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Portfolio optimization

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1. Introduction

An investment in a risky security always carries the burden of possible losses or poor performance. In this project we analyse the advantages of spreading the investment among several securities.

This project is divided into five sections. The risk and return are defined in the second section for the global min vie portfolios.

The affiant portfolio is defended in the fourth section frontier the fifth section present an investment strategy. In the second section we present to rustle of five stokes select from Tadawul.

2. Return and Risk

The return (also called the rate of return) on the investment S is random variable $K : \Omega \to \mathbb{R}$, defined as

$$K = \frac{S(1) - S(0)}{S(0)}$$

By the linearity of mathematical expectation, the expected (or mean) return
is given by

$$\mathbb{E}(K) = \frac{\mathbb{E}\big(S(1) - S(0)\big)}{S(0)}$$

We will use the Greek letter μ for expectations of various random returns

$$\mu = \mathbb{E}(K)$$

By (the measure of) risk we mean the **variance** of the return

$$\mathrm{Var}(K) = \mathbb{E}\,(K - \mu)^2 = \mathbb{E}\,(K^2) - \mu^2,$$

or the **standard deviation** $\sqrt{Var(K)}$.

The variance of the return can be computed from the variance of S (1),

$$\mathrm{Var}(K) = \frac{Var\,(S\,(1) - S\,(0))}{S(0)}$$

$$= \frac{1}{S(0)^2}\ \mathrm{Var}\,(S\,(1) - S\,(0))$$

$$= \frac{1}{S(0)^2}\mathrm{Var}\,(S\,(1))\ .$$

We use the Greek letter σ for standard deviations of various random returns

$$\sigma = \sqrt{Var(K)},$$

Example

Consider three assets with today's prices $S_i(0) = 100$ for $i = 1, 2, 3$ and time 1 prices with the following distributions:

$$S_1(1) = \begin{cases} 120 & \text{with probability } \frac{1}{2}, \\ 90 & \text{with probability } \frac{1}{2}, \end{cases}$$

$$S_2(1) = \begin{cases} 140 & \text{with probability } \frac{1}{2}, \\ 90 & \text{with probability } \frac{1}{2}, \end{cases}$$

$$S_3(1) = \begin{cases} 130 & \text{with probability } \frac{1}{2}, \\ 100 & \text{with probability } \frac{1}{2}. \end{cases}$$

We can see that

$$\sigma_1 = \sqrt{Var(K1)} = 0.15,$$

$$\sigma_2 = \sqrt{Var(K2)} = 0.25,$$

$$\sigma_3 = \sqrt{Var(K2)} = 0.15.$$

Here $\sigma_2 > \sigma_1$ and $\sigma_3 = \sigma_1$, but both the second and third assets are preferable to the first, since at time 1 they bring in more cash. We shall return to this example in the next section.

3. Efficient Subset

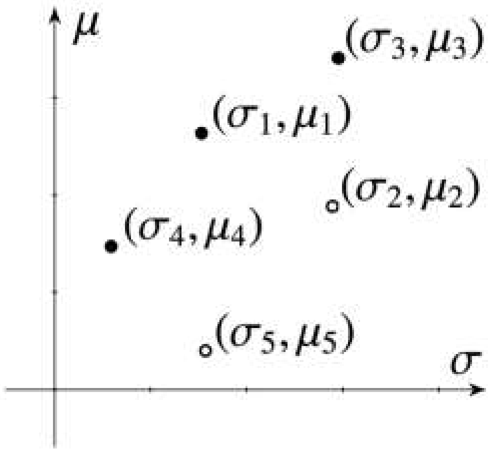


Figure 1.1 Efficient subset.

We say that a security with expected return μ_1 and standard deviation σ_1 dominates another security with expected return μ_2 and standard deviation σ_2 if

$$\mu_1 \geq \mu_2 \quad \text{and} \quad \sigma_1 \leq \sigma_2.$$

The playground of portfolio theory is the (σ, μ)-plane, more precisely the right half-plane, since the standard deviation is non-negative. Each security is represented by a point on this plane. This means that we simplify by assuming that only the expected value and the variance are important in investment decisions.

We assume that the dominant securities are preferred, which geometrically (geographically) means that given any two securities, the one further northwest in the (σ, μ)-plane is preferred. This arrangement does not allow us to compare all pairs: For example, in Figure 1.1 we see that the pairs (σ_1, μ_1) and (σ_3, μ_3) are not comparable.

Given a set A of securities in the (σ, μ)-plane, we consider the subset of all maximal elements with respect to the dominance relation and call it the **efficient subset**. When the set A is finite, the search for the efficient subsets reduces to the elimination of the dominated securities. Figure 1.1 shows a set of five securities, where the efficient subset consists of only three securities numbered 1, 3, and 4.

