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Title

Studies in Topological Groups

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Abstract

In this research project, we investigate the theory of topological groups and Haar measures on locally compact groups. The concept of binary operations in algebra are used as convenient tools for the investigation and study of topological groups as well as other mentioned concepts. Our work is divided into four parts, the first of which includes a review of basic knowledge that we need in group theory and topological spaces. In the second part, we present the notions of topological groups, subgroups, topological quotient groups and product groups. Besides, we discuss the so-called topologically homomorphisms which are required to be continuous. In fact, this is due to the topology which is defined on a group and making them continuous. We distinguish the difference between algebraic and topological structures. One of the ultimate difference between topological spaces and topological groups is the homogeneity axiom which always holds for topological groups. Further, we discover in details the separation axioms which imply to interesting results. The end of this part shows the reason of fail a homogeneous space of being a topological group by applying the metrization of groups. In the third part, we are concerned about the topological properties on groups such as the connectedness and locally compactness. In particular, we are interested in compact and locally compact groups. This part is remarkable since we will characterize a Haar measure on locally compact groups. Finally, in the last part, we recall some roles of measure and integration theories and then we identify a Haar measure. In addition, we state and prove its existence and uniqueness.

Contents

Introduction

The present research project studies the theory of topological groups that is one of interesting and fascinating areas in mathematics and which simply combines, from a purely logical point of view, two fundamental mathematical concepts, the algebraic structure of groups and the topological structure of topological spaces. In considering groups we study in purest form the algebraic operation of multiplication while in considering topological spaces we investigate in just as pure a form the operation of passage to a limit. Since both of these operations are among the fundamental operations of mathematics, they often occur together. Thus the axiomatization of the concept of a topological group is a natural procedure in which precisely these two operations are united and interdependent.

One of the purposes of this theory is to allow us to give topological properties such as homogeneity, connectedness and separation axioms on groups, and deals with interesting topological spaces (torus, circle, $\cdots$) as quotient of topological spaces by topological groups. The concept of a continuous, or what is the same thing, topological group, arose in mathematics from the study of groups of continuous transformations. A group of continuous transformations, for instance, geometric transformation, constitutes in a natural way a topological manifold. It appeared later that for the treatment of the greater part of the problems arising in this connection, it is not necessary to consider a group of transformation, but merely to study the group intrinsically, remembering however that there is defined in it an operation of passage to a limit. Therefore arose a new mathematical concept—topological groups.

Furthermore, since any group $\mathcal{G}$ contains either finite or infinite elements and this is common in topological spaces as well, the question arises here whether it is possible to assign a topology on $\mathcal{G}$ or not. Now, it becomes possible if there is no any restriction placed on the assigned topology. Most infinite groups we encounter in any areas of mathematics are topological groups such as the group of $n \times n$ invertible matrices, the additive and multiplicative groups of the fields $\mathbb{R}$ and $\mathbb{C}$ and their subgroups such as the multiplicative group $\mathbb{S}^1$ of complex numbers of absolute values 1. Note that we will see the homogeneity property makes everything possible on a topological group, because it shows a property locally on a group and then it turns into valid in the entire group. That is, we can find a homeomorphism π on every topological group, say $\mathcal{G}$, such that $\pi(x) = y$ for any $x, y \in \mathcal{G}$, however, unfortunately, the converse is not true since the Sorgenfrey line S is a homogeneous space but it fails to admit any group operation. Due to this property, it turns out to consist this theory many interesting results. For instance, if a subgroup of any topological group is open, then it should be closed, and the equivalence between separation axioms is apparent here. The need to examine the theory of topological groups occurs to answer to the *"David Hilbert's Fifth Problem"* which dropped in 1900 and says that whether every locally Euclidean group admits a Lie group structure. This motivated a great amount of research on locally compact groups. Over the years, mathematicians gave a positive answer to the Hilbert's question and developed much structure theories of locally compact groups to boot. This led Otto Schreier in 1927 to define this theory. In 1940s, the work on the free topological groups of Markov and Graev expanded the study of topological groups in a serious way to non-locally compact groups (see [12]). To nowadays, the theory of topological groups constitutes an important branch of mathematics, it encompasses many essential notions such as the Haar measure, Fourier series, integrals and unitary operators groups. In addition, it involves partially the potential theory, ergodic theory and algebraic topology.

