)\|^2+2t \langle f(x), x - f(x)\rangle+\|f(x)\|^2-R^2 = 0. \quad (2.8)$$

which is a second order equation because $\|f(x) - x\| \neq 0$. The reduced discriminant of this second-order equation is

$$\Delta' = | \langle f(x), x - f(x) \rangle |^2 - \|x - f(x)\|^2(\|f(x)\|^2 - R^2).$$

Since $\|f(x)\| \leq R$, then $\Delta' \geq 0$. Since $\|f(x)\|^2 - R^2 \leq 0$, we deduce that (2.8) has two roots of opposite signs, $t_1 \leq 0 < t_2$

$$t_1 = t_1(x) = \frac{- \langle f(x), x - f(x) \rangle + \sqrt{\Delta'}}{\|x - f(x)\|^2} > 0. \quad (2.9)$$

Since f is continuous, then t_1 is continuous and thus $r(x) = t(x)+(1-t)f(x)$ is a continuous function. It remains to check that $x \in S_R(0)$ implies $r(x) = x$. First, note that $r(x)-tx = (1-t)(f(x)-x)$. Since $f(x) \neq x$, for all $x \in \overline{B_R(0)}$ by assumption, then $r(x) = x \Leftrightarrow t = 1$. For the expression of t_1 stated in (2.9), and as $\|x - f(x)\| \neq 0$ we have,

$$\begin{aligned}
& t_1 = 1 \\
&\Leftrightarrow - \langle f(x), x - f(x) \rangle + \sqrt{\Delta'} = \|x - f(x)\|^2 \\
&\Leftrightarrow \sqrt{\Delta'} = \|x - f(x)\|^2 + \langle f(x), x - f(x) \rangle \\
&\Leftrightarrow \Delta' = \|x - f(x)\|^4 + | \langle f(x), x - f(x) \rangle |^2 + 2\|x - f(x)\|^2 \cdot \langle f(x), x - f(x) \rangle \\
&\Leftrightarrow | \langle f(x), x - f(x) \rangle |^2 - \|x - f(x)\|^2(\|f(x)\|^2 - R^2) \\
&= \|x - f(x)\|^4 + | \langle f(x), x - f(x) \rangle |^2 + 2\|x - f(x)\|^2 \langle f(x), x - f(x) \rangle \\
&\Leftrightarrow \|x - f(x)\|^2 + 2 \langle f(x), x - f(x) \rangle + \|f(x)\|^2 - R^2 = 0 \\
&\Leftrightarrow \|x - f(x)\|^2 + 2 \langle f(x), x \rangle - 2\|f(x)\|^2 + \|f(x)\|^2 - R^2 = 0 \\
&\Leftrightarrow \|x - f(x)\|^2 + 2 \langle f(x), x \rangle - \|f(x)\|^2 - R^2 = 0 \\
&\Leftrightarrow \|x\|^2 + \|f(x)\|^2 - 2 \langle f(x), x \rangle + 2 \langle f(x), x \rangle - \|f(x)\|^2 - R^2 = 0 \\
&\Leftrightarrow \|x\|^2 - R^2 = 0 \\
&\Leftrightarrow \|x\| = R.
\end{aligned}$$

Therefore, $t = 1 \Leftrightarrow r(x) = x \Leftrightarrow \|x\| = R \Leftrightarrow x \in S_R(0)$. Hence, r is a retraction which contradict Theorem 2.5.4. $\square$

Remark 2.5.2. Theorem 2.5.4 is an equivalent version to Brouwer's fixed point theorem 2.5.1.

Theorem 2.5.6 (Poincaré-Böhl Theorem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset and $f, g : \overline{\Omega} \longrightarrow \mathbb{R}^n$ two continuous functions such that*

$$f(x) + \lambda g(x) \neq 0,\ \forall\, \lambda \geq 0,\ \forall\, x \in \partial\Omega \quad \text{and} \quad 0 \notin g(\partial\Omega).$$

Then, $0 \notin f(\partial\Omega)$ *and* $\deg(f, \Omega, 0) = \deg(g, \Omega, 0)$.

Proof. Consider the homotopy

$$H(x,t) = tf(x) + (1-t)g(x),\ t \in [0,1],\ x \in \overline{\Omega}.$$

(a) For $t = 0$, $H(x, 0) = g(x) \neq 0$ for all $x \in \partial\Omega$ by assumption.
(b) For $t \neq 0$, and $x \in \partial\Omega$,

$$H(x,t) = 0 \Leftrightarrow f(x) + \frac{1-t}{t} g(x) = 0,$$

but this contradicts the hypothesis of the theorem with $\lambda = \frac{1-t}{t}$. Hence, $H(x,t) \neq 0$ for all $x \in \partial\Omega$ and for any $t \in [0,1]$, then $\deg(H(x,\cdot), \Omega, 0)$ is well defined and $0 \notin f(\partial\Omega)$. Otherwise, $f(x) = H(x,1) = 0$ for some $x \in \partial\Omega$ contradiction. Therefore, by Property (4),

$$\deg(H(x,0), \Omega, 0)) = \deg(H(x,1), \Omega, 0),$$

proving the claim. $\square$

Remark 2.5.3. This theorem is stronger than Property (5) with $p = 0$. Indeed, if

$$\|f(x) - g(x)\| < \|g(x)\|\ \forall\, x \in \partial\Omega$$

then,

$$f(x) + \lambda g(x) \neq 0\ \forall\, x \in \partial\Omega \ \text{ and }\ \forall\, \lambda \geq 0$$

for otherwise, there is $x \in \partial\Omega$ and $\lambda \geq 0$ such that

$$f(x) + \lambda g(x) = 0 \Leftrightarrow f(x) - g(x) = -(1+\lambda) g(x),$$

then

$$\|f(x) - g(x)\| = (1+\lambda)\|g(x)\| < \|g(x)\|,$$

a contradiction.

Corollary 2.5.2. *Suppose that* $0 \notin \partial\Omega$ *and either*

$$f(x) + \lambda x \neq 0,\ \forall\, \lambda \geq 0,\ \forall\, x \in \partial\Omega$$
$$or\ \ f(x) + \lambda x \neq 0,\ \forall\, \lambda \leq 0,\ \forall\, x \in \partial\Omega'$$

Then,
(a) $f(x) = 0$ *has at least one solution in* Ω.
(b) f *has at least one fixed point* x *in* Ω.

Proof. (1) By applying Pointcaré-Böhl Theorem 2.5.6 with $f = \mathrm{Id}$ and $g = -\mathrm{Id}$, respectively we get

$$\deg(f, \Omega, 0) = \deg(\mathrm{Id}, \Omega, 0) \neq 0$$
$$\deg(f, \Omega, 0) = \deg(-\mathrm{Id}, \Omega, 0) \neq 0.$$

By the existence property of the degree (6), the equation $f(x) = 0$, has at least one solution $x \in \Omega$. (2)

$$\begin{aligned}
&f(x) + \lambda x \neq 0,\ \forall\, \lambda \leq 0,\ \forall\, x \in \partial\Omega \\
&\Leftrightarrow f(x) - x + x + \lambda x \neq 0,\ \forall\, \lambda \leq 0,\ \forall\, x \in \partial\Omega \\
&\Leftrightarrow (f(x) - x) + (1 + \lambda)x \neq 0,\ \forall\, \lambda \leq 0,\ \forall\, x \in \partial\Omega \\
&\Leftrightarrow (f(x) - x) + \lambda' x \neq 0,\ \forall\, \lambda' \leq 1,\ \forall\, x \in \partial\Omega \\
&\Rightarrow (f(x) - x) + \lambda' x \neq 0,\ \forall\, \lambda' \leq 0,\ \forall\, x \in \partial\Omega
\end{aligned}$$

By Part (1), applied to the function $(f - \mathrm{Id})$, there exists $x \in \Omega$ such that

$$f(x) - x = 0 \Leftrightarrow f(x) = x$$

□

Corollary 2.5.3. *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a continuous function such that*

$$\lim_{\|x\| \to \infty} \frac{\langle f(x), x \rangle}{\|x\|} = \infty.$$

then f *is surjective.*

Proof. Let $y_0 \in \mathbb{R}^n$ and $g(x) = f(x) - y_0$. Note that

$$\begin{aligned}
\frac{\langle g(x), x \rangle}{\|x\|} &= \frac{\langle f(x) - y_0, x \rangle}{\|x\|} \\
&= \frac{\langle f(x), x \rangle}{\|x\|} - \frac{\langle y_0, x \rangle}{\|x\|},
\end{aligned}$$

Since

$$\frac{|\langle y_0, x\rangle|}{\|x\|} \leq \frac{\|y_0\|\|x\|}{\|x\|} = \|y_0\| < \infty,$$

then $\frac{\langle y_0,x\rangle}{\|x\|}$ is bounded as $\|x\| \to \infty$, which implies that

$$\frac{\langle g(x), x\rangle}{\|x\|} = \frac{\langle f(x), x\rangle}{\|x\|} - \frac{\langle y_0, x\rangle}{\|x\|} \longrightarrow \infty, \quad \text{as } \|x\| \to \infty$$

by the definition of the limit, there exists $R > 0$ such that $\langle g(x), x\rangle > 0$, for all $\|x\| = R$. Hence,

$$g(x) + \lambda x \neq 0, \forall\, \lambda > 0$$

for otherwise,

$$\langle g(x) + \lambda x, x\rangle = \langle g(x), x\rangle + \lambda\|x\|^2 = 0$$
$$\Rightarrow \langle g(x), x\rangle \leq 0$$

which is a contradiction. By Corollary 2.5.2, there exists $x_0 \in \Omega$ such that $g(x_0) = 0 \Leftrightarrow f(x_0) = y_0$, thus, f is onto. □

Remark 2.5.4. This corollary was already derived from Brouwer's fixed point theorem in Corollary 2.5.1.

