Kingdom of Saudi Arabia
Al-Imam Mohammed Ibn Saud
Islamic University
Faculty of Science

المملكة العربية السعودية
جامعة الإمام محمد بن سعود الإسلامية
كلية العلوم

Department of Mathematics and Statistics

قسم الرياضيات والإحصاء

SOLUTIONS OF FOURTH-ORDER TWO POINTS BVP FOR ORDINARY DIFFERENTIAL EQUATIONS

Research Project by:
Rowaida Abdullah Ali Alrajhi

Supervisor
Prof. Dr. Lazhar Bougoffa

Academic Year: 1438 - 1439

Acknowledgments

I would like to express my sincere thanks and gratitude to Almighty Allah for his blessings who always help and guide me to bring forth to light this work. I am extremely and sincerely thankful to all my family: my father peace and mercy on him, my mother, my brother Mazen and my sisters, for their support and sacrifices and whose enabled me to reach this level of my life.

I am also grateful to my supervisor Prof. Lazhar Bougoffa for suggesting the topic of my project, for his support, fruitful ideas, guidance and assistance.

Finally, I am especially grateful to all members at the Department of Mathematics and Statistics at Al-Imam University for the substantial influence that their courses have had on my research and who taught me to come to this stage of learning. I realize more and more clearly the huge impact that they have had on my academic career.

Abstract

The purpose of this project is to study the boundary value problem for the fourth-order beam equation:

$$u^{(4)}(x) + f(x)u(x) = g(x),\ 0 < x < 1, \tag{1}$$

subject to various boundary conditions:

$$\text{The first set of BC} : u(0) = u'(0) = u''(1) = u'''(1) = 0 \tag{2}$$

which corresponds to a beam clamped at $x = 0$ and free at $x = 1$, or

$$\text{the second set of BC} : u(1) = u'(1) = u''(0) = u'''(0) = 0 \tag{3}$$

that corresponds to a beam clamped at $x = 1$ and free at $x = 0$, where f and g are continuous functions on [0,1].

The existence and uniqueness solution in a Hilbert space are proved. The proof is based on a priori estimate and the density of the range of the operator generated by the studied problem. Also, we investigate the application of the contraction mapping theorem for proving the existence and uniqueness theorems for the classical solution of the above problems and extend our study to the nonlinear fourth-order differential equations.

Contents

Introduction

Linear differential equations subject to some boundary conditions arise in the mathematical description of some physical systems. For example, mathematical models of deflection of beams. These beams, which appear in many structure, deflect under their own weight or under the influence of some external forces. For example, if a load is applied to the beam in a vertical plane containing the axis of symmetry, the beam undergoes a distortion, and the curve connecting the centroids of all cross sections is called the deflection curve or elastic curve. In elasticity it is shown that the deflection of the curve, say $u(x)$ measured from the x-axis, approximates the shape of the beam and satisfies the linear fourth-order differential equation [10]:

$$u^{(4)} + f(x)u = 0, \tag{4}$$

on an interval, say [0, 1], with some boundary conditions. Boundary conditions associated with these types of differential equations depend on how the ends of the beams are supported.

In [1,2,3], the authors considered the following linear boundary value problem:

$$u^{(4)} + f(x)u = g(x), \quad 0 < x < 1, \tag{5}$$

subject to

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{6}$$

where f and g are continuous functions on [0, 1]. An interesting result on the existence and uniqueness theorem can be found in [1]. The reader will find in my project more details about the existence and uniqueness theorems of this type of problems.

This project consists of four chapters and the structure of it is:
In the first chapter, we will present some known notions and results in the form of definitions, examples and properties of a normed, Hilbert and Banach spaces. Also, linear operators are discussed in this chapter.
In the second chapter, we consider the beam equation (5) under various boundary conditions

$$\text{The first set of BC} : u(0) = u'(0) = u''(1) = u'''(1) = 0, \tag{7}$$

or

$$\text{the second set of BC} : u(1) = u'(1) = u''(0) = u'''(0) = 0 \tag{8}$$

and establish a sufficient condition on $f(x)$ that guarantees a unique solution in a Hilbert space by using an a priori estimate and then prove some results on the existence and uniqueness theorems. The proof is based on an a priori estimate and the density of the range of the linear operator generated by the studied problem.
In the third chapter, we will investigate the application of the fixed-point theorem for proving the existence and uniqueness of the classical solution to equation (5) subject to the boundary conditions (7) or (8). The fourth chapter is devoted to study the boundary value problem for nonlinear fourth-order differential equation:

$$u^{(4)} + f(x)u = g(x, u, u''),\ 0 < x < 1, \tag{9}$$

subject to

$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{10}$$

Section 1 deals with the proof of a priori estimate. In section 2, some results on the uniqueness solution is given.

Chapter 1

Preliminaries

In this chapter, we will give some definitions and fundamental theorems on metric, Hilbert, Banach spaces and linear operators which are tremendous importance in this project. For more details, we refer to [4, 5].

1.1 Basic definitions and properties

1.1.1 Metric and Normed spaces

Definition 1. ***[Metric space]*** *A metric d on a set X is a real valued function $d: X \times X \longrightarrow \mathbb{R}^+$ such that for all $x,\, y,\, z \in X$:*

1. $d(x,\, y) \geq 0$ *and* $d(x,\, y) = 0$ *if and only if* $x = y$;
2. $d(x,\, y) = d(y,\, x)$;
3. $d(x,\, z) \leq d(x,\, y) + d(y,\, z)$.

A metric space $(X,\, d)$ is a set X with a metric d defined on $X \times X$.

Example 1. *Define $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ by*

$$d(x,\, y) = |x - y|.$$

Then d is a metric on $\mathbb{R}$.

Theorem 1.1.1. *Let A be a nonempty subset of a metric space (X, d).*
A is closed if and only if for any $u_n \in A$ such that $\lim\limits_{n\to\infty} u_n = u$, then $u \in A$.

Definition 2. ***[Normed space]*** *Let X be a vector space. A norm on X is a real valued function $\|\cdot\| : X \longrightarrow \mathbb{R}^+$ such that for all $x,\, y \in X$:*

1. $\|x\| \geq 0$, *and* $\|x\| = 0$ *if and only if* $x = 0$;
2. $\|\alpha x\| = |\, \alpha\, |\ \|x\|$, $\alpha \in \mathbb{R}$;
3. $\|x + y\| \leq \|x\| + \|y\|$.

A normed space $(X, \|\cdot\|)$ is a set X with a norm $\|\cdot\|$ defined on X.

Remark 1. *Every normed space is a metric space with the metric*

$$d(x, y) = \|x - y\| \quad \text{for all } x, y \in X.$$

Definition 3. *Let $(X, \|\cdot\|)$ be a normed space.*

1. *A sequence $\{u_n\}$ in X is called a Cauchy sequence if*

$$\forall \varepsilon > 0, \exists N > 0 : \|u_n - u_m\| < \varepsilon \text{ for all } n, m \geq N.$$

2. *A sequence $\{u_n\}$ in X converges to $u \in X$ if*

$$\forall \varepsilon > 0, \exists N > 0 : \|u_n - u\| < \varepsilon \text{ for all } n \geq N.$$

Theorem 1.1.2. *Let $(X, \|.\|)$ be a normed space*

1. *The limit of a convergent sequence is unique.*
2. *Every Cauchy sequence is bounded.*
3. *Every convergent sequence is a Cauchy sequence. But the converse needs not be true in every metric space.*

Definition 4. *A metric space X is called complete if every Cauchy sequence in X converges to a point in X. A normed vector space which is complete is called a Banach space.*

Remark 2. *Every Banach space is a normed space but the converse, in general is not true.*

Example 2. The vector space $X = \mathbb{C}[a, b]$ is a normed space with respect to the following norms:

1.

$$\|f\|_1 = \int_a^b |f(x)|\, dx.$$

2.

$$\|f\|_2 = \left(\int_a^b |f(x)|^2\, dx \right)^{1/2}.$$

3.

$$\|f\|_\infty = \sup_{a\leq x\leq b} |f(x)|.$$

1.1.2 Operators

Definition 5. *Let X and Y be two normed spaces. A mapping $T : X \longrightarrow Y$ is called an operator and the value of T at $x \in X$ is denoted by $T(x)$ or Tx.*

1. *T is called a linear operator if*

   (a) $T(x + y) = T(x) + T(y),\ \forall x, y \in X,$

   (b) $T(\alpha x) = \alpha T(x),\ \forall x \in X$ *and* $\alpha \in \mathbb{R}$.