In the early 20^{st} century, people started wondering whether if there is an invariant measure on all topological groups or not. In 1933, this problem was approached significantly by the works of Alfréd Haar [7] and John von Neumman [14]. Haar proved that there exists an invariant measure on any sparable compact group while von Neumann proved the special case of the Hilbert's problem for Euclidean locally compact groups supported by Haar's result. In the next year, von Neumann could prove the uniqueness of an invariant measure, and substantially, neither Haar nor von Neumann proved the existence of invariant measures on any locally compact groups. In fact, the full proof was interoduced by André Weil in his work [18] and the proof was criticized for using the axioms of choice in the form of Tychonoff's theorem. Eventually, Henri Cartan in [2] proved the existence of invariant measure on locally compact groups without using the axioms of choice. Indeed, a Haar measure is analogous to the Lebesgue measure on locally compact groups. More precisely, a Haar measure is a regular Borel measure on a locally compact group such that it is either a left-invariant (respectively, right or both) measure. To illustrate, on the real line $\mathbb{R}$ (more generally, on $\mathbb{R}^n$) a Haar measure coincides with a Lebesgue measure. This research project is organized as follows.

In *Chapter 1* we discover fundamentals in both branches: group theory and general topology. We first recall all about natural structure of groups, subgroups and quotient groups with associated properties. Then we present the notion of morphisms followed by the three isomorphism theorems, and finally, we close with the lattice notion that will help us to prove the Tychonoff's theorem. In the next, we introduce the concept of topology which has been divided into four sections. The four sections may express by two statements. Firstly, we define a topology on a set and continuous functions with concerning materials. Secondly, we exhibit about topological properties such as separation and countability axioms corresponding with metrization of spaces, connectedness and locally compactness. Some proofs of propositions are presented here since this chapter has to be reasonable for what follows of next chapters. These tools are all we need in order to explore the theory of topological groups.

Chapter 2 and *Chapter 3* are devoted to the main definitions, ideas and topological properties in topological groups. We investigate each type of groups which are topological subgroups, quotient groups and product groups. In the meanwhile, the three isomorphism theorems have been discussed. Nevertheless, the second isomorphism theorem does not hold in general as long as the quotient $(\mathbb{Z}+\alpha\mathbb{Z})/\mathbb{Z}$ is not homeomorphic to the quotient $\alpha\mathbb{Z}/(\mathbb{Z}\cap\alpha\mathbb{Z})$ where α is irrational and $\mathbb{Z}$ is a normal subgroup of the additive group $\mathbb{R}$. Besides we define a topological homomorphism that is a nice combination between being a group homomorphim and a continuous function at once. We will also see that many results concerning neighborhood bases are helpful to structure proofs. Since we have mentioned that a homogeneous space may not be a topological group in general, we are now able to justify that the Sorgenfrey line S is not a topological group. We as well study the separation axioms and see that the reverse of implications ($\mathcal{G}$ is $T_4 \Rightarrow \mathcal{G}$ is $T_3 \Rightarrow \mathcal{G}$ is $T_2 \Rightarrow \mathcal{G}$ is $T_1 \Rightarrow \mathcal{G}$ is T_0) holds whenever $\mathcal{G}$ is a topological group. Another interesting result is that $\mathcal{G}$ is always a regular space and hence we will see how the above implications are useful further. Furthermore, we provide several crucial properties and a good background of study the connectedness, compactness and locally compactness for a reader who begins to learn about the theory.

In *Chapter 4* we give a brief introduction into the concept of a Haar measure on locally compact group with identifying it and following by several examples. In the last, we state and prove the existence and essential uniqueness of a Haar measure.