Theorem 2.5.7 (Perron-Frobeninus Theorem). *Let* $A = (a_{ij})_{1\leq i,j\leq 0}$ *be an* $(n \times n)$ *matrix such that* $a_{ij} \geq 0$*, for all* $1 \leq i, j \leq n$*. Then* A *has at least one non-negative eigenvalue corresponding to a non-negative eigenvector (i.e, with non-negative components).*

Proof. Let

$$C = \left\{ x \in \mathbb{R}^n,\ x_i \geq 0,\ \forall\, i \in [1, n] \ \text{ and } \ \sum_{i=1}^{n} x_i = 1 \right\}.$$

Then C is a bounded closed convex subset of $\mathbb{R}^n$. Since $x_i \in [0, 1]$ for all $i \in [1, n]$, then $x \in [0, 1]^n$ and so $C \subset [0, 1]^n$ is bounded, for closeness we take a sequence $x_k \in C$ such that x_k converge to x, as $k \to \infty$. Hence,

$$x_{k_i} \to x_i \quad \forall\, i \in [1, n]$$
$$\Rightarrow \sum_{i=1}^{n} x_{k_i} \to \sum_{i=1}^{n} x_i.$$

Since, $\sum_{i=1}^{n} x_{k_i} = 1$ then
$$\sum_{i=1}^{n} x_{k_i} = 1 \to 1,$$
by the uniqueness of the limit we have
$$\sum_{i=1}^{n} x_i = 1,$$
i.e, $x \in C$, thus C is closed. Now, to show convexity consider $x, y \in C$, then for any $t \in [0,1]$ we have $tx_i \geq 0$ and $(1-t)y_i \geq 0$ for all $i \in [1,n]$, then $tx_i + (1-t)y_i \geq 0$ for all $i \in [1,n]$, also
$$\begin{aligned}\sum_{i=1}^{n} tx_i + (1-t)y_i &= t\sum_{i=1}^{n} x_i + (1-t)\sum_{i=1}^{n} y_i \\ &= t(1) + (1-t)(1) = 1.\end{aligned}$$
Therefore, $tx + (1-t)y \in C$ i.e, C is convex. If there is some $x_0 \in C$ such that $Ax_0 = 0$, then we are done, for $Ax_0 = 0 \cdot x_0$. Otherwise, $Ax \neq 0$ for all $x \in C$, then $(Ax)_{i_0} > 0$ for some $1 \leq i_0 \leq n$. Thus, $\sum_{i=1}^{n}(Ax)_i > 0$ and we define
$$f(x) = \frac{1}{\sum_{i=1}^{n}(Ax)_i} \cdot Ax.$$
Then, f is continuous, as $(f(x))_i = \frac{1}{\sum_{i=1}^{n}(Ax)_i} \cdot (Ax)_i \geq 0$ is continuous for all $i \in [1,n]$. In addition
$$\sum_{i=1}^{n}(f(x))_i = \frac{1}{\sum_{i=1}^{n}(Ax)_i} \cdot \sum_{i=1}^{n}(Ax)_i = 1.$$
Hence,
$$f : C \longrightarrow C$$
By Brouwer's fixed point, there is $x_0 \in C$ such that $f(x_0) = x_0$ i.e,
$$\frac{1}{\sum_{i=1}^{n}(Ax_0)_i} \cdot Ax_0 = x_0$$
$$\Leftrightarrow Ax_0 = \left(\sum_{i=1}^{n}(Ax_0)_i\right) x_0.$$
Hence, A has $\lambda = \sum_{i=1}^{n}(Ax_0)_i > 0$ as eigenvalue with eigenvector $x_0 \in C$ (with non-negative components). $\square$

Chapter 3

Leray-Schauder Degree (the infinite-dimensional case)

In this chapter, we move to the infinite case and we introduce the Leray-Schauder degree and its construction. In addition, we present the main properties associated to this tool. For the results in this chapter, we refer to [1, 4, 6].

3.1 Introduction

First we begin with the definition of completely continuous map and the approximation of it by finite dimensional mapping.

Definition 3.1.1. *Let E, F be two normed spaces and let $\Omega \subset E$. We say that $K : \Omega \longrightarrow F$ is completely continuous if K is continuous, and $\overline{K(A)}$ compact set for all bounded set $A \subset \Omega$.*

Definition 3.1.2. *A compact perturbation of the identity is a map of the form $\mathrm{Id} - K$ where K is a completely continuous map.*

Definition 3.1.3. *Let E be a normed space and let $\Omega \subset E$. Ω is said to be of finite dimension if Ω is contained in a linear subspace of E of finite dimension.*

Theorem 3.1.1. *Let E be a normed space, and $\Omega \subset E$ be a bounded subset and $K : \Omega \longrightarrow E$ a completely continuous mapping. Then for all $\varepsilon > 0$, there* *exist a finite dimensional space F_ε and a continuous mapping $K_\varepsilon : \Omega \longrightarrow F_\varepsilon$ such that*

$$\|K_\varepsilon(x) - K(x)\| < \varepsilon \ \ \forall\, x \in \Omega,$$

and $K_\varepsilon(\Omega) \subset \mathrm{Co}(K(\Omega))$, where Co *refer to the convex hull, which is the smallest convex set containing $K(\Omega)$.*

Proof. Let $\varepsilon > 0$. We have

$$\overline{K(\Omega)} \subset \bigcup_{y \in K(\Omega)} B(y, \varepsilon).$$

Since Ω is bounded, $\overline{K(\Omega)}$ is compact. Then, there exist finite number of points $\{y_1, y_2 \ldots, y_n\} \in K(\Omega)$ such that

$$\overline{K(\Omega)} \subset \bigcup_{i=1}^{n} B(y_i, \varepsilon).$$

Assume, $m_i(x, \varepsilon) = \max\{0, \varepsilon - \|K(x) - y_i\|\}$ where $x \in \Omega$. We set

$$\theta_i(x, \varepsilon) = \frac{m_i(x, \varepsilon)}{\sum_{j=1}^{n} m_j(x, \varepsilon)}\ ,\ x \in \Omega.$$

Since $m_i(\cdot, \varepsilon)$ is continuous then $\theta_i(\cdot, \varepsilon) : \Omega \longrightarrow \mathbb{R}$ is continuous. Let $x \in \Omega$ then there exists $j \in \{1, 2 \ldots, n\}$ such that $K(x) \in B(y_j, \varepsilon)$ i.e, $\|K(x) - y_j\| < \varepsilon$. It follows that $m_j(x, \varepsilon) > 0$. Hence,

$$\sum_{j=1}^{n} m_j(x, \varepsilon) > 0.$$

Therefore, $\theta_i(\cdot, \varepsilon)$ is well defined and continuous. Now we define $K_\varepsilon : \Omega \longrightarrow F_\varepsilon = span\,\{y_1, y_2 \ldots, y_n\}$ as the following

$$K_\varepsilon(x) = \sum_{j=1}^{n} \theta_j(x, \varepsilon) y_j.$$

Since

$$\sum_{j=1}^{n} \theta_j(x, \varepsilon) = 1,$$

we have

$$\|K(x) - K_\varepsilon(x)\| = \left\| \sum_{j=1}^{n} \theta_j(x,\varepsilon)K(x) - \sum_{j=1}^{n} \theta_j(x,\varepsilon)y_j \right\|$$
$$= \sum_{j=1}^{n} \theta_j(x,\varepsilon)\|K(x) - y_j\| < \varepsilon.$$

Since for each $x \in \Omega$, $K_\varepsilon(x)$ is a convex combination of elements y_j of $K(\Omega)$, we conclude that $K_\varepsilon(\Omega) \subset \mathrm{Co}(K(\Omega))$. For the proof of the approximation lemma, we also refer to [2, Proposition (2.2) and Theorem (2.3), Page 117]. □

Lemma 3.1.1. Let X be a metric space, $\Omega \subset X$ a bounded subset, and $K : \Omega \longrightarrow X$ a completely continuous mapping. Then $f = \mathrm{Id} - K$ is a closed map.

Proof. Let $A \subset \Omega$ be a closed subset and $B = f(A)$. We want to show that B is closed. Let $\{y_n\}_n \subset B$ be a sequence which converges to some point $y \in X$. Then, there exists a sequence $\{x_n\}_n \subset A$ such that,

$$y_n = f(x_n),\ \forall\, n$$
$$\Leftrightarrow y_n = x_n - K(x_n).$$

Since Ω is bounded, A is bounded, and since K is completely continuous then, the closure of $K(A)$ is compact. Hence, the closure of $\{K(x_n)\}_n$ is compact, then $\{K(x_n)\}_n$ is sequentially compact by Theorem 1.1.18 and so there is a subsequence $K(x_{n_k} : k \in \mathbb{N})$ which converges to some point $z \in X$. Hence,

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} (y_{n_k} + K(x_{n_k})) = y + z.$$

Since A is closed, $y + z \in A$. $\lim_{k\to\infty} K(x_{n_k}) = K(y + z)$ because K is continuous. By the uniqueness of the limit we have

$$z = K(y + z).$$

Let $x = y + z \in A$, then

$$y = x - z = x - K(y + z) = x - K(x) = (\mathrm{Id} - K)(x) = f(x).$$

Therefore, $y \in f(A) = B$. □

Corollary 3.1.1. *Let X be a metric space, suppose $\Omega \subset X$ is a bounded open subset. Suppose $K : \overline{\Omega} \longrightarrow X$ is completely continuous and $p \notin f(\partial\Omega)$, where $f = \mathrm{Id} - K$. Then* $\mathrm{dist}(p, f(\partial\Omega)) > 0$.

Proof. By Lemma 3.1.1, $f(\partial\Omega)$ is closed in X. Then, $\overline{f(\partial\Omega)} = f(\partial\Omega)$ and $p \notin f(\partial\Omega) \Leftrightarrow p \notin \overline{f(\partial\Omega)}$. Then, $\mathrm{dist}(p, \partial\Omega) > 0$ using Corollary 1.1.2. □

Lemma 3.1.2. Let E be a normed space, $B \subset E$ a closed bounded subset, and $K : B \longrightarrow E$ a completely continuous mapping. Suppose that $K(x) \neq x$, for all $x \in B$. Then there exists $\varepsilon_0 > 0$ for all $\varepsilon_i \in (0, \varepsilon_0)$, where $i = 1, 2$ for any $t \in [0, 1]$ and $x \in B$, such that $x \neq tK_{\varepsilon_1}(x) + (1 - t)K_{\varepsilon_2}(x)$ where, $K_{\varepsilon_i} : B \longrightarrow F_{\varepsilon_i}$ are approximation maps of K as given in Theorem 3.1.1.

Proof. Suppose by contradiction that for any $\varepsilon > 0$ there exist $\varepsilon_i \in (0, \varepsilon)$ for $i = 1, 2$, and some $t_\varepsilon \in [0, 1]$ and $x_\varepsilon \in B$, such that

$$x_\varepsilon = t_\varepsilon K_{\varepsilon_1}(x_\varepsilon) + (1 - t_\varepsilon)K_{\varepsilon_2}(x_\varepsilon).$$

In particular, for $\varepsilon = \frac{1}{j}$ for $j = 1, 2, \dots$, there exist $0 < \varepsilon_i^j < \frac{1}{j}$ for $i = 1, 2$, and there exist $t_j \in [0, 1]$ and $x_j \in B$ such that

$$x_j = t_j K_{\varepsilon_1^j}(x_j) + (1 - t_j)K_{\varepsilon_2^j}(x_j).$$

Since $[0, 1]$ is compact the sequence $(t_j)_j$ has a convergent subsequence still denoted $(t_j)_j$ which converges to $t_0 \in [0, 1]$. Since K is completely continuous, then the closure of $\{K(x_j)\}_{j=1}^{\infty}$ is compact and so it is sequentially compact. By Theorem 1.1.18 it follows that $\{K(x_j)\}_{j=1}^{\infty}$ has a convergent subsequence, say $K(x_{j_k}) \longrightarrow y \in E$, as $k \to \infty$. By Theorem 3.1.1, $\|K(x_{j_k}) - K_{\varepsilon_i^{j_k}}(x_{j_k})\| < \varepsilon_i^{j_k}$ and as $\varepsilon_1^{j_k} \to 0$, $\varepsilon_2^{j_k} \to 0$, we have

$$\begin{aligned} \|K_{\varepsilon_i^{j_k}}(x_{j_k}) - y\| &= \|K_{\varepsilon_i^{j_k}}(x_{j_k}) - K(x_{j_k}) + K(x_{j_k}) - y\| \\ &\leq \|K_{\varepsilon_i^{j_k}}(x_{j_k}) - K(x_{j_k})\| + \|K(x_{j_k}) - y\| \\ &< \varepsilon_i^{j_k} + \|K(x_{j_k}) - y\| \longrightarrow 0, \quad \text{as } k \to \infty. \end{aligned}$$