2. *T is a bounded if $\exists\, k > 0 :\ \|Tx\| \leq k\|x\|, \forall x \in X$.*

3. *T is continuous at $x_0 \in X$ if $\forall \varepsilon > 0\, \exists\, \delta > 0$ such that $\|x - x_0\| < \delta$ implies $\|Tx - Tx_0\| < \varepsilon$.*

4. *T is uniformly continuous if $\forall \varepsilon > 0\, \exists\, \delta > 0$ for any $x,\ y \in X$ such that $\|x - y\| < \delta$ implies $\|Tx - Ty\| < \varepsilon$.*

5. $\|T\| = \sup\left\{ \dfrac{\|Tx\|}{\|x\|} : x \neq 0 \right\} = \sup\{\, \|Tx\| : \|x\| = 1\}$ *is called supremum norm.*

6. *Let T be a linear operator. The null space $\mathcal{N}(T)$ of T is the subspace of X defined by*

   $$\mathcal{N}(T) = \{x \in X : Tx = 0\}.$$

   Note: The null space of T is sometimes called the kernel of T.

7. *Let T be a linear operator. The range space $\mathcal{R}(T)$ of T is the subspace of Y defined by*

   $$\mathcal{R}(T) = \{Tx : x \in X\}.$$

Remark 3. *In the previous definition, when $Y = \mathbb{R}$, if T satisfies (1), then it is called a linear functional.*

Example 3. Let $X = \mathbb{C}[a,\, b]$ and $T: \ X \longrightarrow \mathbb{R}$ be an operator defined by

$$Tf \ = \int_a^b f(t)\, dt.$$

Then T is a linear functional.

Definition 6. *Let X and Y be Banach spaces. An operator $T: \mathcal{D}(T) \subset X \longrightarrow Y$ is said to be a closed operator if for any sequence $\{u_n\} \subset \mathcal{D}(T)$, $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} T(u_n) = w$ imply $u \in \mathcal{D}(T)$ and $w = T(u)$.*

Theorem 1.1.3. *Let X and Y be normed spaces and $T: \ X \longrightarrow Y$ be a linear operator. Then the following statements are equivalent*

1. *T is continuous.*

2. *T is bounded.*

Example 4. Let $P[0,\, 1]$ be the space of $\mathbb{C}[0,\, 1]$ equipped with supremum norm. Define the operator $T: \ P[0,\, 1] \longrightarrow \mathbb{C}[0,\, 1]$ by

$$Tf = \frac{df}{dt}.$$

Then we have

(a) T is linear.

(b) T is closed.

(c) T is not continuous.

Indeed,

(a) If f, g are in $P[0,\, 1]$, and $\alpha \in \mathbb{R}$, then

- $T(f \,+\, g) \;=\; \dfrac{d(f+g)}{dt} \;=\; \dfrac{df}{dt} \,+\, \dfrac{dg}{dt} \;=\; T(f) \,+\, T(g).$

- $T(\alpha\, f) \;=\; \dfrac{d(\alpha\, f)}{dt} \;=\; \alpha\, \dfrac{df}{dt} \;=\; \alpha T(f).$

(b) We have $\lim_{n\to\infty} f_n(t) = f(t)$ *and* $\lim_{n\to\infty} Tf_n(t) = g(t)$ *uniformly. Then we get*

$$\begin{aligned}\int_0^t g(s)\,ds &= \int_0^t \lim_{n\to\infty} \frac{df_n(s)}{ds}\,ds = \lim_{n\to\infty}\int_0^t \frac{df_n(s)}{ds}\,ds \\ &= \lim_{n\to\infty}\,[f_n(t) - f_n(0)] = f(t) - f(0).\end{aligned}$$

Hence $\int_0^t g(s)\,ds = f(t) - f(0)$ and $g = \dfrac{df}{dt}$.

(c) The function $f_n(t) = \dfrac{t^n}{n}$ is continuous on $[0,\,1]$, since it is a polynomial. We have $\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \dfrac{t^n}{n} = 0$ for all $t \in [0,1]$ and $\lim_{n\to\infty} T(f_n(t)) = \lim_{n\to\infty} t^{n-1} = 1$ when $t = 1$. Hence $\lim_{n\to\infty} T(f_n(t)) \neq 0$.

Remark 4. *A closed linear operator need not be bounded.*

1.2 Hilbert Space

Definition 7. *An inner product on a vector space X is a mapping of $X \times X$ into the scalar field $\mathbb{R}$ or $\mathbb{C}$; that is with every pair of vectors x and y, there is an associated scalar which is written $\langle x,\, y\rangle$ where it is called the inner product of x and y, such that for all vectors x, y, z and scalar α we have*

1. $\langle x + y,\, z\rangle = \langle x,\, z\rangle + \langle y,\, z\rangle$;
2. $\langle \alpha\, x,\, y\rangle = \alpha\,\langle x,\, y\rangle$;
3. $\langle x,\, y\rangle = \overline{\langle y,\, x\rangle}$;
4. $\langle x,\, x\rangle \geq 0$ *and* $\langle x,\, x\rangle = 0$ *if and only if* $x = 0$.

$(X, \langle\cdot,\,\cdot\rangle)$ *is called the inner product space or pre-Hilbert space.*

Remark 5. *In view of definition (7), we have*

1. $\langle x,\, y\rangle = \langle y,\, x\rangle$ *in* $\mathbb{R}$.
2. *For $x = y$, and in view of (3), $\langle x,\, x\rangle = \overline{\langle x,\, x\rangle}$ implies $\langle x,\, x\rangle$ is a real number.*

3. *The conditions (1) - (2) imply the following formulas*

   (a) $\langle \alpha\, x + \beta\, y,\, z \rangle = \alpha\, \langle x,\, z \rangle + \beta\, \langle y,\, z \rangle$.

   (b) $\langle x,\, \alpha\, y \rangle = \bar{\alpha}\, \langle x,\, y \rangle$.

   (c) $\langle x,\, \alpha\, y + \beta\, z \rangle = \bar{\alpha}\, \langle x,\, y \rangle + \bar{\beta}\, \langle x,\, z \rangle$.

Theorem 1.2.1. ***[Cauchy-Schwarz inequality]*** *Let X be an inner product space and $x,\, y\, \in\, X$. Then,*

$$\big|\, \langle x,\, y \rangle \,\big|^2 \,\leq\, \langle\, x,\, x \rangle\, \langle y,\, y \rangle.$$

Theorem 1.2.2. *Every inner-product space X is a normed space with respect to the norm*

$$\|x\| = (\langle x,\, x \rangle)^{\frac{1}{2}}\,, \quad \forall x\, \in\, X.$$

Proof.

Let x , $y\, \in\, X$ and $\alpha \in \mathbb{R}$

1. $\|x\| = (\langle x,\, x \rangle)^{\frac{1}{2}} \geq\, 0, \forall\, x\, \in X$, then $\langle x,\, x \rangle\, \geq\, 0$ and $\|x\| = (\langle x,\, x \rangle)^{\frac{1}{2}} = 0$ if and only if $x = 0$.

2. $\|\alpha\, x\| = (\langle \alpha\, x,\, \alpha\, x \rangle)^{\frac{1}{2}} = (|\alpha|^2\, \langle x,\, x \rangle)^{\frac{1}{2}} = |\alpha|\, \|x\|$.