4. Weights

When we design a portfolio, usually its initial value is the starting point of our considerations, and it is given. The decision on the number of shares in each asset will follow from the decision on the division of our wealth, which is our primary concern and is expressed by means of the **weights**

defined by

$$w_1 = \frac{x_1 S_1(0)}{V(x_1,x_2)(0)}, \quad w_2 = \frac{x_2 S_2(0)}{V(x_1,x_2)(0)}.$$

If the initial wealth $V(0)$ and the weights w_1,w_2, $w_1+w_2 = 1$, are given, then the funds allocated to a particular stock are $w_1V(0)$, $w_2V(0)$, respectively, and the numbers of shares we buy are

$$x_1 = \frac{w_1 V(0)}{S_1(0)}, \quad x_2 = \frac{w_2 V(0)}{S_2(0)}.$$

At the end of the period the securities prices change, which gives the final value of the portfolio as a random variable

$$V(x_1,x_2)(1) = x_1 S_1(1) + x_2 S_2(1).$$

To express the return on a portfolio we employ the weights rather than the numbers of shares since this is more convenient.

The return on the investment in **two assets** depends on the method of allocation of the funds (the weights) and the corresponding returns. The vector of weights will be denoted by $w = (w_1,w_2)$, or in matrix notation

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

and the return of the corresponding portfolio by K_w.

5. Portfolio Return and Risk

The return K_w on a portfolio consisting of **two securities** is the weighted average

$$K_w = w_1K_1 + w_2K_2, \qquad (1)$$

where w_1 and w_2 are the weights and K_1 and K_2 the returns on the two components.

In reality, the numbers of shares have to be integers. This, however, puts a constraint on possible weights since not all percentage splits of our wealth can be realised. To simplify matters we make the assumption that our stock position, that is, the number of shares, can be any real number. When the number of shares of given stock is positive, then we say that we have a long position in the stock. We shall assume that we can also hold a negative number of shares of stock. This is known as **short-selling**. Short-selling is a mechanism by which we borrow stock at time 0 and sell it immediately; we then need to buy it back at time 1 to return it to the lender. This mechanism gives us additional money at time 0 that can be invested in a different security.

Theorem 1

The expected return and the variance of the return on a portfolio are given by

$$\mu_w = E(K_w) = w_1\mu_1 + w_2\mu_2, \qquad (2)$$

$$\sigma^2_w = \mathrm{Var}\,(K_w) = w^2_1\sigma^2_1 + w^2_2\sigma^2_2 + 2w_1w_2\sigma_{12}. \qquad (3)$$

6. Minimum Variance Portfolio

We return to the case of two risky securities, S_1 and S_2. We wish to minimize the variance σ^2_w – or, equivalently, the standard deviation σ_w. We start with a theorem where the problem is solved when there are no restrictions on short-selling.

Theorem 2

If short-selling is allowed, then the portfolio with minimum variance has the weights $w_{min} = (w_1,w_2)$ with

$$w_1 = \frac{a}{a+b}, \qquad w_2 = \frac{b}{a+b},$$

where

$$a = \sigma^2_2 - \rho_{12}\sigma_1\sigma_2,$$

$$b = \sigma^2_1 - \rho_{12}\sigma_1\sigma_2,$$

unless both $\rho_{12} = 1$ and $\sigma_1 = \sigma_2$.

Theorem 3

The expected return $\mu_w = E(K_w)$ and variance $\sigma^2_w = Var(K_w)$ of a portfolio with weights **w** are given by

$$\mu_w = w^T\mu,$$

$$\sigma^2_w = w^TCw.$$

Theorem 4

The portfolio with the smallest variance in the attainable set has weights

$$w_{min} = \frac{C^{-1}\mathbf{1}}{\mathbf{1}^T C^{-1}\mathbf{1}}. \qquad (3)$$

The minimum variance portfolio has the surprising property that its covariance with any other portfolio is constant. This property will prove useful later on, when discussing the shape of the attainable set in the (σ, μ) plane.