Thus,

$$
\begin{aligned}
\|x_{j_k} - y\| &= \|t_{j_k} K_{\varepsilon_1^{j_k}}(x_{j_k}) + (1 - t_{j_k}) K_{\varepsilon_2^{j_k}}(x_{j_k}) - y\| \\
&= \|t_{j_k} (K_{\varepsilon_1^{j_k}}(x_{j_k}) - K_{\varepsilon_2^{j_k}}(x_{j_k})) + K_{\varepsilon_2^{j_k}}(x_{j_k}) - y\| \\
&\leq t_{j_k} \|K_{\varepsilon_1^{j_k}}(x_{j_k}) - K_{\varepsilon_2^{j_k}}(x_{j_k})\| + \|K_{\varepsilon_2^{j_k}}(x_{j_k}) - y\| \\
&\leq \|K_{\varepsilon_1^{j_k}}(x_{j_k}) - K_{\varepsilon_2^{j_k}}(x_{j_k})\| + \|K_{\varepsilon_2^{j_k}}(x_{j_k}) - y\| \\
&= \|K_{\varepsilon_1^{j_k}}(x_{j_k}) - K(x_{j_k}) + K(x_{j_k}) - K_{\varepsilon_2^{j_k}}(x_{j_k})\| + \|K_{\varepsilon_2^{j_k}}(x_{j_k}) - y\| \\
&\leq \|K_{\varepsilon_1^{j_k}}(x_{j_k}) - K(x_{j_k})\| + \|K(x_{j_k}) - K_{\varepsilon_2^{j_k}}(x_{j_k})\| + \|K_{\varepsilon_2^{j_k}}(x_{j_k}) - y\| \\
&< \varepsilon_1^{j_k} + \varepsilon_2^{j_k} + \|K_{\varepsilon_2^{j_k}}(x_{j_k}) - y\| \longrightarrow 0, \quad \text{as } k \to \infty,
\end{aligned}
$$

i.e, $x_{j_k} \longrightarrow y$, since B closed then $y \in B$. Therefore,

$$K(y) = K(\lim_{k \longrightarrow \infty} (x_{j_k})) = \lim_{k \longrightarrow \infty} K(x_{j_k}) = y,$$

i.e, $K(y) = y$ which is a contradiction. □

Lemma 3.1.3. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded subset, $1 \leq m \leq n$, let $f : \overline{\Omega} \longrightarrow \mathbb{R}^m$ continuous function and $g = \mathrm{Id} - f$. If $p \notin (\mathrm{Id} - f)(\partial\Omega)$, then

$$\deg(g, \Omega, p) = \deg(g_m, \Omega \cap \mathbb{R}^m, p),$$

where g_m is the restriction of g on $\overline{\Omega} \cap \mathbb{R}^m$.

We are now in position to define the Leray-Schauder degree.

Definition 3.1.4. *Let $K : \Omega \longrightarrow X$ be completely continuous mapping, where $\Omega \subset X$ is an open bounded subset, $f = \mathrm{Id} - K$ and $p \notin f(\partial\Omega)$. The Leray-Schauder degree of f at p with respect to Ω is defined by*

$$\deg(f, \Omega, p) = \deg(\mathrm{Id} - K_\varepsilon, \Omega \cap F_\varepsilon, p)$$

where K_ε is an approximation map of K as defined in Theorem 3.1.1 and $\varepsilon > 0$ is small enough.

Remark 3.1.1. The degree is well defined and does not depend on the choice of K_ε and F_ε. Indeed, suppose $p = 0$ in the definition and $0 \notin (\mathrm{Id} - K)(\partial\Omega)$. By Lemma 3.1.2 with $B = \partial\Omega$, there exists $\varepsilon_0 > 0$ such that

$$x \neq tK_{\varepsilon_1}(x) + (1 - t)K_{\varepsilon_2}(x) \;\; \forall\, x \in \partial\Omega,$$

for any $\varepsilon_i \in (0, \varepsilon_0)$ and $K_{\varepsilon_i} : \overline{\Omega} \longrightarrow F_{\varepsilon_i}$ for $i = 1, 2$, are approximation maps of K as in Theorem 3.1.1, we take

$$0 < \varepsilon_i < \min\left(\varepsilon_0, \frac{\operatorname{dist}(0, (\mathrm{Id} - K)(\partial\Omega))}{2}\right),$$

for $i = 1, 2$. Set $K_\varepsilon = tK_{\varepsilon_1}(x) + (1-t)K_{\varepsilon_2}(x)$, then

$$\begin{aligned}
\|K_\varepsilon(x) - K(x)\| &= \|tK_{\varepsilon_1}(x) + (1-t)K_{\varepsilon_2}(x) - K(x)\| \\
&= \|tK_{\varepsilon_1}(x) + (1-t)K_{\varepsilon_2}(x) - tK(x) - (1-t)K(x)\| \\
&= \|t(K_{\varepsilon_1}(x) - K(x)) + (1-t)(K_{\varepsilon_2}(x) - K(x))\| \\
&\leq t\|K_{\varepsilon_1}(x) - K(x)\| + (1-t)\|K_{\varepsilon_2}(x) - K(x)\| \\
&\leq \|K_{\varepsilon_1}(x) - K(x)\| + |K_{\varepsilon_2}(x) - K(x)\| \\
&< \varepsilon_1 + \varepsilon_2.
\end{aligned}$$

Hence, $K_\varepsilon : \overline{\Omega} \longrightarrow F_{\varepsilon_1} \cup F_{\varepsilon_2}$ is an approximation $(\varepsilon_1 + \varepsilon_2)$-approximation of K in Theorem 3.1.1 as $\dim(F_{\varepsilon_1}) < \infty$ and $\dim(F_{\varepsilon_2}) < \infty$, and so, $\dim(F_{\varepsilon_1} \cup F_{\varepsilon_2}) < \infty$. In addition, $0 \notin (\mathrm{Id} - K_\varepsilon)(\partial\Omega)$, otherwise there exists $x_0 \in \partial\Omega$ such that $x_0 = K_\varepsilon(x_0)$ and thus

$$\begin{aligned}
&\|K(x_0) - K_\varepsilon(x_0)\| < \varepsilon_1 + \varepsilon_2 < \operatorname{dist}(0, (\mathrm{Id} - K)(\partial\Omega)) \\
\Leftrightarrow\ &\|K(x_0) - x_0\| < \operatorname{dist}(0, (\mathrm{Id} - K)(\partial\Omega)) \leq \|K(x_0) - x_0\|,
\end{aligned}$$

a contradiction. Thus Brouwer's degree $\deg(\mathrm{Id} - K_\varepsilon, \Omega \cap \operatorname{span}\{F_{\varepsilon_1} \cup F_{\varepsilon_2}\}, 0)$ is well defined. Hence, we can define

$$\deg(\mathrm{Id} - K, \Omega, 0) = \deg(\mathrm{Id} - K_\varepsilon, \Omega \cap F_\varepsilon, 0),$$

where $F_\varepsilon = \operatorname{span}\{F_{\varepsilon_1} \cup F_{\varepsilon_2}\}$. Since $K_\varepsilon(x) \neq x \quad \forall\, x \in \partial\Omega$, then by the homotopy property (4) in Theorem 2.3.1, we have

$$\deg(\mathrm{Id} - K_{\varepsilon_1}, \Omega \cap \operatorname{span}\{F_{\varepsilon_1} \cup F_{\varepsilon_2}\}, 0) = \deg(\mathrm{Id} - K_{\varepsilon_2}, \Omega \cap \operatorname{span}\{F_{\varepsilon_1} \cup F_{\varepsilon_2}\}, 0).$$

But $K_{\varepsilon_i} : \overline{\Omega} \cap \operatorname{span}\{F_{\varepsilon_1} \cup F_{\varepsilon_2}\} :\longrightarrow F_i$ for $i = 1, 2$. so by Lemma 3.1.3 we have

$$\deg(\mathrm{Id} - K_{\varepsilon_1}, \Omega \cap \operatorname{span}\{F_{\varepsilon_1} \cup F_{\varepsilon_2}\}, 0) = \deg(\mathrm{Id} - K_{\varepsilon_1}, \Omega \cap F_{\varepsilon_1}, 0),$$

and

$$\deg(\mathrm{Id} - K_{\varepsilon_2}, \Omega \cap \operatorname{span}\{F_{\varepsilon_1} \cup F_{\varepsilon_2}\}, 0) = \deg(\mathrm{Id} - K_{\varepsilon_2}, \Omega \cap F_{\varepsilon_2}, 0).$$

Thus,

$$\deg(\mathrm{Id} - K_{\varepsilon_1}, \Omega \cap F_{\varepsilon_1}, 0) = \deg(\mathrm{Id} - K_{\varepsilon_2}, \Omega \cap F_{\varepsilon_2}, 0),$$

i.e, Definition 3.1.4 does not depend on the choice of K_ε and F_ε. Finally we take p in the general case, if $p \notin (\mathrm{Id} - K)(\partial\Omega)$, we have $\deg(\mathrm{Id} - K, \Omega, p) = \deg(\mathrm{Id} - K - p, \Omega, 0)$ as we have showed in Brouwer's degree.

3.2 Properties of Leray-Schauder Degree

Let X be normed space and $\Omega \subset X$ an open and bounded subset. In this section, we present some of the main properties of the Leray-Schauder degree for maps of the form $f = \mathrm{Id} - K$, i.e, for compact perturbations of the identity on Ω and $p \notin f(\partial\Omega)$. In fact Leray-Schauder degree satisfies most of the properties of Brouwer's degree.

Theorem 3.2.1. *If f is compact perturbation of the identity map on Ω which is open bounded subset of X, and $p \notin f(\partial\Omega)$, then Leray-Schauder degree has the following properties:*

(1) If $p \notin \partial\Omega$ then, deg$(\mathrm{Id}, \Omega, p) = \begin{cases} 1, & p \in \Omega \\ 0, & p \notin \overline{\Omega}, \end{cases}$ where Id *is the identity map.*

*(2) (**Continuity with respect to** p) If $p_1 \notin f(\partial\Omega)$ and $d_1 = \mathrm{dist}(p_1, f(\partial\Omega))$. Let $p_2 \in X$ be such that $\|p_1 - p_2\| < d_1$, then $p_2 \notin f(\partial\Omega)$ and*

$$\deg(f, \Omega, p_1) = \deg(f, \Omega, p_2).$$

*(3) (**Invariance by homotopy of the degree**) Let $H_t(x) : \overline{\Omega} \times [0,1] \longrightarrow X$ be a completely continuous map, $p_t : [0,1] \longrightarrow X$ a continuous map and $p_t \notin (\mathrm{Id} - H_t)(\partial\Omega)$ for all $t \in [0,1]$. Then $\deg(\mathrm{Id} - H_t, \Omega, p)$ does not depend on the parameter t.*

(4) Let $p \notin f(\partial\Omega)$ and $f, g : \overline{\Omega} \longrightarrow X$ be a compact perturbation of the identity maps on Ω, such that for each $x \in \partial\Omega$,

$$\|f(x) - g(x)\| < d = \mathrm{dist}(p, f(\partial\Omega))$$

then,

$$p \notin g(\partial\Omega) \quad \text{and} \quad \deg(f, \Omega, p) = \deg(g, \Omega, p).$$