3. We have

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y,\, x + y \rangle \\
&= \langle x,\, x \rangle + \langle x,\, y \rangle + \langle y,\, x \rangle + \langle y,\, y \rangle \\
&= \|x\|^2 + \langle x,\, y \rangle + \overline{\langle x,\, y \rangle} + \|y\|^2 \\
&= \|x\|^2 + 2\, Re\langle x,\, y \rangle + \|y\|^2 \\
&\leq \|x\|^2 + 2\, |\langle x,\, y \rangle| + \|y\|^2.
\end{aligned}$$

   Using Cauchy-Schwarz inequality we get

$$\begin{aligned}
\|x + y\|^2 &\leq \|x\|^2 + 2\, \|x\|\|y\| + \|y\|^2 \\
&\leq (\|x\| + \|y\|)^2\,.
\end{aligned}$$

Therefore $\|x+y\| \;\le\; \|x\| + \|y\|$.

Definition 8. *An inner product space X is called a Hilbert space if the normed space induced by the inner product is complete.*

Theorem 1.2.3. ***[Parallelogram law]*** *For any two elements x and y belong to any inner product space, we have*

$$\|x+y\|^2 + \|x-y\|^2 = 2\,(\|x\|^2 + \,\|y\|^2).$$

Theorem 1.2.4. *A Banach space is a Hilbert space if and only if its norm satisfies the parallelogram law.*

Definition 9. *Let X be an inner product space and $x,\, y \,\in\, X$.*

1. *The two vectors x and y are called orthogonal, denoted by $x \;\perp\; y$, if $\langle x,\, y\rangle = 0$.*
2. *x is orthogonal to the subset A of X and we write $x \perp A$ if $\langle x, y\rangle = 0$ for each $y \in A$.*
3. *The set $A^{\perp} = \{x \in X : \langle x, y\rangle = 0$ for each $y \in A\ \}$.*

Theorem 1.2.5. *Let X be any closed subspace of a Hilbert space H. Then*

$$H = X \oplus Y,\ \textit{ with } Y = X^{\perp}.$$

Theorem 1.2.6. *Let X be an inner product space and $x,\, y \,\in\, X$. If $x \;\perp\; y$, then*

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

1.2.1 Operators in Hilbert spaces

Definition 10. *Let X be a Hilbert space and $T : X \longrightarrow X$ be a bounded linear operator on X. Then the adjoint operator T^* is defined by*

$$\langle Tx,\, y\rangle \;=\; \langle x,\, T^*y\rangle, \forall x,\, y \,\in\, X.$$

Theorem 1.2.7. *Let T, S be bounded linear operators on a Hilbert space into itself and α scalar.*
Then we have

1. $I^* = I$, *where I is the identity operator.*

2. $(T + S)^* = T^* + S^*$.

3. $(\alpha T)^* = \bar{\alpha} T^*$.

4. $(T S)^* = S^* T^*$.

5. $T^{**} = T$.

6. $\|T^*\| = \|T\|$.

7. $\|T^*T\| = \|T\|^2$.

8. *If T is invertible so is T^* and* $(T^*)^{-1} = (T^{-1})^*$.

1.3 Notation and useful results

Throughout this project, we will denote by

- $L^2[0, 1]$ the space of square-integrable functions, i.e.,

$$L^2[0,1] = \left\{ f : [0,1] \longrightarrow \mathbb{R} : \int_0^1 |f(x)|^2 dx < \infty \right\}.$$

- $\mathbb{C}[0, 1]$ the space of all continuous functions defined on $[0, 1]$.

- $\mathbb{C}^n[0, 1]$, $n \geq 1$, the space of all continuous functions defined on $[0, 1]$ and having continuous derivatives of orders less than or equal to n.

Proposition 1.3.1. *[$\varepsilon-$**inequality**]*

$$2uv \leq \varepsilon u^2 + \frac{1}{\varepsilon} v^2, \ \ u, v \geq 0, \ \ \varepsilon > 0.$$

Lemma 1.3.2. *Let I be an open interval of $\mathbb{R}$ and let f be a continuous function on I. If*

$$\int_I f(x)g(x)dx = 0,\ \forall g \in \mathbb{C}_0^\infty(I),$$

where $\mathbb{C}_0^\infty(I)$ is the space of all functions with compact support in I having continuous derivatives of any order, then $f = 0$ on I.

Chapter 2

An Existence and uniqueness theorem for the solution of the BVP for fourth-order equation

2.1 Introduction

Consider the linear boundary value problem for fourth-order differential equation [1, 2, 3]:

$$u^{(4)} + f(x)u = g(x),\ 0 < x < 1, \tag{2.1}$$

subject to

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{2.2}$$

where f and g are continuous functions on [0, 1]. This problem is used in different areas of physics, engineering and mathematics such as plate deflection theory. The analytical solution for problem (2.1) with (2.2) is given by Timoshenko and Woinowsky-Krieger [3] provided the functions f and g are continuous.
In [1], the author established a sufficient condition $\sup\limits_{0\leq x\leq 1} \mid f(x) \mid < \pi^4$ that guarantees a unique solution for problem (2.1) with (2.2). Also, the existence and uniqueness theorem for the positivity (or negativity) solution is given in [6]. In [7], the authors established sufficient conditions on f that guarantee a unique solution of this problem in a Hilbert space by using an a priori estimate. Accurate analytic solutions in series forms are obtained by a new variation of the Duan-Rach modified Adomian decomposition method (DRMA). Also, a comparison of the two approximate solutions by the ADM with the Green function approach is presented in [7].

In this chapter, we consider the beam equation (2.1) under various boundary conditions:

$$\text{The first set of BC} : u(0) = u'(0) = u''(1) = u'''(1) = 0, \tag{2.3}$$

which corresponds to a beam clamped at $x = 0$ and free at $x = 1$, or

$$\text{the second set of BC} : u(1) = u'(1) = u''(0) = u'''(0) = 0, \tag{2.4}$$

that corresponds to a beam clamped at $x = 1$ and free at $x = 0$.

We first establish a sufficient condition on f that guarantees a unique solution in a Hilbert space by using a priori estimate and then prove some results on the existence and uniqueness theorem.

2.2 A priori Estimate

Now we shall give some lemmas which we want to use later on.
The solution of (2.1) with (2.3) or (2.1) with (2.4) will be considered as a solution of the functional equation

$$L\,u\;=\;g,\quad u\in U, \tag{2.5}$$

where $L\;:\;\mathcal{D}(L)\;\subset\;U\;\longrightarrow\;L^2[0,1]$:

$$L\,u\;=\;u^{(4)}\;+\;f(x)\,u. \tag{2.6}$$

Here U is a normed space defined as

$$U\;=\;\{u\;:\;u,\;\frac{d^iu}{dx^i}\in L^2[0,1],\;i=1,2,3,4\}, \tag{2.7}$$

with respect to the norm

$$\|u\|_U^2\;=\;\int_0^1\left[u^2+\left(\frac{du}{dx}\right)^2+\left(\frac{d^2u}{dx^2}\right)^2+\left(\frac{d^3u}{dx^3}\right)^2+\left(\frac{d^4u}{dx^4}\right)^2\right]dx\;<\;\infty \tag{2.8}$$

and $\mathcal{D}(L)$ is the domain of the operator L which consists of all function $u\in U$ satisfying the boundary conditions (2.3) or (2.4).
We recall that the inner product is defined on $L^2[0,1]$ as follows:

$$\langle u,v\rangle=\int_0^1 u(x)v(x)dx.$$

Lemma 2.2.1. *The set U defined in (2.7) is a Hilbert space with respect to the inner product $\langle u,v\rangle_U\;=\;\langle u,v\rangle+\sum_{i=1}^{4}\langle u^{(i)},\,v^{(i)}\rangle$ and its induced norm $\|.\|$.*

Proof.

First, we will prove that U is a subspace of $L^2[0,1]$. Let $\alpha,\,\beta\in\mathbb{R}$ and $u,\,v\;\in U$, we will show that $\alpha\,u+\beta\,v\in U$. Indeed, we have $u,\;v,\;u^{(i)}$ and $v^{(i)}\in L^2[0,1]$ for $i=$

1, 2, 3, 4. Since $L^2[0,1]$ is a vector space, then $\alpha\, u + \beta\, v \in L^2[0,1]$ and $\alpha\, u^{(i)} + \beta\, v^{(i)} \in L^2[0,1]$ for $i = 1, 2, 3, 4$, then $\alpha\, u + \beta\, v \in U$. Hence U is a subspace of $L^2[0,1]$.