Chapter 1

Fundamentals of Group Theory and Topology

In this chapter, we collect many ideas, facts and illustrated examples from the group theory and topology with investigating some results that we gathered from (e.g. [5], [6] and [13]). The aim for presenting this chapter is to make our justifications in next chapters clear and follow our knowledge progressively.

1.1 Theory of Groups

This section is dedicated to the exposition of the fundamentals of theory of groups.

Definition 1.1.1 (Group)**.** *A non-empty set G together with a binary operation $*$ which assigns to each ordered pair (x, y) of elements of G an element denoted by $x * y$ in G, is called a **group** if the following axioms hold.*

(i) $(x * y) * z = x * (y * z)$*, for every* $x, y, z \in G$ *(associativity),*

(ii) there exists an element e *in* G *such that* $e * x = x * e = x$*, for each* $x \in G$ *(identity element),*

(iii) there exists an element $x^{-1} \in G$ *such that* $x^{-1} * x = x * x^{-1} = e$*, for each* $x \in G$ *(inverse element).*

We may denote a group with $*$ as $(G, *)$. For simplicity, we shall write the binary operation between any pair in any group, say G, as $x * y = xy$, for every $x, y \in G$.

Remark 1.1.1.

- A group G is said to be an **abelian group** if the commutative property holds, i.e.,

$$xy = yx, \text{ for all } x, y \in G.$$

- The identity element e and the inverse element of each element in a group G are unique.

Recall that the order $n = o(G)$ of a group G is the number of its elements. A group G is called a **cyclic group** if there is an element x in G such that $G = \langle x \rangle = \{x^k : k \in \mathbb{Z}\}$. In this case, we say that x is a **generator** of G. A cyclic group can be either finite or infinite. If G is a finite cyclic group, then we call it a cyclic group of order n. If x is a generator of G, then so is x^{-1}. Further, every cyclic group is an abelian group.

Definition 1.1.2 (Subgroup)**.** *A non-empty subset H of a group G is called a **subgroup** of G, written as $H \leq G$, if the following conditions are satisfied.*

(i) for all h and g in H, hg in H,

(ii) for each h in H, h^{-1} in H.

Proposition 1.1.1. *Let H be a non-empty subset of a group G. H is a subgroup of G if and only if for all h and g in H, hg^{-1} in H.*

Proposition 1.1.2. *Let H be a subgroup of a group G and $x, y \in G$. Then*

(i) $xH = H$ if and only if $x \in H$,

(ii) $xH = yH$ if and only if $x^{-1}y \in H$ if and only if $y \in xH$.

Remark 1.1.2.

- Every subgroup of a cyclic group is cyclic.
- If H and K are subgroups of a group G, then so is their intersection. But their union is a subgroup if and only if one of them is contained in other.

Definition 1.1.3 (Normal Subgroup)**.** *A subgroup N of a group G is called a **normal subgroup** provided that for all $g \in G$, we have $gN = Ng$. We denote it by $N \trianglerighteq G$.*

When G is abelian, then every subgroup is normal.

Proposition 1.1.3. *A subgroup N of a group G is normal if and only if for all $g \in G$, $g^{-1}Ng = N$.*

We call gN a **left coset** obtained by multiplying N by an element g from the left. Similarly for the **right coset**. Now let N be a normal subgroup of a group G. Consider the family Q of all left (or right) cosets of N in G, i.e.,

$$Q = \{gN : g \in G\},$$

then Q forms a group with the binary operation defined as

$$xNyN = xyN, \text{ for all } x, y \in G.$$

Definition 1.1.4 (Quotient Group)**.** *The group Q which defined above is called the **quotient group** of G by N and denoted by G/N.*

When G is abelian, then so is G/N. Further, every quotient group of a cyclic group is cyclic. If we consider N a subgroup of a group G, then the number of distinct cosets of N in G is called the **index** of N in G and denoted as $[G : N]$. It can be either finite or infinite.