*(5) (**Existence property**) Let f be a compact perturbation of the identity on Ω, and $p \notin f(\partial\Omega)$ such that, $\deg(f, \Omega, p) \neq 0$. Then there exists $x \in \Omega$*

such that $f(x) = p$.
*(6) (**Shifting property**) If* $p \notin f(\partial\Omega)$ *where* f *is a compact perturbation of the identity on* Ω *and let* $q \in X$*, then*

$$\deg(f, \Omega, p) = \deg(f - q, \Omega, p - q)$$

*(7) (**Domain decomposition**) Let* $(\Omega_i)_{i\in I} \subset \Omega$ *be a family of disjoint open subsets of* Ω *such that either*
(a) $\bigcup_{i\in I} \Omega_i = \Omega$ *and* $p \notin f(\partial\Omega)$ *or*
(b) $\bigcup_{i\in I} \Omega_i \subset \Omega$ *and* $p \notin f(\overline{\Omega} \setminus \bigcup_{i\in I} \Omega_i)$*. Then*

$$\deg(f, \Omega, p) = \sum_{i\in I} \deg(f, \Omega_i, p),$$

where only a finite number of terms is nonzero in the sum.
*(8) (**Excision property**) Let* $B \subset \Omega$ *compact subset and* $p \notin f(B) \cup f(\partial\Omega)$*. Then,*

$$\deg(f, \Omega, p) = \deg(f, \Omega\setminus B, p)$$

*(9) (**Multiplicity property of the degree**) Let* $f : U \longrightarrow X$ *and* $g : V \longrightarrow Y$ *be two compact perturbation of the identity maps on* U *and* V *respectively, where* U *and* V *are open bounded subsets of* X *and* Y*, respectively. Let* $p \notin f(\partial U)$ *and* $q \notin g(\partial V)$*. Then,*

$$\deg(f \times g, U \times V, (p, q)) = \deg(f, U, p) \cdot \deg(g, V, q)$$

where $(f \times g)$ *defined by*

$$(f \times g)(x_1, x_2) = (f(x_1), g(x_2)), \forall\, (x_1, x_2) \in X \times Y.$$

*(10) (**Invariance on connected components**) Let* f *be a compact perturbation of the identity on* Ω*, and* Ω *a connected component of* $X\setminus f(\partial\Omega)$*. Then* $\deg(f, \Omega, \cdot)$ *is constant in* Ω*.*

Proof. (1) Since $\mathrm{Id} = \mathrm{Id} - 0$ then we take $K_\varepsilon(x) = 0$ for all $x \in \Omega$ and $F_\varepsilon = \mathrm{span}\,\{p\}$, then by Definition 3.1.4

$$\deg(\mathrm{Id}, \Omega, p) = \deg(\mathrm{Id} - 0, \Omega \cap F_\varepsilon, p) = \deg(\mathrm{Id}, \Omega \cap F_\varepsilon, p).$$

Using Property (1) of Brouwer's degree (Theorem 2.3.1), we get

$$\deg(\mathrm{Id}, \Omega \cap F_\varepsilon, p) = \begin{cases} 1, & p \in \Omega \cap F_\varepsilon \\ 0, & p \notin \overline{\Omega \cap F_\varepsilon} \end{cases}$$

Since $p \in \Omega \Leftrightarrow p \in \Omega \cap F_\varepsilon$, because $F_\varepsilon = \text{span}\,\{p\}$, i.e, $p \in F_\varepsilon$, thus if $p \in \Omega$. we have

$$\deg(\text{Id}, \Omega, p) = \deg(\text{Id}, \Omega \cap F_\varepsilon, p) = 1.$$

Suppose, $p \notin \overline{\Omega}$, then $p \notin \overline{\Omega \cap F_\varepsilon}$, because $\overline{\Omega \cap F_\varepsilon} \subset \overline{\Omega} \cap \overline{F_\varepsilon} \subset \overline{\Omega}$. Hence,

$$\deg(\text{Id}, \Omega, p) = \deg(\text{Id}, \Omega \cap F_\varepsilon, p) = 0.$$

(2) By Corollary 3.1.1 we have $d_1 = \text{dist}(p_1, f(\partial\Omega)) > 0$. Suppose by contradiction that $p_2 \in f(\partial\Omega)$. Then there exists $x_0 \in \partial\Omega$ such that $p_2 = f(x_0)$, since

$$\|p_1 - p_2\| < d_1$$
$$\Rightarrow \|p_1 - f(x_0)\| < d_1 \leq \|p_1 - f(x_0)\|,$$

a contradiction. Since f is a compact perturbation of the identity on Ω, we can write $f = \text{Id} - K$ where K is completely continuous. Let $K_\varepsilon : \overline{\Omega} \longrightarrow F_\varepsilon$ be an approximation map of K as in Theorem 3.1.1 i.e, $\|K_\varepsilon(x) - K(x)\| < \varepsilon$ and $\dim(F_\varepsilon) < \infty$, we take ε small enough such that $\varepsilon < \frac{d_1}{2}$ and F_ε containing p_1 and p_2. Since $\partial(\Omega \cap F_\varepsilon) \subset \partial\Omega \cap F_\varepsilon$ then

$$f(\partial(\Omega \cap F_\varepsilon)) \subset f(\partial\Omega \cap F_\varepsilon) \subset f(\partial\Omega),$$

and so $p_1 \notin f(\partial\Omega)$ implies $p_1 \notin f(\partial(\Omega \cap F_\varepsilon))$. Set $f_\varepsilon = \text{Id} - K_\varepsilon$, then for all $x \in \partial(\Omega \cap F_\varepsilon) \subset \partial\Omega$,

$$\begin{aligned}
\|f_\varepsilon(x) - p_1\| &= \|f_\varepsilon(x) - f(x) + f(x) - p_1\| \\
&\geq \Big| \|f_\varepsilon(x) - f(x)\| - \|f(x) - p_1\| \Big| \\
&= \|f(x) - p_1\| - \|f_\varepsilon(x) - f(x)\| \\
&\geq d_1 - \varepsilon > 0,
\end{aligned}$$

i.e, $p_1 \notin \text{Id} - K_\varepsilon(\partial(\Omega \cap F_\varepsilon))$. By Definition 3.1.4. we have

$$\deg(f, \Omega, p_1) = \deg(\text{Id} - K_\varepsilon, \Omega \cap F_\varepsilon, p_1).$$

Since $\|p_1 - p_2\| < d_1$, by Property (3) of Brouwer's degree (Theorem 2.3.1), we have

$$\deg(\text{Id} - K_\varepsilon, \Omega \cap F_\varepsilon, p_1) = \deg(\text{Id} - K_\varepsilon, \Omega \cap F_\varepsilon, p_2).$$

Therefore, $\deg(f,\Omega,p_1)=\deg(\mathrm{Id}-K_\varepsilon,\Omega\cap F_\varepsilon,p_2)=\deg(f,\Omega,p_2)$.
(3) Set $f_t=\mathrm{Id}-H_t$, by Corollary 3.1.1 we have $r_t=\mathrm{dist}(p_t,f_t(\partial\Omega))>0$ for all $t\in[0,1]$. We claim that there exists $r>0$ such that

$$\mathrm{dist}(p_t,f_t(\partial\Omega))\geq r>0,\ \forall\, t\in[0,1].$$

On the contrary, suppose there exists a sequence $(t_n)_n\subset[0,1]$ and $(x_n)_n\subset\partial\Omega$ such that

$$\lim_{n\to\infty}\|f_{t_n}(x_n)-p_{t_n}\|=0. \tag{3.1}$$

Since $[0,1]$ is compact, there exists a convergent subsequence $t_n\to t_0\in[0,1]$, which implies $p_{t_n}\to p_{t_0}$, as $n\to\infty$. Since H_{t_n} is completely continuous and $(x_n)_n\subset\partial\Omega$, then $H_{t_n}(x_n)$ is relatively compact and so by Theorem 1.1.22

$$\lim_{n\to\infty}H_{t_n}(x_n)=y,\ \text{ for some }\ y\in X. \tag{3.2}$$

By (3.1) and (3.2), we have

$$\begin{aligned}\|x_n-(y+p_{t_0})\|&=\|x_n-H_{t_n}(x_n)+H_{t_n}(x_n)-y-p_{t_0}\|\\&=\|f_{t_n}(x_n)-p_{t_0}+H_{t_n}(x_n)-y\|\\&\leq\|f_{t_n}(x_n)-p_{t_0}\|+\|H_{t_n}(x_n)-y\|\longrightarrow 0,\ \text{as } n\to\infty.\end{aligned}$$

Since $\partial\Omega$ is closed, $\lim_{n\to\infty}x_n=y+p_{t_0}\in\partial\Omega$. Note that $f_{t_0}(x_n)\to p_{t_0}$, by the uniqueness of the limit and the continuity of f_{t_0}, we have

$$f_{t_0}(y+p_{t_0})=p_{t_0}.$$

Then, $p_{t_0}\in f_{t_0}(\partial\Omega)$, contradiction. Thus, using Definition 3.1.4 we have

$$\deg(\mathrm{Id}-H_t,\Omega,p_t)=\deg(\mathrm{Id}-H_t^\varepsilon,\Omega\cap F_t^\varepsilon,p_t),$$

for $\varepsilon>0$ small enough. The last Brouwer's degree does not depend on t, i.e, $\deg(\mathrm{Id}-H_t,\Omega,p_t)$ does not depend on t.
(4) Let $H(x,t)=tf(x)+(1-t)g(x)$, where $x\in\partial\Omega$, $t\in[0,1]$. H is a homotopy, and for all $x\in\partial\Omega$,

$$\begin{aligned}\|H(x,t)-p\|&=\|tf(x)+(1-t)g(x)-p+f(x)-f(x)\|\\&=\|(f(x)-p)-(1-t)(f(x)-g(x))\|\\&\geq\Big|\|f(x)-p\|-\|(1-t)(f(x)-g(x))\|\Big|\\&=\|f(x)-p\|-(1-t)\|(f(x)-g(x))\|\\&>\|f(x)-p\|-(1-t)d\\&\geq d-(1-t)d=td\geq 0.\end{aligned}$$

Therefore, for all $t \in [0,1]$, $p \notin H(\cdot,t)(\partial\Omega)$. By homotopy property (3) we deduce that

$$\deg(f,\Omega,p) = \deg(g,\Omega,p).$$

(5) We first show that if $f = \mathrm{Id} - K$ where K is completely continuous and $\deg(\mathrm{Id} - K,\Omega,0) \neq 0$ then K has a fixed point on Ω. Let K_n be an approximation map of K such that $\|K_n(x) - K(x)\| < \frac{1}{n}$ for all $x \in \overline{\Omega}$ and n large enough, where $K_n(\overline{\Omega}) \subset F_n$ and $\dim(F_n) < \infty$. By Definition 3.1.4 we have

$$\deg(\mathrm{Id} - K,\Omega,0) = \deg(\mathrm{Id} - K_n,\Omega \cap F_n,0).$$

By hypothesis $\deg(\mathrm{Id} - K_n,\Omega \cap F_n,0) \neq 0$, and by Property (6) of Theorem 2.3.1 there exits $x_n \in \Omega \cap F_n$ such that $K_n(x_n) = x_n$. Then

$$\|K(x_n) - K_n(x_n)\| < \frac{1}{n}$$

$$\Leftrightarrow \|K(x_n) - x_n\| < \frac{1}{n}.$$

Since $\{x_n\}_n \subset \overline{\Omega}$, and $\overline{\Omega}$ is bounded then $\{x_n\}_n$ is bounded and as K is completely continuous, there exits a subsequence $K(x_{n_k} : k \in \mathbb{N})$ which converges to some point $y \in X$, as $k \to \infty$. Then

$$\begin{aligned}
&\|K(x_{n_k}) - x_{n_k}\| < \frac{1}{n_k} \longrightarrow 0, \quad \text{as} \quad k \to \infty \\
&\Rightarrow \lim_{k\to\infty} K(x_{n_k}) - (x_{n_k}) = 0 \\
&\Rightarrow \lim_{k\to\infty} x_{n_k} = y.
\end{aligned}$$

Since K is continuous, then $\lim_{k\to\infty} K(x_{n_k}) = K(y)$, and as $\lim_{k\to\infty} K(x_{n_k}) = y$, then by the uniqueness of the limit we have $K(y) = y$, i.e, K has a fixed point. For the general case we use $\deg(\mathrm{Id} - K - p,\Omega,0) = \deg(\mathrm{Id} - K,\Omega,p)$ by Brouwer's degree.