Now, we will prove that $\langle \cdot\,,\,\cdot \rangle_U$ is an inner product on U. Let $u, v, w \in U$ and $\alpha \in \mathbb{R}$, then

1.

$$\begin{aligned}\langle u+v, w\rangle_U &= \langle u, w\rangle + \langle u', w'\rangle + \langle u'', w''\rangle + \langle u''', w'''\rangle + \langle u^{(4)}, w^{(4)}\rangle \\ &+ \langle v, w\rangle + \langle v', w'\rangle + \langle v'', w''\rangle + \langle v''', w'''\rangle + \langle v^{(4)}, w^{(4)}\rangle \\ &= \langle u, w\rangle_U + \langle v, w\rangle_U.\end{aligned}$$

2.

$$\begin{aligned}\langle u, v\rangle_U &= \langle u, v\rangle + \langle u', v'\rangle + \langle u'', v''\rangle + \langle u''', v'''\rangle + \langle u^{(4)}, v^{(4)}\rangle \\ &= \langle v, u\rangle + \langle v', u'\rangle + \langle v'', u''\rangle + \langle v''', u'''\rangle + \langle v^{(4)}, u^{(4)}\rangle \\ &= \langle v, u\rangle_U\end{aligned}$$

3.

$$\begin{aligned}\langle \alpha\, u, v\rangle_U &= \langle \alpha\, u, v\rangle + \langle \alpha\, u', v'\rangle + \langle \alpha\, u'', v''\rangle + \langle \alpha\, u''', v'''\rangle + \langle \alpha\, u^{(4)}, v^{(4)}\rangle \\ &= \alpha\,\langle u, v\rangle + \alpha\,\langle u', v'\rangle + \alpha\,\langle u'', v''\rangle + \alpha\,\langle u''', v'''\rangle + \alpha\,\langle u^{(4)}, v^{(4)}\rangle \\ &= \alpha\,\langle u, v\rangle_U\end{aligned}$$

4.

$$\begin{aligned}\langle u, u\rangle_U &= \langle u, u\rangle + \langle u', u'\rangle + \langle u'', u''\rangle + \langle u''', u'''\rangle + \langle u^{(4)}, u^{(4)}\rangle \\ &\geq 0.\end{aligned}$$

Then $\langle u, u\rangle_U \geq 0$ and $\langle u, u\rangle_U = 0$ if and only if $\langle u, u\rangle_{L^2} = \langle u^{(i)}, u^{(i)}\rangle_{L^2} = 0$ for $i = 1, 2, 3, 4$ if and only if $u = 0$. Hence U is an inner product space.

Now, let $\{u_n\}_{n=1}^{\infty}$ be a Cauchy sequence in U, that is for every $\varepsilon > 0$ there exists a natural number $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have

$$\|u_n - u_m\|_U^2 \;=\; \|u_n - u_m\|_{L^2}^2 + \sum_{i=1}^{4} \|u_n^{(i)} - u_m^{(i)}\|_{L^2}^2 < \varepsilon. \tag{2.9}$$

Then $\{u_n\}_{n=1}^{\infty}$ and $\{u_n^{(i)}\}_{n=1}^{\infty}$, $i = 1, 2, 3, 4$ are Cauchy sequences in $L^2[0, 1]$. Since $L^2[0, 1]$ is a Hilbert space, there exist $u, v_i \in L^2[0, 1]$, $i = 1, 2, 3, 4$ such that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} u_n^{(i)} = v_i$ in $L^2[0, 1]$. Take $\varphi \in \mathbb{C}^1[0, 1]$ with $\varphi(0) = \varphi(1) = 0$, then we have

$$\int_0^1 u_n' \,\varphi(x)\, dx = -\int_0^1 u_n\, \varphi'(x)\, dx, \text{ where } \varphi(0) = \varphi(1) = 0. \tag{2.10}$$

Now, taking the limit

$$\int_0^1 v_1\, \varphi(x)\, dx = -\int_0^1 u\, \varphi'(x)\, dx = \int_0^1 u'\, \varphi(x)\, dx. \tag{2.11}$$

It follows that

$$\int_0^1 (v_1 - u')\, \varphi(x)\, dx = 0, \text{ for any } \varphi \in \mathbb{C}^1[0, 1]. \tag{2.12}$$

This implies that $v_1 = u'$. Similarly, for $\lim_{n\to\infty} u_n^{(i)} = u^{(i)}$ in $L^2[0, 1]$ for $i = 2, 3, 4$. Consequently, we get $\lim_{n\to\infty} u_n = u$ in U. Hence U is complete. Therefore U is a Hilbert space.

We also need the following integral inequalities involving the function and its derivative.

Lemma 2.2.2. ***[Wirtinger's Inequalities]***

1. *Suppose $u \in \mathbb{C}^1[a, b]$ with $u(a) = 0$ or $u(b) = 0$. Then*

$$\int_a^b (u(x))^2\, dx \leq \frac{4(b-a)^2}{\pi^2} \int_a^b (u'(x))^2\, dx.$$

2. Suppose $u \in \mathbb{C}^1[a,\, b]$ with $u(a) = u(b) = 0$. Then

$$\int_a^b (u(x))^2\, dx \;\le\; \frac{(b-a)^2}{\pi^2} \int_a^b (u'(x))^2\, dx.$$

Proof.

See [8].

Theorem 2.2.3. *Let u be a solution of Problem (2.1) with (2.3) and $0 < \alpha_1 \le f(x) \le \alpha_2$ for all $x \in [0\,,\, 1]$, such that $\alpha_1 = \min\limits_{0 \le x \le 1} f(x) > \dfrac{\pi}{2k}$, $k > \sqrt{2}$ and $\alpha_2 = \max\limits_{0 \le x \le 1} f(x) < \dfrac{\pi}{2\sqrt{2}}$. Then for $g \in L^2\,[0\,, 1]$, there exists a constant $c > 0$ such that the obtained a priori estimate*

$$||\; u \;||_U \le \; c \;||\; g \;||_{L^2} \tag{2.13}$$

holds.

Proof.

Let $Lu = g$, where $Lu = u^{(4)} + f(x)\, u$.

Firstly, consider the scalar product $\langle Lu,\, u\rangle$. Employing integration by parts, and taking into account that $u(0) = u'(0) = 0$ and $u''(1) = u'''(1) = 0$, we obtain

$$\langle Lu\,,\, u\rangle = \int_0^1 (u''(x))^2\, dx + \int_0^1 f(x)\, u^2(x)\, dx. \tag{2.14}$$

The scalar product $\langle Lu, u\rangle$ can be estimated by means of the $\varepsilon-$inequality

$$|\; \langle Lu, u\rangle \;| \le \frac{\varepsilon_1}{2} \int_0^1 (g(x))^2\, dx \;+\; \frac{1}{2\,\varepsilon_1} \int_0^1 (u(x))^2\, dx. \tag{2.15}$$

Since f is a positive and continuous function on $[0, 1]$. Thus $0 < \alpha_1 \le f(x) \le \alpha_2$ for all $x \in [0,\, 1]$, where $\alpha_1 = \min\limits_{0 \le x \le 1} f(x)$ and $\alpha_2 = \max\limits_{0 \le x \le 1} f(x)$. Then

$$\int_0^1 (u''(x))^2\, dx + \alpha_1 \int_0^1 (u(x))^2\, dx \;\le\; \frac{\varepsilon_1}{2} \int_0^1 (g(x))^2\, dx + \frac{1}{2\varepsilon_1} \int_0^1 (u(x))^2\, dx. \tag{2.16}$$