7.Minimum variance line

To find the efficient frontier, we have to recognise and eliminate the dominated portfolios. To this end we fix a level of expected return, denote it by m, and consider all portfolios with $\mu_w = m$. All of these are redundant except the one with the smallest variance. The family of such portfolios, parameterised by m, is called the **minimum variance line**. More precisely, portfolios on the minimum variance line are solutions of the following problem:

$$\min \mathbf{w}^{\mathbf{T}}\mathbf{C}\mathbf{w},$$

$$\text{subject to:} \quad \boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{\mu} = m, \qquad (4)$$

$$\mathbf{w}^{\mathbf{T}}\mathbf{1} = 1.$$

Theorem 5

For a given level of return m, let M be a 2×2 matrix of the form

$$M = \begin{bmatrix} \mu^T C^{-1}\mu & \mu^T C^{-1}\mathbf{1} \\ \mu^T C^{-1}\mathbf{1} & \mathbf{1}^T C^{-1}\mathbf{1} \end{bmatrix} .$$

If C and M are invertible, then the solution of problem (4) is given by

$$w = \frac{1}{det(M)} C^{-1} (\det(M_1)\mu + \det(M_2)\,\mathbf{1}) , \qquad (5)$$

where

$$M_1 = \begin{bmatrix} m & \mu^T C^{-1}\mathbf{1} \\ 1 & \mathbf{1}^T C^{-1}\mathbf{1} \end{bmatrix} , M_2 = \begin{bmatrix} \mu^T C^{-1}\mathbf{1} & m \\ \mathbf{1}^T C^{-1}\mathbf{1} & 1 \end{bmatrix}.$$

Proof We introduce the Lagrange multiplier $\lambda = (\lambda 1, \lambda 2)$, and the Lagrangian

$$L(w) = \nabla(w^T C w) - \lambda_1 \nabla(w^T \mu - m) + \lambda_2 \nabla(w^T \mathbf{1} - 1) = 0.$$

Using Lemma 4.3 we can compute

$$L(w) = 2Cw - \lambda_1 \mu - \lambda_2 \mathbf{1} = 0.$$

We solve this system for w:

$$w = \frac{1}{2}\lambda_1 C^{-1}\mu + \frac{1}{2}\lambda_2 C^{-1}\mathbf{1}. \qquad (6)$$

Since $w^T\mu = \mu^T w$ and $w^T\mathbf{1} = \mathbf{1}^T w$, substituting (6) into the constraints from (4), we obtain a system of linear equations

$$\frac{1}{2}\lambda_1\mu^T C^{-1}\mu + \frac{1}{2}\lambda_2\mu^T C^{-1}\mathbf{1} = m,$$

$$\frac{1}{2}\lambda_1\mathbf{1}^T C^{-1}\mu + \frac{1}{2}\lambda_2 1^T C^{-1}\mathbf{1} = 1.$$

We can solve the above system for λ_1 and λ_2 to obtain (note the relevance of the assumption that M is invertible, which ensures that $\det(M) \neq 0$)

$$\frac{1}{2}\lambda_1 = \frac{det\,(M_1)}{det\,(M)}, \qquad \frac{1}{2}\lambda_2 = \frac{det\,(M_2)}{det\,(M)}.$$

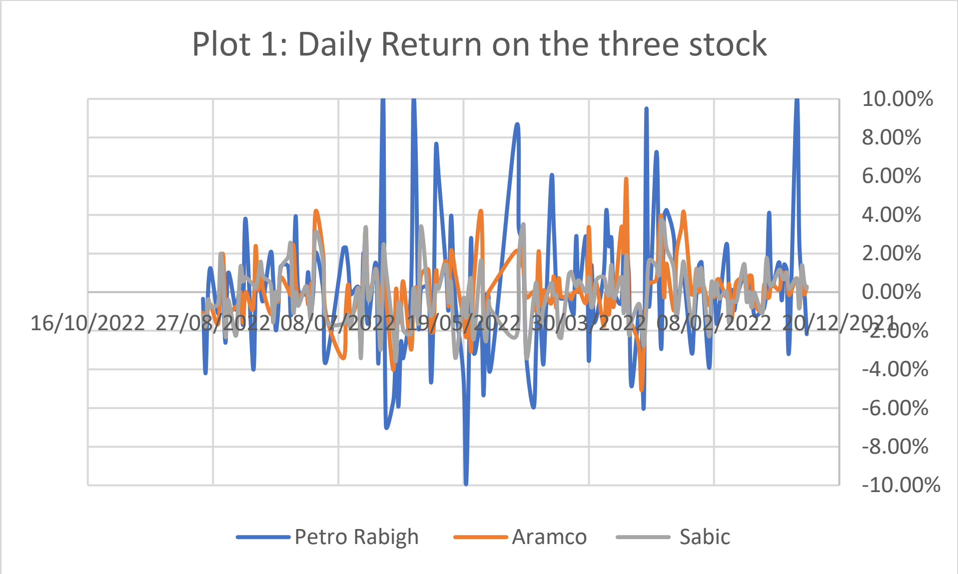
Substituting the above back into (6) gives (5).

We have found a candidate for the solution of (4). this ensures that we have found a global minimum.

8. Application to Market Data.

In this project, we used data from 3 companies, namely Petro Rabigh, Aramco, and SABIC, all of which are in one industry, petrochemicals, and the data was collected for 3 companies from 02/01/2022 to 01/09/2022.