(6) For $f = \mathrm{Id} - K$ where K is completely continuous, let $K_\varepsilon : \overline{\Omega} \longrightarrow F_\varepsilon$ be an approximation map of K , i.e, for all $x \in \partial\Omega$, $\|K_\varepsilon(x) - K(x)\| < \varepsilon$, where $\varepsilon > 0$ is small enough and we take $F = \mathrm{span}\,\{F_\varepsilon, p, p-q\}$, then $\dim(F) < \infty$ as $\dim(F_\varepsilon) < \infty$. Then by Definition 3.1.4 we have

$$\deg(f,\Omega,p) = \deg(\mathrm{Id} - K_\varepsilon,\Omega \cap F,p).$$

Since

$$\|(K_\varepsilon + q)(x) - (K+q)(x)\| = \|K_\varepsilon(x) - K(x)\| < \varepsilon,$$

and $p \notin f(\partial\Omega)$ implies $p-q \notin (f-q)(\partial\Omega)$, also as f is a compact perturbation of the identity on Ω then $f-q$ is a compact perturbation of the identity on Ω. By Definition 3.1.4,

$$\deg(\mathrm{Id} - K_\varepsilon - q, \Omega \cap F, p-q) = \deg(f-q, \Omega, p-q).$$

Using Property (11) of Theorem 2.3.1, we have

$$\deg(\mathrm{Id} - K_\varepsilon - q, \Omega \cap F, p-q) = \deg(\mathrm{Id} - K_\varepsilon, \Omega \cap F, p).$$

Therefore, $\deg(f, \Omega, p) = \deg(\mathrm{Id} - K_\varepsilon, \Omega \cap F, p) = \deg(f-q, \Omega, p-q)$.
(7) For $f = \mathrm{Id} - K$, let $K_\varepsilon : \overline{\Omega} \longrightarrow F_\varepsilon$ be an approximation of K, where $\varepsilon > 0$ is small enough. Then by Definition 3.1.4

$$\deg(f, \Omega, p) = \deg(\mathrm{Id} - K_\varepsilon, \Omega \cap F_\varepsilon, p),$$

since $(\Omega_i)_i \subset \Omega$ then $(\Omega_i)_i \cap F_\varepsilon \subset \Omega \cap F_\varepsilon$ are disjoint subsets, by Property (7) of Theorem 2.3.1

$$\begin{aligned}
\deg(f, \Omega, p) &= \deg(\mathrm{Id} - K_\varepsilon, \Omega \cap F_\varepsilon, p) \\
&= \sum_{i \in I} \deg(\mathrm{Id} - K_\varepsilon, \Omega_i \cap F_\varepsilon, p) \\
&= \sum_{i \in I} \deg(f, \Omega_i, p).
\end{aligned}$$

We follow the same steps we did in part (a) to obtain the desired result.
(8) $f = \mathrm{Id} - K$, let $K_\varepsilon : \overline{\Omega} \longrightarrow F_\varepsilon$ be an approximation of K, where $\varepsilon > 0$ is small enough and since B is compact then B is closed and bounded, so we take $F = \mathrm{span}\{F_\varepsilon, B\}$, thus $\dim(F) < \infty$. Then by Definition 3.1.4,

$$\deg(f, \Omega, p) = \deg(\mathrm{Id} - K_\varepsilon, \Omega \cap F, p).$$

Since $p \notin f(B)$, then $p \notin (\mathrm{Id} - K_\varepsilon)(B)$, as $B \subset \Omega \cap F$ is closed subset, by Property (8) of Theorem 2.3.1 we deduce that

$$\begin{aligned}
\deg(f, \Omega, p) &= \deg(\mathrm{Id} - K_\varepsilon, \Omega \cap F, p) \\
&= \deg(\mathrm{Id} - K_\varepsilon, (\Omega \cap F) \backslash B, p) \\
&= \deg(f, \Omega \backslash B, p).
\end{aligned}$$

(9) We argue by approximation and use the multiplicity property of Brouwer's topological degree (Property (11) of Theorem 2.3.1).
(10) Define $g : \Omega \longrightarrow \mathbb{Z}$ as $g(p) = \deg(f, \Omega, p)$. We show that g is continuous. Indeed, for fixed $p \in \Omega$ we set $d = \text{dist}(p, f(\partial\Omega)) > 0$, by Corollary 3.1.1. Let $q \in \Omega$, if $\|p - q\| < \varepsilon$, for all $\varepsilon > 0$, in particular for $d = \text{dist}(p, f(\partial\Omega)) > 0$. Define $f_q : \overline{\Omega} \longrightarrow X$ by

$$f_q(x) = f(x) - (q - p),$$

for all $x \in \overline{\Omega}$. Then f_q is compact perturbation of the identity on Ω, by Property (6) we have

$$\deg(f, \Omega, q) = \deg(f - (q - p), \Omega, q - (q - p)) = \deg(f_q, \Omega, p).$$

Since $\|f(x) - f_q(x)\| = \|p - q\| < d$ then by Property (4)

$$\deg(f, \Omega, p) = \deg(f_q, \Omega, p).$$

Thus,

$$\deg(f, \Omega, p) = \deg(f, \Omega, q),$$

i.e, $\|g(p) - g(q)\| = 0 < \varepsilon$, and so g is continuous. Since Ω is connected and g is continuous then $g(\Omega)$ is connected, it follows that $g(\Omega) = \{g(p)\}$ for every $q \in \Omega$, otherwise if $g(\Omega) = \{g(p_1), g(p_2)\} \in \mathbb{Z}$, then $g(\Omega)$ is disconnected contradiction. □

Corollary 3.2.1. *Suppose that f, g are compact perturbation of the identity, $f|_{\partial\Omega} = g|_{\partial\Omega}$, and $p \notin f(\partial\Omega)$. Then,* $\deg(f, \Omega, p) = \deg(g, \Omega, p)$.

Proof. Since for every $x \in \partial\Omega$,

$$\|f(x) - g(x)\| = 0 < \text{dist}(p, f(\partial\Omega)),$$

then by Property (4) we deduce

$$\deg(f, \Omega, p) = \deg(g, \Omega, p).$$

□

3.3 Schauder's Fixed Point Theorem

Theorem 3.3.1 (**Schauder's Fixed Point Theorem: First Version**). *Let X be a normed space and $C \subset X$ a bounded convex subset and its interior containing 0. If $K : C \longrightarrow C$ is a completely continuous map, then K has a fixed point.*

Proof. Let $\text{int}(C) = \Omega$, the interior of C. (1) First we want to show that, if $x \in C$, then $tx \in \Omega$ where $0 \leq t < 1$. Since $0 \in \Omega$, then by the definition of the interior there exists $\varepsilon > 0$ such that $B(0, \varepsilon) \subset C$. We prove that $B(tx, (1-t)\varepsilon) \subset C$. Indeed, let $y \in B(tx, (1-t)\varepsilon)$, then by Corollary 1.2.1 there exists some $u \in X$ such that $\|u\| < 1$ and $y = tx - (1-t)\varepsilon u$. Since, $y = tx + (1-t)(-\varepsilon u)$, $-\varepsilon u \in B(0, \varepsilon) \subset C$, $t \in [0,1]$ and $x \in C$, then by convexity of C we have $y \in C$. Thus, we found $r = (1-t)\varepsilon > 0$ such that $B(tx, r) \subset C$, and so $tx \in \Omega$.
(2) Now, if $K(x) = x$ for some $x \in \partial\Omega$ then the proof is done. Suppose that K has no fixed point in $\partial\Omega$. We set $H_t(x) = x - tK(x)$, for $x \in \overline{\Omega}$, $t \in [0,1]$. $H_t(x)$ is homotopy compact perturbation of the identity. We show $0 \notin H_t(\partial\Omega)$, for all $t \in [0,1]$. Suppose by contradiction that there is some $x_0 \in \partial\Omega$ and some $t_0 \in [0,1]$ such that

$$x_0 = t_0 K(x_0).$$

Since K has no fixed point in $\partial\Omega$, then $t_0 \neq 1$. Since $K(x_0) \in C$ and $0 \leq t_0 < 1$, then by the first part (1) we have

$$x_0 = t_0 K(x_0) \in \Omega,$$

contradiction since $x_0 \in \partial\Omega$. Hence, $0 \notin H_t(\partial\Omega)$, for all $t \in [0,1]$, by Schauder's Properties (1) and (3) Theorem 3.2.1 we have

$$1 = \deg(\text{Id}, \Omega, 0) = \deg(\text{Id} - K, \Omega, 0).$$

Since $\deg(\text{Id} - K, \Omega, 0) \neq 0$, then by Property (5) Theorem 3.2.1 there exists $x \in \Omega$ such that $x = K(x)$. □

Theorem 3.3.2 (**Schauder's Fixed Point Theorem: Second Version**). *Let X be a normed space and $C \subset X$ a closed bounded and convex subset. If $K : C \longrightarrow C$ is a completely continuous map, then K has a fixed point.*

Proof. Let $(K_n)_n : C \longrightarrow X$ be a sequence of approximations of K such that

$$\|K_n(x) - K(x)\| < \frac{1}{n},$$

for each $n = 1, 2, \ldots$ with range in a finite dimensional space F_n and $K_n(C) \subset C_o(K(C))$. Since, $K(C) \subset C$ and C is convex then, $C_o(K(C)) \subset C$. This implies that $K_n(C) \subset C_o(K(C)) \subset C$ and so $K_n(C) \subset C \cap F_n$. Since, $K_n : C \cap F_n \longrightarrow C \cap F_n$, is continuous by Brouwer's fixed point there exists $x_n \in C \cap F_n$ such that $K_n(x_n) = x_n$. Since $(x_n)_n \subset C$ and C is bounded, $(x_n)_n$ is bounded in F_n. In addition, since $dim(F_n) < \infty$, by Bolzano-Weierstrass Theorem there exists a subsequence $(x_{n_k})_k$ which converges to some limit $x \in C$ as C closed. Then,

$$\begin{aligned} \|K(x) - x\| &= \|K(x) - K(x_{n_k}) + K(x_{n_k}) - K_{n_k}(x_{n_k}) + K_{n_k}(x_{n_k}) - x\| \\ &\leq \|K(x) - K(x_{n_k})\| + \|K(x_{n_k}) - K_{n_k}(x_{n_k})\| \\ &\quad + \|K_{n_k}(x_{n_k}) - x\|. \end{aligned}$$

As $k \to \infty$, we have

$$\begin{aligned} &\|K(x) - K(x_{n_k})\| \longrightarrow 0 \quad \text{(as } K \text{ is continuous)}, \\ &\|K(x_{n_k}) - K_{n_k}(x_{n_k})\| < \frac{1}{n_k} \to 0, \\ &\|K_{n_k}(x_{n_k}) - x\| = \|x_{n_k} - x\| \to 0. \end{aligned}$$

Therefore, $\|K(x) - x\| = 0$ which equivalent to $K(x) = x$. □

Corollary 3.3.1 (Schauder's Fixed Point Theorem: Third Version). *Let X be a normed space and $C \subset X$ be a compact, convex subset, and $f : C \longrightarrow C$ a continuous map. Then K has a fixed point in C.*

Proof. We just check that f is completely continuous. Let $B \subset C$ be a bounded subset. Then $f(B) \subset f(C) \subset C$. Since C is compact and f is continuous, $f(C)$ is compact set. Hence $f(B)$ is relatively compact, i.e, f is completely continuous. Therefore by Theorem 3.3.2 f has a fixed point in C. □

Corollary 3.3.2 (Schaefer's Nonlinear Alternative). *Let X be a normed space and $K : X \longrightarrow X$ a completely continuous map. Then, either*
(a) tK has a fixed point, for all $t \in [0, 1]$,
or (b) the set $S = \{x \in X : \exists\, t \in [0, 1],\, x = tK(x)\}$ is unbounded.