Now, using Wirtinger's inequalities, we have

$$\int_0^1 (u'(x))^2\,dx \;\le\; \frac{4}{\pi^2}\int_0^1 (u''(x))^2\,dx,\; u'(0) \;=\; 0, \tag{2.17}$$

and

$$\int_0^1 (u'''(x))^2\,dx \;\le\; \frac{4}{\pi^2}\int_0^1 (u^{(4)}(x))^2\,dx,\; u'''(1) = 0. \tag{2.18}$$

From (2.1), we have

$$u^{(4)} \;=\; g(x) \;-\; f(x)\,u, \tag{2.19}$$

and therefore, it follows that

$$\int_0^1 (u^{(4)}(x))^2\,dx \le 2\,(\int_0^1 (g(x))^2\,dx \;+\; \alpha_2^2\int_0^1 (u(x))^2\,dx). \tag{2.20}$$

Applying Wirtinger's inequality to obtain

$$\int_0^1 (u^{(4)}(x))^2\,dx \le 2\,(\int_0^1 (g(x))^2\,dx \;+\; \frac{4\alpha_2^2}{\pi^2}\int_0^1 (u'(x))^2\,dx),\; u(0) = 0. \tag{2.21}$$

Adding (2.16), (2.17) and (2.18) to (2.21), we get

$$(\alpha_1 - \frac{1}{2\varepsilon_1})\int_0^1 (u(x))^2\,dx + (1-\frac{8\alpha_2^2}{\pi^2})\int_0^1 (u'(x))^2\,dx + (1-\frac{4}{\pi^2})\int_0^1 (u''(x))^2\,dx$$
$$+\int_0^1 (u'''(x))^2\,dx + (1-\frac{4}{\pi^2})\int_0^1 (u^{(4)}(x))^2\,dx \le (\frac{\varepsilon_1}{2}+2)\int_0^1 (g(x))^2 dx. \tag{2.22}$$

Choosing $\varepsilon_1 = \dfrac{k}{\pi}$ where $k > \sqrt{2}$ so that $\alpha_1 - \dfrac{1}{2\varepsilon_1} > 0$ and $1 - \dfrac{8\alpha_2^2}{\pi^2} > 0$. Let $c_1 = \min(\alpha_1 - \dfrac{1}{2\varepsilon_1}, 1 - \dfrac{8\alpha_2^2}{\pi^2}, 1 - \dfrac{4}{\pi^2})$. Hence the inequality (2.13) holds, where $c = c_2^{\frac{1}{2}}$ and $c_2 = \dfrac{2+\frac{\varepsilon_1}{2}}{c_1}$.

Remark 6. *The same estimate can be obtained for Problem (2.1) with (2.4).*

2.3 Existence and Uniqueness Theorem

In this section, we need the following lemmas to show the existence and uniqueness of the solutions.

Lemma 2.3.1. *Let U be a Hilbert space and let $L : \mathcal{D}(L) \subset U \longrightarrow L^2[0,1]$ be a linear operator. Then $\mathcal{R}(L) = L^2[0,1]$ if and only if $\mathcal{R}(L)$ is closed and $\mathcal{R}(L)^{\perp} = \{0\}$.*

Proof.
See [11].

Lemma 2.3.2. *The linear operator $L : \mathcal{D}(L) \subset U \longrightarrow L^2[0,\, 1]$ defined in (2.6) is closed.*

Proof.
Let $\{u_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{D}(L)$ such that $\lim\limits_{n\to\infty} u_n = u$ and

$$\lim_{n\to\infty} Lu_n = \lim_{n\to\infty} \{u_n^{(4)}(x) + f(x)\, u_n(x)\} = v.$$

We want to prove that $Lu = u^{(4)} + f(x)\, u = v$. Let $\varphi \in \mathbb{C}^4[0,\, 1]$ with $\varphi(0) = \varphi(1) = \varphi^{(i)}(0) = \varphi^{(i)}(1) = 0,$ for $i = 1, 2, 3$.
We have

$$\begin{aligned} \int_0^1 (u_n^{(4)}(x) + f(x)u_n(x))\varphi(x)dx &= \int_0^1 u_n^{(4)}(x)\varphi(x)dx + \int_0^1 f(x)u_n(x)\varphi(x)dx \\ &= \int_0^1 u_n(x)\varphi^{(4)}(x)dx + \int_0^1 f(x)u_n(x)\varphi(x)dx. \quad (2.23) \end{aligned}$$

Now, taking the limit, we obtain

$$\begin{aligned} \int_0^1 v(x)\, \varphi(x)\, dx &= \int_0^1 u(x)\, \varphi^{(4)}(x)\, dx + \int_0^1 f(x)\, u(x)\, \varphi(x)\, dx \\ &= \int u^{(4)}(x)\, \varphi(x)\, dx + \int_0^1 f(x)\, u(x)\, \varphi(x)\, dx \\ &= \int_0^1 (u^{(4)}(x) + f(x)\, u(x))\, \varphi(x)\, dx. \end{aligned} \quad (2.24)$$

It follows that

$$\int_0^1 [v(x) - (u^{(4)}(x) + f(x)\, u(x))]\, \varphi(x)\, dx = 0 \text{ for any } \varphi \in \mathbb{C}^4[0,\, 1]. \quad (2.25)$$

Therefore $v = u^{(4)} + f(x)\, u = Lu$ and $u \in \mathcal{D}(L)$. Hence L is closed.

Theorem 2.3.3. *Let U be a Hilbert space and $L : \mathcal{D}(L) \subset U \longrightarrow L^2[0,1]$ a linear closed operator. Assume that $\mathcal{R}(L)^{\perp} = \{0\}$ and for some constant $c > 0$ the a priori estimate defined in (2.13) holds. Then for each $g \in L^2[0,1]$ the equation $Lu = g, \;\; u \in U$ has a unique solution.*

Proof.

In order to prove the existence, we prove that $\mathcal{R}(L)$ is closed. In fact, let $\{g_n\}$ be a sequence in $\mathcal{R}(L)$ that converges to g and let $\{u_n\} \in \mathcal{D}(L)$ be a sequence with $Lu_n = g_n$. By the inequality (2.13), we have $\|u_n - u_m\|_U \leq c\,\|g_n - g_m\|_{L^2}$. This implies that $\{u_n\}$ is a Cauchy sequence in U. Since U is a Hilbert space, $\{u_n\}$ converges to some $u \in U$. Now, using the fact that L is a closed linear operator, we conclude that $u \in \mathcal{D}(L) \subset U$ and

$$Lu = g \in \mathcal{R}(L).$$

Hence $\mathcal{R}(L)$ is closed. By using Theorem 1.2.5, we get

$$L^2[0,1] = \mathcal{R}(L) \oplus \mathcal{R}(L)^{\perp}.$$

We know that $\mathcal{R}(L)^{\perp} = \{0\}$ and also by Lemma 2.3.1, we obtain $\mathcal{R}(L) = L^2[0,1]$. Then L is onto, which implies that $\forall g \in L^2[0,1] \; \exists\, u \in \mathcal{D}(L)$ such that

$$Lu = g.$$

The uniqueness of the solution follows immediately from the a priori estimate. Indeed, let u_1 and u_2 be two solutions, then we have

$$Lu_1 = g \text{ and } Lu_2 = g.$$

Hence, we get $Lu_1 - Lu_2 = 0$. This implies that $0 \leq \|u_1 - u_2\|_U \leq c\,\|Lu_1 - Lu_2\|_{L^2}$. Thus $u_1 = u_2$.

Chapter 3

Existence and uniqueness theorem for the classical solution

3.1 Introduction

Theorems concerning the existence and uniqueness and properties of fixed points are known as fixed-point theorems. Such theorems are important tools for proving the existence and uniqueness of the solution to various mathematical models (differential equations, integral differential equations) representing phenomena arising in different fields.

Fixed Point Theorems of metric spaces provide us exact or approximate solutions of boundary value problems.

In this chapter, we will investigate the application of the fixed-point theorem for proving the existence and uniqueness of the classical solution to problem (2.1) with (2.3) and problem (2.1) with (2.4).