The following are some examples of groups, subgroups, and quotient groups that are known in algebra.

Example 1.1.1.

1. **The centre, centralizer and normalizer groups:**
   Let G be a group and A any non-empty subset of G.

(a) The **centre** $Z(G)$ of G defined as

$$Z(G) = \{x \in G : xy = yx \text{ for all } y \in G\}$$

is a normal subgroup of G. When G is abelian, then so is $Z(G)$ and vice versa. if $G/Z(G)$ is cyclic group, then G is abelian.

(b) The **normalizer** $N_G(A)$ and the **centralizer** $Z_G(A)$ of A in G defined as

$$N_G(A) = \{x \in G : xA = Ax\} \quad \text{and} \quad Z_G(A) = \{x \in G : xa = ax \text{ for all } a \in A\}$$

are subgroups of G. Furthermore, $Z_G(A) \leq N_G(A)$. However, $N_G(A)$ may not be normal. If $N_G(A) = G$, then A is normal in G and vice versa.

Whenever G is abelian, then the above subgroups are equal.

2. The set of complex numbers $\mathbb{C}$ with addition forms an abelian group, and the subsets $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ form abelian subgroups of $\mathbb{C}$. If we consider these groups without zero number, then they form multiplicative groups except $\mathbb{Z}$.

3. The set of integers modulo n with addition, i.e., $(\mathbb{Z}_n, +)$, is an abelian quotient group. If we consider it without zero class, then it forms an abelian group with multiplication.

Next, we give some fundamental concepts in the study of morphisms such as homomorphisms.

Definition 1.1.5 (Group Homomorphism)**.** *Let* $(G_1, *)$ *and* $(G_2, \circ)$ *be two groups. A mapping* $\psi : G_1 \to G_2$ *is called a* ***group homomorphism*** *if for all* $x, y \in G_1$, $\psi(x * y) = \psi(x) \circ \psi(y)$.

Remark 1.1.3. Let H_1 and H_2 be two groups and $\phi : H_1 \to H_2$ any group homomorphism.

- If ϕ is bijective (one-to-one and onto), then it is said to be an **isomorphism**.
- If ϕ maps a group H_1 to itself, then it is called an **endomorphism**, and when it is an isomorphism, then so is called an **automorphism**.
- If ϕ is onto, then it is said to be an **epimorphism**, and H_2 is called a homomorphic image of H_1.
- If ϕ is one-to-one, then it is said to be a **monomorphism**, and H_1 is called embeddable in H_2, denoted as $H_1 \hookrightarrow H_2$.

Proposition 1.1.4. *Let* G_1 *and* G_2 *be two groups and* $\psi : G_1 \to G_2$ *a homomorphism. Then*

(i) $\psi(e_1) = e_2$ *where* e_1 *and* e_2 *are the identity elements of* G_1 *and* G_2*, respectively.*

(ii) $\psi(x^{-1}) = [\psi(x)]^{-1}$*, for every* $x \in G_1$.

(iii) $\psi(x^n) = [\psi(x)]^n$*, for every* $x \in G_1$ *and* $n \in \mathbb{N}$.

Remark 1.1.4. When two groups, say G_1 and G_2, are isomorphic, i.e., there is an isomorphism mapping ϕ between them, then we have the following properties.

- $o(G_1) = o(G_2)$,
- If G_1 is abelian, then so is G_2 and vice versa,
- $o(x) = o(\phi(x))$, for every $x \in G_1$.

Given two groups G_1 and G_2, recall that the **kernal** of a homomorphism $\phi : G_1 \to G_2$ is defined as

$$\ker(\phi) = \{x \in G_1 : \phi(x) = e_2\},$$

where e_2 is the identity element of G_2, and the **image of** ϕ is $Im(\phi) = \phi(G_1)$. We know that ϕ is a monomorphism if and only if $\ker(\phi) = \{e_1\}$ where e_1 is the identity element of G_1, and it is an epimorphism whenever $Im(\phi) = G_2$ and vice versa.