Proof. Assume that (b) does not hold. Then, there exists a positive constant R such that for $x \in X$ and $t \in [0,1]$,

$$x = tK(x) \Rightarrow \|x\| < R \tag{3.3}$$

Consider the radial retraction given in Example 1.2.1 $r : X \longrightarrow \overline{B}(0,R)$

$$r(x) = \begin{cases} x, & \|x\| \leq R \\ R\frac{x}{\|x\|}, & \|x\| \geq R. \end{cases}$$

The composition $r \circ tK : X \longrightarrow \overline{B}(0,R)$ is completely continuous for all $t \in [0,1]$. Indeed, let $A \in X$ be bounded subset, then $\overline{tK(A)}$ is relatively compact. Since $tK(A) \subset \overline{tK(A)}$, then $r(tK(A)) \subset r(\overline{tK(A)})$. Since r is continuous and tK is completely continuous then $r(\overline{tK(A)})$ is compact. Hence, $r(tK(A))$ is relatively compact, i.e, $r \circ tK$ is completely continuous. According to the second version of Schauder's fixed theorem (Theorem 3.3.2), there exists $x_0 \in \overline{B}(0,R)$ such that $(r \circ tK)(x_0) = x_0$. We claim that $tK(x_0) \in \overline{B}(0,R)$. Let $t \neq 0$, otherwise $\|tK(x_0)\| = 0 < R$. On the contrary, let $\|tK(x_0)\| > R$ in which case

$$x_0 = r(tK(x_0)) = R\frac{tK(x_0)}{\|tK(x_0)\|}$$
$$\Leftrightarrow x_0 = \frac{R}{\|K(x_0)\|}K(x_0).$$

Since $R < \|tK(x_0)\| \leq \|K(x_0)\|$, for all $t \in [0,1]$, then for $t_0 = \frac{R}{\|K(x_0)\|} < 1$ and by the hypothesis (3.3),

$$\|x_0\| < R.$$

In addition $\|x_0\| = \frac{R}{\|K(x_0)\|}\|K(x_0)\| = R$, which is a contradiction. Then $\|tK(x_0)\| \leq R$, which implies $x_0 = r(tK(x_0)) = tK(x_0)$, i.e, tK has a fixed point. □

Corollary 3.3.3. *Let X be a normed space and $K : X \longrightarrow X$ a completely continuous map. Assume that there exists $R > 0$ such that for all $t \in [0,1]$,*

$$x = tK(x) \Rightarrow \|x\| \leq R.$$

Then, K has a fixed point such that $\|x\| \leq R$.

Proof. The hypothesis of the corollary is just the negation of the assumption (b) in Corollary 3.3.2. □

The following theorem have two different proofs, one can be obtained by the previous corollaries and one by Leray-Schauder's degree only.

Theorem 3.3.3 (**A boundary condition result**). *Let* $R > 0$ *and* $K : \overline{B}(0, R) \longrightarrow X$ *be a completely continuous map such that*

$$x \neq tK(x), \, \forall \, x \in \partial B(0, R), \; \forall \, t \in [0, 1]. \tag{3.4}$$

Then, K *has a fixed point in* $B(0, R)$*.*

Proof. (1) **First Proof.** By assumption, for all $t \in [0, 1]$ and $x \in \overline{B}(0, R)$

$$\begin{aligned} x = tK(x) &\Rightarrow x \notin \partial B(0, R) \\ &\Rightarrow x \in B(0, R) \\ &\Rightarrow \|x\| < R. \end{aligned}$$

By Corollary 3.3.1, K has a fixed point in $B(0, R)$.
(2) **Second Proof.** We show that the Leray-Schauder degree $\deg(\mathrm{Id} - tK, B(0, R), 0)$ is well defined, by hypothesis

$$\begin{aligned} &x \neq tK(x), \, \forall \, x \in \partial B(0, R) \\ &\Leftrightarrow (\mathrm{Id} - tK)(x) \neq 0, \, \forall \, x \in \partial B(0, R), \end{aligned}$$

which means that $0 \notin (\mathrm{Id} - tK)(\partial B(0, R))$. Thus, $\deg(\mathrm{Id} - tK, B(0, R), 0)$ is well defined and by homotopy Property (3) and Property (1) Theorem 3.2.1, we have

$$\deg(\mathrm{Id} - tK, B(0, R), 0) = \deg(\mathrm{Id} - K, B(0, R), 0) = \deg(\mathrm{Id}, B(0, R), 0) = 1.$$

Since $\deg(\mathrm{Id} - K, B(0, R), 0) \neq 0$ then by Existence Property of the degree (5) Theorem 3.2.1, there exists $x \in B(0, R)$ such that $K(x) = x$. □

Theorem 3.3.4 (**Rothe's Fixed Point Theorem**). *Let* $B(0, R)$ *be an open ball in a normed space* X *and* $K : X \longrightarrow X$ *a completely continuous map such that* $K(\partial B(0, R)) \subset \overline{B}(0, R)$*. Then,* K *has a fixed point in* $\overline{B(0, R)}$*.*

Proof. (1) **First Proof.** We just show that the condition $K(\partial B(0,R)) \subset \overline{B}(0,R)$ of the theorem implies the condition (3.4). Indeed, if $x_0 = t_0 K(x_0)$ for some $x_0 \in \partial B$ and $t_0 \in (0,1)$ because if $t_0 = 1$, then the proof is done and $t_0 = 0$ cannot occur as $0 \notin \partial B(0,R)$. Then,

$$\begin{aligned} K(x_0) = \frac{x_0}{t_0} \in \overline{B}(0,R) &\Leftrightarrow \left\| \frac{x_0}{t_0} \right\| \leq R \\ &\Leftrightarrow \frac{1}{t_0} \leq 1 \quad (\text{as} \quad \|x_0\| = R) \\ &\Leftrightarrow t_0 \geq 1, \end{aligned}$$

contradiction, as $t_0 \in (0,1)$.
(2) **Second Proof.** Assume that $K(x) \neq x$, for all $x \in \partial B(0,R)$, otherwise we are done. Then, $tK(x) \neq x$, for all $x \in \partial B(0,R)$, for every $t \in [0,1]$. Otherwise, $R = \|x_0\| = t_0 \|K(x_0)\|$ for some $x_0 \in \partial B$, $t_0 \in [0,1)$, as if $t_0 = 1$ we have $K(x) \neq x$. Hence, by assumption $R = t_0 \|K(x_0)\| < \|K(x_0)\| \leq R$, contradiction. The Leray-Schauder degree $\deg(\mathrm{Id} - tK, B(0,R), 0)$ is well defined, by the homotopy property in Theorem 3.2.1, $\deg(\mathrm{Id} - K, B(0,R), 0) = \deg(\mathrm{Id}, B(0,R), 0) = 1$. By the existence property of the degree in Theorem 3.2.1, $(\mathrm{Id} - K)$ has a zero, i.e, K has a fixed point. □

Theorem 3.3.5 (Leray-Schauder Fixed Point Theorem). *Let X be a normed space and $C \subset X$ be a non-empty bounded open subset. Let $K : \overline{C} \longrightarrow X$ be a completely continuous map satisfying the boundary condition:*

$$x \neq tK(x), \forall\, x \in \partial C, \ \forall\, t \in [0,1].$$

Then, K has a fixed point in C.

Proof. Consider the homotopy $H(x,t) = (\mathrm{Id} - tK)(x)$, for $(x,t) \in C \times [0,1]$. By assumption $(\mathrm{Id} - tK)(x) \neq 0$, for all $x \in \partial C$ and $t \in [0,1]$. Then, $\deg(\mathrm{Id} - tK, C, 0)$ is well defined. By the homotopy property in Theorem 3.2.1,

$$\deg(\mathrm{Id} - K, C, 0) = \deg(\mathrm{Id}, C, 0) = 1.$$

By the existence property of the degree, K has a fixed point in C. □

Remark 3.3.1. (1) When $C = B(0,R)$, we recapture Theorem 3.3.3.
(2) C is not necessarily convex.

Chapter 4

Applications

In this last chapter, we use Leray-Schauder topological degree to investigate an initial value problem for a first-order differential equation and then discuss the solvability of a second-order differential equation subject to Dirichlet boundary conditions. We will need some technical results. The first one is an important tool in determining a priori estimates. The second auxiliary result concerns the Green's function of a linear boundary value problem. Then two compactness criteria are proved. They are based on Ascoli-Arzéla Lemma which is presented without proof.

4.1 Preliminaries

Lemma 4.1.1. (**Grönwall's Lemma**) Let $u\colon I \subset \mathbb{R} \longrightarrow E$ be a continuous function. Suppose that there exist two constants k and k' ($k' \geq 0$) such that

$$u(x) \leq k + k' \int_{x_0}^{x} u(s)ds, \quad \forall\, x \in [x_0, a]. \tag{4.1}$$

Then,

$$u(x) \leq k e^{k'(a-x_0)}.$$

Proof. Let $v(x) = k + k' \int_{x_0}^{x} u(s)ds$. Then, $v(x_0) = k$ and $v'(x) = k'u(x)$. By

(4.1), we have

$$\begin{aligned} u(x) \le v(x) &\Rightarrow k'u(x) \le k'v(x) \\ &\Rightarrow v'(x) \le k'v(x) \\ &\Rightarrow v'(x) - k'v(x) \le 0 \\ &\Rightarrow e^{-\int_{x_0}^{x} k'ds}(v'(x) - k'v(x)) \le 0. \end{aligned}$$

The latter inequality is equivalent to

$$\left(v(x)e^{-k'(x-x_0)}\right)' \le 0.$$

Integrating between x_0 and x, we obtain the estimates

$$u(x) \le v(x) \le ke^{k'(x-x_0)} \le ke^{k'(a-x_0)}, \quad \forall\, x \in [x_0, a].$$

□

The proof of the following compactness criterion can be found in most textbooks of Topology, e.g., [7].