Definition 11. *Let (X, d) be a metric space and $T : X \longrightarrow X$ be a map. A solution of $T(x) = x$ is called a fixed point of T.*

Definition 12. *Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is a contraction mapping if there exists a constant c with $0 < c < 1$, such that*

$$d(T(x), T(y)) \leq c\, d(x, y)$$

for all $x, y \in X$.

Theorem 3.1.1. *[4, 5]* ***[Fixed-Point Theorem]*** *Let X be a complete metric space and T be a contraction on X. Then there exists a unique $x \in X$ such that $T(x) = x$.*

3.2 The conversion of the BVP (2.1) with (2.3) and (2.1) with (2.4) into Fredholm Integral Equations

In this section, we will reformulate the BVP (2.1) with (2.3) and (2.1) with (2.4) as a fixed point problem for integral equations.

We need the following lemmas.

Lemma 3.2.1. *The solution of the BVP*

$$u'' \; = \; F(x)\,, \quad 0 < x < 1, \tag{3.1}$$
$$u(0) \; = \; u'(0) \; = \; 0 \tag{3.2}$$

is given by $u(x) = \int_0^x G_1(x,y)\,F(y)\,dy$, *where* $G_1(x,y) = x - y$ *is the "Green Function".*

Proof.

We integrate both sides of the equation (3.1) and taking into account the boundary condition (3.2), we obtain

$$u(x) \; = \; \int_0^x \int_0^y F(s)\,ds\,dy. \tag{3.3}$$

Using integration by parts, we get

$$u(x) \; = \; \int_0^x (x \; - \; y)\,F(y)\,dy. \tag{3.4}$$

Hence the solution u is given by

$$u(x) \; = \; \int_0^x G_1(x,y)\,F(y)\,dy, \text{ where } G_1(x,y) \; = \; x \; - \; y. \tag{3.5}$$

Lemma 3.2.2. *The solution of the BVP*

$$v'' \; = \; F(x)\,, \quad 0 < x < 1, \tag{3.6}$$
$$v(1) \; = \; v'(1) \; = \; 0 \tag{3.7}$$

is given by $v(x) = \int_x^1 G_2(x,y)\,F(y)\,dy$, *where* $G_2(x,\,y) = y - x$ *is the "Green Function".*

Proof.

We integrate both sides of the equation (3.6) and taking into account the boundary condition (3.7), we obtain

$$v(x) \; = \; \int_x^1 \int_y^1 F(s)\,ds\,dy. \tag{3.8}$$

Using integration by parts, we get

$$v(x) = \int_x^1 (y - x)\, F(y)\, dy. \tag{3.9}$$

Hence the solution v is given by

$$v(x) = \int_x^1 G_2(x,y) F(y)\, dy, \text{ where } G_2(x,y) = y - x. \tag{3.10}$$

The given problems (2.1) with (2.3) and (2.1) with (2.4) can be converted into the following coupled systems.

$$\begin{cases} u'' = v, \; u(0) = 0, \; u'(0) = 0, \\ \\ v'' = g(x) - f(x)\, u\,, \; v(1) = v'(1) = 0 \end{cases} \tag{3.11}$$

and

$$\begin{cases} u'' = v, \; u(1) = 0, \; u'(1) = 0, \\ \\ v'' = g(x) - f(x)\, u\,, \; v(0) = v'(0) = 0, \end{cases} \tag{3.12}$$

respectively.

We conclude the following results:

Corollary 3.2.3. *The problem (2.1) with (2.3) can be converted into the following Fredholm integral equation*

$$u(x) = \int_0^1 G(x,y)\, f(y)\, u(y)\, dy + h(x), \tag{3.13}$$

where

$$h(x) = -\int_0^1 G(x,\, y) g(y)\, dy$$

and

$$G(x,y) = \begin{cases} y(x-y)^2 & \textit{if } 0 \le y \le x \le 1, \\ \\ x(x-y)^2 & \textit{if } 0 \le x \le y \le 1. \end{cases}$$

Proof.

By using Lemma 3.2.1 and by replacing $F(y)$ with $v(y)$, we obtain

$$u(x) \;=\; \int_0^x G_1(x,y)\,v(y)\,dy, \text{ where } G_1(x,y) \;=\; x \;-\; y, \tag{3.14}$$

and by using Lemma 3.2.2 and by replacing $F(y)$ with $g(y) \;-\; f(y)\,u(y)$, we obtain

$$v(x) = \int_x^1 G_2(x,y)\,[g(y) - f(y)\,u(y)]\,dy, \text{ where } G_2(x,y) = y - x. \tag{3.15}$$

From (3.14) and (3.15) with change of variable, we obtain

$$u(x) = \int_0^x \int_y^1 -(x \;-\; z)^2\,g(z)\,dz\,dy + \int_0^x \int_y^1 (x \;-\; z)^2\,f(z)\,u(z)\,dz\,dy. \tag{3.16}$$

Using integration by parts, we get

$$u(x) \;=\; \int_0^x y(x-y)^2[f(y)u(y)-g(y)]dy + \int_x^1 x(x-y)^2[f(y)u(y)-g(y)]dy. \tag{3.17}$$

Therefore,

$$u(x) \;=\; \int_0^1 G(x,y)\,f(y)\,u(y)\,dy \;+\; h(x), \text{ where} \tag{3.18}$$

$$h(x) \;= -\int_0^1 G(x,\,y)g(y)\,dy$$

and

$$G(x,y) \;=\; \begin{cases} y(x-y)^2 & \text{if } 0 \le y \le x \le 1, \\ \\ x(x-y)^2 & \text{if } 0 \le x \le y \le 1. \end{cases}$$

3.3 Existence and Uniqueness Theorem

Rewrite problem (2.1) with (2.3) as $T(u) \;=\; u$, where

$$Tu \;=\; \int_0^1 G(x,y)\,f(y)\,u(y)\,dy \;+\; h(x). \tag{3.19}$$

Theorem 3.3.1. *Suppose that $G : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ and $h : [0,1] \longrightarrow \mathbb{R}$ are continuous functions, and assume that*

$$K = \sup_{0 \le x \le 1} \int_0^1 \Big| G(x,\, y)\, f(y) \Big| dy \, < \, 1. \tag{3.20}$$

Then there is a unique continuous solution of problem (2.1) with (2.3).

Proof.

Let $T : \mathbb{C}[0,\, 1] \longrightarrow \mathbb{C}[0,\, 1]$ defined by

$$Tu(x) = \int_0^1 G(x,y)\, f(y)\, u(y)\, dy \, + \, h(x). \tag{3.21}$$

We prove that T is a contraction on the normed space $\mathbb{C}[0,1]$ with its uniform norm. We have for all $u,\, v \, \in \mathbb{C}[0,\, 1]$

$$\begin{aligned} |\, Tu(x) - Tv(x) \,| \quad &= \quad \Big| \int_0^1 G(x,y)\, f(y)\, [u(y) \, - \, v(y)]\, dy \Big| \\ &\le \quad \int_0^1 \Big| G(x,y)\, f(y) \Big| \Big| \, u(y) \, - \, v(y) \Big| \, dy \\ &\le \quad \sup_{0 \le x \le 1} \int_0^1 \Big| G(x,\, y)\, f(y) \Big| dy \; \|u - v\|_\infty . \end{aligned}$$

Hence

$$\|Tu - Tv\|_\infty \quad \le \quad K \|u - v\|_\infty . \tag{3.22}$$

Thus, T is a contraction on $\mathbb{C}[0,1]$. Now applying Theorem 3.1.1, there exists one solution $u \, \in \mathbb{C}[0,\, 1]$ such that $Tu \, = \, u$.