We use the following formula to compute the daily returns.



Plot 1 shows the daily returns on the three stocks, and it is clear that the returns on Petro Rabigh have the highest volatility compared with the other stocks.

The expected values of return, variance and standard deviation (μ, σ^2 and σ respectively) and covariance, and correlation matrixes are calculated.

	Petro Rabigh	Aramco	SABIC
μ	0.15%	0.10%	-0.09%
σ^2	0.10%	0.02%	0.02%
σ	3.16%	1.54%	1.47%

correlation matrix	Petro Rabigh	Aramco	Sabic
Petro Rabigh	1.00	0.26	0.30
Aramco	0.26	1.00	0.48
Sabic	0.30	0.48	1.00

covariance matrix	Petro Rabigh	Aramco	Sabic
Petro Rabigh	0.10%	0.01%	0.01%
Aramco	0.01%	0.02%	0.01%
Sabic	0.01%	0.01%	0.02%

From our calculation it is clear that Petro Rabigh has the highest return and risk. Moreover, the expected return on SABIC has a negative value. Further, all pairs of stock are poorly positive correlated

We use **Theorem 2** to find the GMV portfolio for each pair stock.

Case 1	[Petro Rabigh, Aramco]	W = [11.40%,88.60%]	R = 0.10%	σ = 1.50%
Case 2	[Petro Rabigh, SABIC]	W = [10.09%,89.91%]	R = -0.07%	σ = 1.45%
Case 3	[Aramco, SABIC]	W = [45.43%,54.57%]	R = -0.01%	σ = 1.29%

Form above it is clear that the successful investment is Case 1, which has return of 0.10% and risk of 1.50%. We note that this investment has a similar return on Aramco, but with lower risk.

We use **Theorem 4** to find the GMV portfolio consisting of the three stocks.

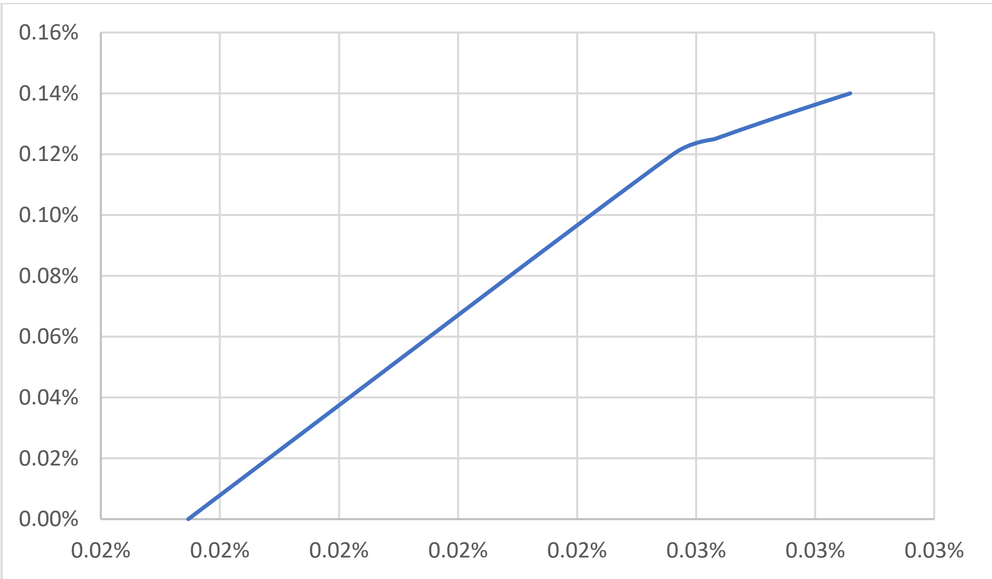
W = [3.85% , 43.89% , 52.25%]

R = 0.000012%

σ = 1.28%

We use **Theorem 5** to find the MV portfolio consisting of the three stocks for given return.

R	W=[Petro Rabigh, Aramco, SABIC]	σ
0.12%,	[16.23% , 92.38% , -8.61%]	1.57%
0.125%,	[16.75% , 94.42% , -11.17%]	1.59%
0.13%,	[17.27% , 96.46% , -13.73%]	1.61%
0.135%,	[17.79% , 98.50% , -16.30%]	1.64%
0.14%,	[18.31% , 100.54% , -18.86%]	1.66%



We see here in the graph that (as expected) the higher return on a portfolio leads to a higher risk .

References

Maciej J. Capiński, *AGH University of Science and Technology, Krakow*, Ekkehard Kopp, *University of Hull*

Tadawul (saudiexchange) , https://www.saudiexchange.sa/wps/portal/tadawul/markets/equities/market-watch?locale=ar

Excel functions, https://support.microsoft.com