Lemma 4.1.2 (Ascoli-Arzéla Lemma). Let E, F be two metric spaces such that E is compact and F is complete, and $H \subset C(E, F)$ be a bounded subset. We have

$$H \text{ relatively compact } \Leftrightarrow \begin{cases} H \text{ equicontinuous.} \\ \forall\, x \in E,\ H(x) \text{ is relatively compact in } F. \end{cases}$$

Here $H(x) = \{f(x) :\ f \in H\}$. Recall

Definition 4.1.1. *A subset $H \subset C(E, F)$ is equicontinuous if for all $\varepsilon > 0$, there exists $\alpha = \alpha(\varepsilon) > 0$, such that*

$$\forall\, t, s \in E :\ d_E(x, x') < \alpha \Rightarrow d_F(f(x), f(x')) < \varepsilon, \quad \forall\, f \in H.$$

When F is a normed space with finite dimension, the boundedness and the relatively compactness are equivalent concepts in Topology. Thus, we have

Corollary 4.1.1. *If F is a Banach space with $\dim(F) < \infty$, then for every bounded subset $H \subset C(E, F)$, we have the equivalence*

$$H \text{ relatively compact} \Leftrightarrow H \text{ equicontinuous.}$$

Ascoli-Arzéla Lemma may adapted to C^1 spaces.

Lemma 4.1.3. Let E and F be two metric spaces such that E is compact and F is complete and let $H \subset C^1(E, F)$ be a bounded subset. We have

$$H \text{ relatively compact } \Leftrightarrow \begin{cases} H \text{ and } H_1 \text{ are equicontinuous.} \\ \forall\, x \in E,\ H(x) \text{ is relatively compact in } F \\ \forall\, x \in E,\ H_1(x) \text{ is relatively compact in } F, \end{cases}$$

where $H_1(x) = \{f'(x) :\ f \in H\}$.

The following lemma can be checked by direct integration (see, e.g., [3]).

Lemma 4.1.4. Let $h : [0,1] \longrightarrow \mathbb{R}$ be a continuous function. Then, the linear problem

$$\begin{cases} -u''(x) &= h(x),\ 0 < x < 1 \\ u(0) = u(1) &= 0, \end{cases}$$

has a unique solution given by

$$u(x) = \int_0^1 G(x,s)h(s)ds,$$

where

$$G(x,s) = \begin{cases} x(1-s), & 0 \le x \le s \le 1 \\ s(1-x), & 0 \le s \le x \le 1, \end{cases}$$

is the Green's function of the corresponding homogeneous problem. Moreover, for all $x, s \in [0,1]$,

$$0 \le \int_0^1 G(x,s)ds \le \frac{1}{8} \text{ and } 0 \le \int_0^1 \left|\frac{\partial G}{\partial x}(x,s)\right| ds \le \frac{1}{2}. \tag{4.2}$$

Lemma 4.1.5. Let $x_a < a$ be real numbers and $f : I = [x_0, a] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a continuous function. The Banach space $X = C([0,1], \mathbb{R}^n)$ is equipped with the sup-norm $\|u\|_X = \sup\limits_{x\in I} \|u(x)\|_n$, where $\|x\|_n = \|x\|_{\mathbb{R}^n}$.

Then, the mapping $K : X \longrightarrow X$ given by

$$Ku(x) = x_0 + \int_{x_0}^x f(s, u(s))ds, \quad x \in I.$$

is completely continuous.

Proof. Since f is continuous, then the map

$$x \mapsto \int_{x_0}^{x} f(s, u(s))ds$$

is continuous (even differentiable). Then K is well defined.

Step 1. K **is continuous.** Using Theorem 1.1.5, we show that K is sequentially continuous. Let $(u_n)_n$ be a sequence which converges to a limit u in X, that is

$$\lim_{n\to\infty} \|u_n - u\|_X = \lim_{n\to\infty} \sup_{x\in I} \|u_n(x) - u(x)\|_n = 0.$$

Let $x \in I$ be fixed. Then, $\lim\limits_{n\to\infty} u_n(s) = u(s)$, for all $s \in [x_0, x]$. Since f is continuous, $\lim\limits_{n\to\infty} f(s, u_n(s)) = f(s, u(s))$, for all $s \in [x_0, x]$ and the convergence is uniform over the compact interval $[x_0, x]$. Hence, $\lim\limits_{n\to\infty} Ku_n(x) = Ku(x)$, for all $x \in I$. Since I is a compact interval, the convergence is uniform, that is

$$\lim_{n\to\infty} \|Ku_n - Ku\|_X = 0,$$

proving the convergence of Ku_n to Ku in X.

Step 2. For every bounded subset $B \subset X$**,** $K(B)$ **is bounded in** X**.** Let $M > 0$ be such that

$$\|u\|_X \leq M, \quad \forall\, u \in B,$$

that is

$$\|u(x)\|_n \leq M, \quad \forall\, u \in B,\ \forall\, x \in I.$$

Then for all $u \in B$ and $x \in I$, $(x, u(x)) \in I \times \overline{B}(0, M)$ and $f(x, u(x)) \in f(I \times \overline{B}(0, M))$. Since f is continuous and $I \times \overline{B}(0, M)$ is compact in $I \times \mathbb{R}^n$, by Theorem 1.1.17, $f(I \times \overline{B}(0, M))$ is compact in $\mathbb{R}^n$, hence bounded. Then, there exists $M' > 0$ such that

$$\|f(x, u(x))\|_n \leq M', \quad \forall\, x \in I,\ \forall\, u \in B.$$

Then, for all $x \in I$ and $u \in B$,

$$\begin{array}{rcl} \|Ku(x)\|_n & \leq & |x_0| + \| \int_{x_0}^{x} f(s, u(s))ds\|_n \\ & \leq & |x_0| + M'|x - x_0| \\ & \leq & |x_0| + M'(|x_0| + |a|). \end{array}$$

Taking the supremum over $x \in I$, we get

$$\|Ku\|_X \le |x_0| + M'(|x_0| + |a|), \quad \forall\, u \in B,$$

proving that $K(B) \subset B(0, |x_0| + M'(|x_0| + |a|)$.

Step 3. For every bounded subset $B \subset X$, $K(B)$ is equi-continuous. Let $M' > 0$ be given by step 2. For all $x, x' \in I$,

$$\begin{array}{rcl} \|Ku(x) - Ku(x')\|_n & \le & \| \int_x^{x'} f(s, u(s)) ds\|_n \\ & \le & \int_x^{x'} \|f(s, u(s))\|_n ds \\ & \le & M'|x - x'| \\ & \le & \varepsilon \end{array}$$

whenever $|x - x'| \le \delta = \frac{\varepsilon}{M'}$, independently of $u \in B$, proving the equi-continuity of $K(B)$.

By Ascoli-Arzéla Lemma (Corollary 4.1.1) applied with $H = K(B) \subset C([0,1], \mathbb{R}^n)$, we conclude that K is completely continuous. □

Lemma 4.1.6. Let $f : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$, $G : [0,1]^2 \longrightarrow \mathbb{R}$ and $\frac{\partial G}{\partial x} : [0,1] \longrightarrow \mathbb{R}$ are all continuous functions. Let $X = C^1([0,1], \mathbb{R})$, equipped with the norm $\|u\|_X = \max(\|u\|_0, \|u'\|_0)$, where $\|u\|_0 = \sup\limits_{x \in I} |u(x)|$. Then the mapping $K : X \longrightarrow X$ given by

$$Ku(x) = \int_0^1 G(x,s) f(s, u(s), u'(s)) ds, \quad x \in [0,1].$$

is K is completely continuous.

Proof. Since G and f are continuous, Ku is continuous. In addition by Lemma 4.1.4, G has partial derivatives. Then, Ku is differentiable and for all $x \in [0,1]$,

$$(Ku)'(x) = \int_0^1 \frac{\partial G}{\partial x}(x,s) f(s, u(s), u'(s)) ds.$$

Hence, Ku is differentiable for all $u \in X$, $Ku \in X$.
Step 1. K is continuous. The proof that K is sequentially continuous is the same as in the proof of Lemma 4.1.5, Step 1.

Step 2. For every bounded subset $B \subset X$, $K(B)$ is bounded in X. Arguing as in Lemma 4.1.5, there exists some constant $R > 0$ such that

$$\|u\|_X \leq R \quad \forall\, u \in B,$$

that is,

$$|u(x)| \leq R, \quad \text{and} \quad |u'(x)| \leq R.$$

Then, $(s, u(s), u'(s)) \in [0,1] \times [-R, R]^2$ which is compact in $I \times \mathbb{R}^2$. Since f is continuous, $f(I \times [-R, R]^2)$ is compact in $\mathbb{R}$, hence bounded. Then there exists $M_R > 0$ such that, for all $u \in B$ and $s \in [0,1]$,

$$M_R = \max\{|f(x,y,z)| : \ 0 \leq x \leq 1,\ |y| \leq R,\ |z| \leq R\}.$$

Using the estimates of the Green's function in (4.2), we have for all $x \in [0,1]$ and $u \in B$,

$$\begin{aligned} |Ku(x)| &\leq \int_0^1 |G(x,s)||f(s,u(s),u'(s))|ds \leq M_R \\ |(Ku)'(x)| &\leq \int_0^1 \left|\frac{\partial G}{\partial x}\right| |f(s,u(s),u'(s))|ds \leq M_R. \end{aligned}$$

Taking the supremum over $x \in [0,1]$, we get

$$\|Ku\|_X \leq M_R,$$

proving that $K(B) \subset \overline{B(0, M_R)}$. In particular, we deduce that the sets $K(\overline{B(0,R)})(x) = \{Ku(x) : \ u \in \overline{B(0,R)}\}$ and $K(\overline{B(0,R)})_1(x) = \{(Ku)'(x) : u \in \overline{B(0,R)}\}$ are bounded in $\mathbb{R}$, hence relatively compact.

Step 3. $K(\overline{B(0,R)})$ and $K(\overline{B(0,R)})_1$ are equi-continuous. Let $M_R > 0$ be given by step 2. Since the functions G and $\frac{\partial G}{\partial x}$ are uniformly continuous over the compact set $[0,1] \times [0,1]$, for all $\varepsilon > 0$, there exists $\alpha > 0$ such that

$$\begin{aligned} |x - x'| < \alpha &\implies |G(x,s) - G(x',s)| < \tfrac{\varepsilon}{M_R},\ \forall\, s, x, x' \in [0,1], \\ |x - x'| < \alpha &\implies |\tfrac{\partial G}{\partial x}(x,s) - \tfrac{\partial G}{\partial x}(x',s)| < \tfrac{\varepsilon}{M_R},\ \forall\, s, x, x' \in [0,1]. \end{aligned}$$

Hence,

$$\begin{aligned} |Ku(x) - Ku(x')| \leq \int_0^1 |G(x,s) - G(x',s)||f(s,u(s),u'(s)|ds < \tfrac{\varepsilon}{M_R} M_R = \varepsilon, \\ |(Ku)'(x) - (Ku)'(x')| \leq \int_0^1 |\tfrac{\partial G}{\partial x}(x,s) - \tfrac{\partial G}{\partial x}(x',s)||f(s,u(s),u'(s)|ds < \tfrac{\varepsilon}{M_R} M_R = \varepsilon. \end{aligned}$$

Independently of $u \in \overline{B(0,R)}$, proving the claim.