Similarly rewrite problem (2.1) with (2.4) as $T(u) \, = \, u$, where

$$Tu \; = \int_0^1 G(x,y)\, f(y)\, u(y)\, dy \, + \, h(x), \tag{3.23}$$

$$h(x) \; = - \int_0^1 G(x,\, y) g(y)\, dy$$

and

$$G(x,y) \;=\begin{cases} y(x-y)^2 & \text{if } 0\le y\le x\le 1, \\ \\ x(x-y)^2 & \text{if } 0\le x\le y\le 1. \end{cases}$$

Remark 7. *For problem (2.1) with (2.3) and (2.1) with (2.4), we find the estimate:*

$$\sup_{0\le x\le 1}\int_0^1 \Big|G(x,\,y)\,f(y)\,\Big|dy \;<\; \frac{1}{12}\,\|f\|_\infty. \tag{3.24}$$

Thus, there is a unique solution of problem (2.1) with (2.3) for any continuous function f *with* $\|f\|_\infty \le 12$.

Chapter 4

BVP for nonlinear fourth-order equation

4.1 Introduction

In [9], Yang extended the previous boundary value problem for fourth-order equation and considered the following nonlinear problem:

$$u^{(4)} = g(x, u, u''), \ 0 < x < 1, \tag{4.1}$$

subject to

$$u(0) = u(1) = u''(0) = u''(1) = 0, \tag{4.2}$$

and established the following result on the existence and uniqueness theorem:

Theorem 4.1.1. *Suppose that $g(x, u, v)$ is continuous on $[0, 1] \times \mathbb{R} \times \mathbb{R}$ and there are constants $a, b, c \geq 0$ such that*

$$|g(x, u, v)| \leq a|u| + b|v| + c, \ \textit{where } \frac{a}{\pi^4} + \frac{b}{\pi^2} < 1. \tag{4.3}$$

Then problem (4.1) subject to (4.2) has a solution.

Motivated by this work of Yang [9], we consider the general nonlinear problem:

$$u^{(4)} + f(x)u = g(x, u, u''), \ 0 < x < 1, \tag{4.4}$$

subject to the set of the homogeneous mixed boundary conditions (4.2).

4.2 A priori estimate

In this section, we establish a result concerning a priori estimate for solutions of the boundary value problem (4.4) subject to (4.2).

Lemma 4.2.1. *Suppose that $g(x, u, v)$ is continuous on $[0, 1] \times \mathbb{R} \times \mathbb{R}$, $f(x)$ is continuous on $[0, 1]$ and there are constants $a, b, c \geq 0$ such that*

$$|g(x, u, v)| \leq a|u| + b|v| + c, \ \textit{where } \frac{a + \|f\|_\infty}{\pi^4} + \frac{b}{\pi^2} < 1. \tag{4.5}$$

Then there exists a constant $M > 0$ such that for any $x \in [0, 1]$ and any solution

$u(x)$ to problem (4.4) subject to (4.2), we have

$$\|u\|_\infty + \|u''\|_\infty = \sup_{0\le x\le 1} |u(x)| + \sup_{0\le x\le 1} |u''(x)| \le M. \tag{4.6}$$

Proof.

Set $v = u''$. Then, we have the following coupled nonlinear system

$$\begin{aligned} u'' &= v,\ u(0) = u(1) = 0, && (4.7)\\ v'' &= g(x,u,v) - f(x)u,\ v(0) = v(1) = 0. && (4.8)\end{aligned}$$

From (4.7), we have $uu'' = uv$, and first using Cauchy-Schwarz inequality and then Wirtinger's inequality, we obtain

$$\begin{aligned} \int_0^1 (u')^2 dx &= -\int_0^1 uv dx \\ &\le \left(\int_0^1 u^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 v^2 dx\right)^{\frac{1}{2}} \\ &\le \frac{1}{\pi}\left(\int_0^1 (u')^2 dx\right)^{\frac{1}{2}} \frac{1}{\pi}\left(\int_0^1 (v')^2 dx\right)^{\frac{1}{2}}. \end{aligned}$$

Consequently

$$\left(\int_0^1 (u')^2 dx\right)^{\frac{1}{2}} \le \frac{1}{\pi^2}\left(\int_0^1 (v')^2 dx\right)^{\frac{1}{2}}. \tag{4.9}$$

From (4.8), we have $vv'' = vg(x,u,v) - f(x)uv$, then

$$\begin{aligned} \int_0^1 (v')^2 dx &= -\int_0^1 vg(x,u,v)dx + \int_0^1 f(x)uvdx \\ &\le \int_0^1 [a|uv| + b|v^2| + c|v|]dx + \max_{0\le x\le 1} |f(x)| \int_0^1 |uv|dx \\ &= \int_0^1 [a|uv| + b|v^2| + c|v|]dx + \|f\|_\infty \int_0^1 |uv|dx. \end{aligned} \tag{4.10}$$

Now, using Cauchy-Schwarz inequality and $\varepsilon-$inequality, we have

$$\int_0^1 (v')^2 dx \le (a+\|f\|_\infty)\left(\int_0^1 u^2 dx\right)^{\frac{1}{2}} \left(\int_0^1 v^2 dx\right)^{\frac{1}{2}} + (b+\frac{\varepsilon}{2})\int_0^1 v^2 dx + \frac{c^2}{2\varepsilon}. \tag{4.11}$$

From Wirtinger's inequality and (4.9), we have

$$\left(\int_0^1 u^2dx\right)^{\frac{1}{2}} \leq \frac{1}{\pi}\left(\int_0^1 (u')^2dx\right)^{\frac{1}{2}} \leq \frac{1}{\pi^3}\left(\int_0^1 (v')^2dx\right)^{\frac{1}{2}} \tag{4.12}$$

and

$$\left(\int_0^1 v^2dx\right)^{\frac{1}{2}} \leq \frac{1}{\pi}\left(\int_0^1 (v')^2dx\right)^{\frac{1}{2}}. \tag{4.13}$$

Therefore, (4.11) becomes

$$\int_0^1 (v')^2dx \leq \frac{a+\|f\|_\infty}{\pi^4}\int_0^1 (v')^2dx + \frac{b}{\pi^2}\int_0^1 (v')^2dx + \frac{\varepsilon}{2\pi^2}\int_0^1 (v')^2dx + \frac{c^2}{2\varepsilon}. \tag{4.14}$$

Since a, b and $\|f\|_\infty$ satisfy the assumptions, we can choose $\varepsilon > 0$ sufficiently small such that

$$1 - \left(\frac{a+\|f\|_\infty}{\pi^4} + \frac{b}{\pi^2} + \frac{\varepsilon}{2\pi^2}\right) = k > 0.$$

Then it follows from (4.14) that

$$\int_0^1 (v')^2dx \leq \frac{c^2}{2\varepsilon k} = c_1, \tag{4.15}$$

and consequently from (4.9) and (4.15), we obtain

$$\left(\int_0^1 (u')^2dx\right)^{\frac{1}{2}} \leq \frac{\sqrt{c_1}}{\pi^2}. \tag{4.16}$$

In particular, (4.15) and (4.16) give us

$$|v(x)| = \left|\int_0^x v'dx\right| \leq \left(\int_0^1 (v')^2dx\right)^{\frac{1}{2}} \leq \sqrt{c_1} \tag{4.17}$$

and

$$|u(x)| = \left|\int_0^x u'dx\right| \leq \left(\int_0^1 (u')^2dx\right)^{\frac{1}{2}} \leq \frac{\sqrt{c_1}}{\pi^2}. \tag{4.18}$$

Then the estimate (4.6) follows from estimates (4.17) and (4.18).

4.3 A uniqueness Theorem

In this section, we will prove the uniqueness of the solution.

Theorem 4.3.1. *Suppose that f is continuous on $[0,1]$ and $g(x,u,v)$ is a Lipschitz function in u and v, that is, there are two constants $k_1, k_2 > 0$ such that*

$$|g(x,u_1,v_1) - g(x,u_2,v_2)| \leq k_1|u_1 - u_2| + k_2|v_1 - v_2| \tag{4.19}$$

for all $u_i, v_i \in \mathbb{C}[0,1], i = 1,2,$ where $\dfrac{k_1 + \|f\|_\infty}{\pi^4} + \dfrac{k_2}{\pi^2} < 1.$

Then, there exists at most one solution $u \in \mathbb{C}[0,1]$ of problem (4.4) with (4.2).

Proof.

Suppose there are two solutions u_1 and u_2 such that $u_1 \neq u_2$. Then we have

$$\begin{cases} u_1^{(4)} + f(x)u_1 = g(x,u_1,u_1''), \\ \\ u_1(0) = u_1(1) = u_1''(0) = u_1''(1) = 0 \end{cases} \quad \text{and} \quad \begin{cases} u_2^{(4)} + f(x)u_2 = g(x,u_2,u_2''), \\ \\ u_2(0) = u_2(1) = u_2''(0) = u_2''(1) = 0. \end{cases}$$

Let $w = u_1 - u_2$. Then

$$\begin{cases} w^{(4)} + f(x)w = g(x,u_1,u_1'') - g(x,u_2,u_2''), \ 0 < x < 1 \\ \\ w(0) = w(1) = w''(0) = w''(1) = 0. \end{cases} \tag{4.20}$$

Set $z = w''$. We have the following coupled problem

$$w'' = z, \ w(0) = w(1) = 0, \tag{4.21}$$

$$z'' = -f(x)w + g(x,u_1,u_1'') - g(x,u_2,u_2''), \ z(0) = z(1) = 0. \tag{4.22}$$

From (4.21), we have $ww'' = wz$. Proceeding as in Lemma (4.2.1), we have

$$\left(\int_0^1 (w')^2 dx\right)^{\frac{1}{2}} \leq \frac{1}{\pi^2}\left(\int_0^1 (z')^2 dx\right)^{\frac{1}{2}}. \tag{4.23}$$

From (4.22), we have $zz'' = -f(x)zw + z\left[g(x, u_1, u_1'') - g(x, u_2, u_2'')\right]$. Since g is a Lipschitz function, we get

$$\int_0^1 (z')^2 dx \leq (\|f\|_\infty + k_1) \int_0^1 |zw| dx + k_2 \int_0^1 |z|^2 dx. \tag{4.24}$$

Now, using Cauchy-Schwarz inequality and $\varepsilon-$inequality, we obtain

$$\int_0^1 (z')^2 dx \leq (\|f\|_\infty + k_1)(\int_0^1 z^2 dx)^{\frac{1}{2}}(\int_0^1 w^2)^{\frac{1}{2}} dx + k_2 \int_0^1 |z|^2 dx. \tag{4.25}$$

From Wirtinger's inequality and (4.23), we have

$$\left(\int_0^1 w^2 dx\right)^{\frac{1}{2}} \leq \frac{1}{\pi}\left(\int_0^1 (w')^2 dx\right)^{\frac{1}{2}} \leq \frac{1}{\pi^3}\left(\int_0^1 (z')^2 dx\right)^{\frac{1}{2}}$$

and

$$\left(\int_0^1 z^2 dx\right)^{\frac{1}{2}} \leq \frac{1}{\pi}\left(\int_0^1 (z')^2 dx\right)^{\frac{1}{2}}.$$

Therefore (4.25) becomes

$$\left[1 - \left(\frac{\|f\|_\infty + k_1}{\pi^4} + \frac{k_2}{\pi^2}\right)\right] \int_0^1 (z')^2 dx \leq 0. \tag{4.26}$$

Since

$$1 - \left(\frac{\|f\|_\infty + k_1}{\pi^4} + \frac{k_2}{\pi^2}\right) > 0. \tag{4.27}$$

Then it follows from (4.26)

$$\int_0^1 (z')^2 dx \leq 0 \tag{4.28}$$

and consequently from (4.23) and (4.28), we obtain

$$\left(\int_0^1 (w')^2 dx\right)^{\frac{1}{2}} \leq 0. \tag{4.29}$$

In particular, (4.28) and (4.29) give us

$$|z(x)| = \left|\int_0^x z' dx\right| \leq \left(\int_0^1 (z')^2 dx\right)^{\frac{1}{2}} \leq 0 \tag{4.30}$$

and

$$|w(x)| = \left|\int_0^x w' dx\right| \leq \left(\int_0^1 (w')^2 dx\right)^{\frac{1}{2}} \leq 0. \tag{4.31}$$

Then,

$$\sup_{0\leq x\leq 1} |w(x)| + \sup_{0\leq x\leq 1} |z(x)| = \|w\|_\infty + \|z\|_\infty \leq 0. \tag{4.32}$$

Hence

$$0 \leq \|w\|_\infty + \|w''\|_\infty \leq 0. \tag{4.33}$$

This is a contradiction. Therefore, $u_1 = u_2$.

Example 5. We consider the following BVP:

$$\begin{cases} u^{(4)} + (x^2+1)u = \sqrt{\lambda + u^2 + (u'')^2},\ 0 < x < 1,\ \lambda \in \mathbb{R}^+ \\ \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

We have

$$\begin{aligned} |g(x,u_1,u_1'') - g(x,u_2,u_2'')| &= \left|\sqrt{\lambda + u_1^2 + (u_1'')^2} - \sqrt{\lambda + u_2^2 + (u_2'')^2}\right|, \text{ for all } u_1, u_2 \in \mathbb{R} \\ &= \frac{|(\lambda + u_1^2 + (u_1'')^2) - (\lambda + u_2^2 + (u_2'')^2)|}{\sqrt{\lambda + u_1^2 + (u_1'')^2} + \sqrt{\lambda + u_2^2 + (u_2'')^2}} \\ &= \frac{|(u_1^2 - u_2^2) + ((u_1'')^2 - (u_2'')^2)|}{\sqrt{\lambda + u_1^2 + (u_1'')^2} + \sqrt{\lambda + u_2^2 + (u_2'')^2}} \\ &= \frac{|u_1 + u_2||u_1 - u_2| + |u_1'' + u_2''||u_1'' - u_2''|}{\sqrt{\lambda + u_1^2 + (u_1'')^2} + \sqrt{\lambda + u_2^2 + (u_2'')^2}} \\ &\leq |u_1 - u_2| + |u_1'' - u_2''|. \end{aligned}$$

Thus, g is a Lipschitz function with $k_1 = k_2 = 1$, and the condition $\frac{\|f\|_\infty + k_1}{\pi^4} + \frac{k_2}{\pi^2} < 1$ is satisfied. So that, Theorem 4.3.1 implies that this problem has at most one solution.

References

[1] R.A. Usmani, A uniqueness theorem for a boundary value problem, Proceedings of the American Mathematical Society, Volume 77, Number(3), 329-335, (1979).

[2] E. L. Reiss, A. J. Callegari and D. S. Ahluwalia, Ordinary differential equations with applications, Holt, Rinehart and Winston, New York, (1976).

[3] S. Timoshenko and S. Woinowsky-Krieger, Theory of plates and shells. McGraw-Hill, New York, (1959).

[4] A. Siddiqi, Functional Analysis and Applications, Springer Singapore, (2018).

[5] J. Hunter and B. Nachtergaele, Applied Analysis, World Scientific Publishing Company; Reprint edition (February 28, 2001).

[6] P. Dràbek and G. Holubovà, Positive and negative solutions of one- dimensional beam equation, Applied Mathematics Letters, 51:1-7, (2016).

[7] L. Bougoffa, R. Rach and Abdul-Majid Wazwaz, On solutions of boundary value problem for fourth-order beam equations, Mathematical Modelling and Analysis, Volume 21, Issue 3, pp. 304-318, (2016).

[8] J. B. Diaz and F. T. Metcalf, Variations of Wirtinger's inequality, 1967 Inequalities. Proc. Sympos. Wright-Patterson Air Force Base, Ohio. 79–103, Academic Press, New York, (1965).

[9] Y. Yang, Fourth-order two point boundary value problem, Proceeding of the American Mathematical Society, Volume 104, Number1, 175-180, (1988).

[10] S.H. Saker, R.P. Agarwal, and D. O'Regan, Properties of solutions of fourth-order differential equations with boundary conditions, Journal of Inequalities and Applications, 2013:278, (2013).

[11] K. Atkinson and W. Han, Theoretical Numrical Analysis: A functional Analysis Framework, Third Edition, Springer, (2001).