By Ascoli-Arzéla Lemma (Lemma 4.1.3) applied with $H = K(\overline{B(0,R)})$, we conclude that K is completely continuous. □

4.2 An Initial Value Problem

Let $I = [x_0, a]$ and $f\colon I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a continuous function. Consider the initial value problem for system of n first-order differential equations:

$$\begin{cases} u'(x) &= f(x, u(x)), \quad x \in I \\ u(x_0) &= u_0. \end{cases} \tag{4.3}$$

Suppose that f verifies the following growth condition

$$\exists\, k > 0, \quad \|f(x,y)\|_n \leq k(1 + \|y\|_n), \quad \text{over } I \times \mathbb{R}^n, \tag{4.4}$$

where $\|x\|_n = \|x\|_{\mathbb{R}^n}$.

Theorem 4.2.1. *Problem (4.3) has at least a global solution $u \in C^1(I, \mathbb{R}^n)$.*

Proof. **(1) Fixed point setting.** We seek classical solutions in the Banach space $X = C(I, \mathbb{R}^n)$ endowed with the supremum norm:

$$\|u\|_X = \sup_{x \in I} \|u(x)\|_n, \quad \forall\, u \in X.$$

Note that since $\mathbb{R}^n$ is Banach, then so is X. A simple integration between x_0 and x shows that problem (4.3) is equivalent to the nonlinear integral equation

$$u(x) = u_0 + \int_{x_0}^{x} f(s, u(s))ds,\ x \in I.$$

This suggests to define the mapping $K\colon X \longrightarrow X$ by

$$Ku(x) = u_0 + \int_{x_0}^{x} f(s, u(s))ds,\ x \in I.$$

Then, u is solution of problem (4.3) if and only if u is a fixed point of K. We are going to use the Leray-Schauder degree to prove the existence of a solution for the equation $(Id - K)u = 0$, that is a fixed point of the mapping K.

(2) For all $t \in [0,1]$, tK is completely continuous. This follows from Lemma 4.1.5.

(3) A priori estimates and definition of a degree. We prove that all possible fixed points of the mapping tK lie in a closed ball of X, independently of the parameter $t \in [0,1]$. Indeed, using (4.4)

$$\begin{array}{rcl} \|u(x)\|_n & = & \|tKu(x)\|_n \\ & \le & \|u_0 + \int_{x_0}^{x} f(s,u(s))ds\|_n \\ & \le & \|u_0\|_n + \int_{x_0}^{x} \|f(s,u(s))\|_n ds \\ & \le & \|u_0\|_n + k\int_{x_0}^{x} (1+\|u(s)\|_n)\, ds \\ & \le & k' + k\int_{x_0}^{x} \|u(s)\|_n ds, \end{array}$$

where $k' = \|u_0\|_n + k(a-x_0)$. By Grönwall's Lemma 4.1.1, we have

$$\forall\, x \in I, \quad \|u(x)\|_n \le k' \exp(kx) \le k' \exp(ka) = C,$$

Then, for all $R > C$, $\|u\|_X < R$, that is $u \in B_R(0)$. This means that the parameterized family of equations $tKu = u$ have no solution on the boundary of the open ball $B_R(0)$. This with part (2) imply that the Leray-Schauder topological degree $\deg(Id - tK, B_R(0), 0)$ is well defined for all $t \in [0,1]$).

(4) Conclusion. By the invariance under homotopy of Leray-Schauder degree, we have

$$\deg(Id - tK, B_R(0), 0) = \deg(Id, B_R(0), 0) = 1 \neq 0.$$

Hence, $\deg(Id - K, B_R(0), 0) \neq 0$. By the existence property of the degree, the equation $(Id - K)(u) = 0$ has one solution $u \in X$. Therefore, problem (4.3) has at least one solution $u \in B_R(0) \subset C(I, \mathbb{R}^n)$. Since f is continuous, $u \in C^1(I, \mathbb{R}^n)$. □

4.3 A Boundary Value Problem

Consider the Dirichlet boundary value problem on the interval $I = (0,1)$:

$$\begin{cases} -u''(x) = f(x, u(x), u'(x)), \ x \in I \\ u(0) = u(1) = 0, \end{cases} \tag{4.5}$$

where the nonlinear function $f = f(x,y,z) : I \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies the following hypotheses:

(H1) There exist positive constants a, b, c $(a + c < 1)$ such that for all $x \in I$ and $y, z \in \mathbb{R}$, we have

$$yf(x, y, z) \leq ay^2 + b|y| + c|yz|.$$

(H2) There exist positive constants d, e such that for all $x \in I$ and $y, z \in \mathbb{R}$, we have

$$|f(x, y, z)| \leq dz^2 + e.$$

Theorem 4.3.1. *Under Assumptions (H1)-(H2), problem (4.5) has at least one solution* $u \in C^2([0, 1], \mathbb{R})$*.*

Proof. **(1) Fixed point setting.**
Let $X = C^2([0, 1], \mathbb{R})$ be the Banach space of continuously differentiable functions equipped with the norm

$$\|u\| = \max(\|u\|_0, \|u'\|_0),$$

where $\|u\|_0 = \sup_{0 \leq x \leq 1} |u(x)|$. For $0 \leq t \leq 1$, define the map $K_t : X \longrightarrow X$ by $K_t u(x) = t \int_0^1 G(x, s) f(s, u(s), u'(s)) ds$. By Lemma 4.1.4, u is solution of problem (4.5) if and only if u is a fixed point of K_1. According to Lemma 4.1.6, for all $t \in [0, 1]$, K_t is completely continuous.

(2) A priori estimates. Let u be a fixed point of the map K_t. Multiply the equation satisfied by u and integrate by parts using the homogeneous boundary conditions. We get

$$\int_0^1 |u'(x)|^2 dx = t \int_0^1 uf(x, u(x), u'(x)) dx.$$

By applying Assumption (H1), we get

$$\int_0^1 |u'(x)|^2 dx \leq \int_0^1 au^2(x) + b|u(x)| + c|u(x)u'(x)| dx. \tag{4.6}$$

By Cauchy-Schwartz inequality and the homogeneous boundary conditions, we have

$$|u(x)| = \left| \int_0^x u'(s) ds \right| \leq \int_0^1 |u'(s)| ds \leq \left(\int_0^1 |u'(s)|^2 ds \right)^{\frac{1}{2}} \left(\int_0^1 1^2 ds \right)^{\frac{1}{2}}.$$

$$\Rightarrow |u(x)| \leq \left(\int_0^1 |u'(s)|^2 ds\right)^{\frac{1}{2}}, \quad \forall\, x \in [0,1] \tag{4.7}$$

Squaring and integrating yield the so-called Poincaré inequality

$$\int_0^1 |u(x)|^2 dx \leq \int_0^1 |u'(x)|^2 dx.$$

By using (4.6) and combining Cauchy-Schwartz and Poincaré inequalities, we obtain the estimates

$$\begin{aligned}
\int_0^1 |u'(x)|^2 dx &\leq a\int_0^1 |u(x)|^2 dx + b\int_0^1 |u(x)| dx + c\int_0^1 |u(x)u'(x)| dx \\
&\leq a\int_0^1 |u'(x)|^2 dx + b\|u\|_0 + c\int_0^1 |u(x)u'(x)| dx \\
&\leq a\int_0^1 |u'(x)|^2 dx + b\|u\|_0 + c\left(\int_0^1 |u(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 |u'(x)|^2 dx\right)^{\frac{1}{2}} \\
&\leq a\int_0^1 |u'(x)|^2 dx + b\|u\|_0 + c\left(\int_0^1 |u'(x)|^2 dx\right) \\
&\leq (a+c)\int_0^1 |u'(x)|^2 dx + b\|u\|_0.
\end{aligned}$$

Since $0 < a + c < 1$, we get

$$\int_0^1 |u'(x)|^2 dx \leq \frac{b}{1-a-c}\|u\|_0. \tag{4.8}$$

By (4.7), we have $\|u\|_0 \leq \left(\int_0^1 |u'(x)|^2 dx\right)^{\frac{1}{2}}$. This with (4.8) yield

$$\left(\int_0^1 |u'(x)|^2 dx\right)^{\frac{1}{2}} \leq \frac{b}{1-a-c} := R_0.$$

Hence,

$$\|u\|_0 \leq R_0. \tag{4.9}$$

On the other hand, Hypothesis (H2) yields the estimate

$$|u''(x)| \leq d|u'(x)|^2 + e, \text{ for all } x \in I.$$

Integrating and using (4.8), we get

$$\int_0^1 |u''(s)| ds \leq dR_0\|u\|_0 + e$$

Finally, since $u \in C^1(I)$, by Rolle's Theorem and the homogeneous boundary conditions, there exists some $x_0 \in I$ such that $u'(x_0) = 0$. Hence,

$$\forall\, x \in I,\ |u'(x)| = \left| \int_{x_0}^{x} u''(s)ds \right| \leq dR_0^2 + e := R_1.$$

Taking the supremum over $x \in I$, hence $\|u'\|_0 \leq R_1$ and finally,

$$\|u\|_X \leq R := \max(R_0, R_1).$$

(4) Conclusion. By considering the open ball $B = B(0, R+1)$ in the space $C^1([0,1], \mathbb{R})$, the map K_t has no fixed point on the boundary of ball B. Then, the Leray-Schauder degree $\deg(Id - K_t, B, 0)$ is well defined and equals, by homotopy, $\deg(Id, B, 0) = 1$. By the existence property of the Leray-Schauder degree, we conclude the existence of a fixed point for operator K_1, which is a solution of problem (4.5). □

Remark 4.3.1. (1) In Theorem 4.2.1 and Theorem 4.3.1, it was also possible to apply directly Theorem 3.3.5.
(2) However, if one applies instead Schauder's fixed point Theorem 3.3.1 or Theorem 3.3.2, one can check that the condition $0 < k < \frac{1}{a - x_0}$ should be added in problem (4.3) and a condition on the constants e, d appearing in Hypothesis (H2) is required in problem (4.5).

Remark 4.3.2. In Theorem 4.2.1 and Theorem 4.3.1, the obtained solution lies in a ball. So, it could be the trivial solution (center of the ball). To avoid such a trivial solution, we only need to add the restriction
(1) $f(x_0) \neq 0$ in problem (4.3),
(2) $f(t, 0, 0)$ not identically zero in problem (4.5).

Bibliography

[1] Fonseca, Irene; Gangbo, Wilfrid. Degree theory in analysis and applications. Oxford Lecture Series in Mathematics and its Applications, 2. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[2] Granas, Andrzej; Dugundji, James. Fixed Point Theory. Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[3] Hsieh, Po-Fang; Sibuya, Yasutaka. Basic theory of ordinary differential equations. Universitext. Springer-Verlag, New York, 1999.

[4] Krawcewicz, Wieslaw; Wu, Jianhong. Theory of Degrees with Applications to Bifurcations and Differential Equations. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1997.

[5] J.R. Munkres, Topology (Second edition), Pearson, 2000.

[6] O'Regan, Donal; Cho, Yeol Je; Chen, Yu-Qing. Topological degree theory and applications. Series in Mathematical Analysis and Applications, 10. Chapman & Hall/CRC, Boca Raton, FL, 2006.

[7] H.L. Royden and P.M. Fitzpatrick, Real Analysis (Fourth edition), Pearson, 2010.

[8] M. Spivak, Calculus on Manifolds, Addison-Wesley Puplishing Company, 1965.

[9] Parameter-dependent integral, Encyclopedia of Mathematics, URL: http://encyclopediaofmath.org/index.php?title=Parameter-dependent_integral&oldid=44764