The following is an example of some other groups that we are already familiar with.

Example 1.1.2 (The Matrix Groups). Let $\mathbb{K}$ be a scalar field ($\mathbb{C}$ or $\mathbb{R}$), then the set of $n \times n$ matrices with entries in $\mathbb{K}$ is denoted by $M_n(\mathbb{K})$.

1. **The general linear group** $GL_n(\mathbb{K})$ is the set of all invertible matrices in $M_n(\mathbb{K})$, i.e.,

$$GL_n(\mathbb{K}) = \{A \in M_n(\mathbb{K}) : det(A) \neq 0\}.$$

   $GL_n(\mathbb{K})$ forms a group under the matrix multiplication. Recall that the map $det : GL_n(\mathbb{K}) \to \mathbb{R}^*$ is a homomorphism map, and the kernal of this map is a normal subgroup of $GL_n(\mathbb{K})$ which is called the **special linear group** $SL_n(\mathbb{K})$. We define it as

$$SL_n(\mathbb{K}) = \{A \in GL_n(\mathbb{K}) : det(A) = 1\}.$$

2. **The orthogonal matrices group** $\mathcal{O}_n$ is the set of all orthogonal matrices in $M_n(\mathbb{R})$, i.e.,

$$\mathcal{O}_n = \{A \in M_n(\mathbb{R}) : A^t A = I\}.$$

   Clearly, $\mathcal{O}_n$ forms a subgroup of $GL_n(\mathbb{R})$. The **special orthogonal matrices** $\mathcal{SO}_n$ is a normal subgroup of $\mathcal{O}_n$ and defined as

$$\mathcal{SO}_n = \{A \in \mathcal{O}_n : det(A) = 1\}.$$

3. **The unitary matrices group** $\mathcal{U}_n$ is the set of all unitary matrices in $M_n(\mathbb{C})$, i.e.,

$$\mathcal{U}_n = \{A \in M_n(\mathbb{C}) : A^* A = I\}$$

   where A^* is the adjoint of the matrix A. Clearly, $\mathcal{U}_n$ forms a subgroup of $GL_n(\mathbb{C})$. The **special unitary matrices** $\mathcal{SO}_n$ is a normal subgroup of $\mathcal{U}_n$ and defined by

$$\mathcal{SU}_n = \{A \in \mathcal{U}_n : det(A) = 1\}.$$

We represent the theorem that gives the relationship between homomorphisms and quotient groups often called the Fundamental Theorem of Homomorphism and then we state other related theorems, i.e., second and third isomorphism theorems.

Proposition 1.1.5 (First Isomorphism Theorem). *Given two groups G_1 and G_2, let $\psi : G_1 \to G_2$ be an epimorphism. Then every homomorphic image of G_1 is isomorphic to the quotient group $G_1/ker(\psi)$. That is,*

$$G_1/ker(\psi) \cong \psi(G_1).$$

Definition 1.1.6 (Canonical Map)**.** *Let H be a normal subgroup of a group G. An epimorphism map*

$$\mu\colon G \to G/H$$
$$x \mapsto xH$$

is called the ***canonical map****.*

Proposition 1.1.6 (Second Isomorphism Theorem)**.** *If H is a normal subgroup and K a subgroup of a group G, then we have the following.*

(i) HK is a subgroup of G,

(ii) $H \cap K$ is a normal subgroup of K, and

$$\frac{HK}{H} \cong \frac{K}{H \cap K}.$$

Proposition 1.1.7 (Third Isomorphism Theorem)**.** *Let H and K be two normal subgroups of a group G such that $H \leq K$, then*

$$G/K \cong \frac{G/H}{K/H}.$$

Example 1.1.3.

1. Every infinite cyclic group is isomorphic to $(\mathbb{Z}, +)$.

2. Every finite cyclic group is isomorphic to $(\mathbb{Z}_n, +)$.

Let G be any group. The set of all automorphisms $Aut(G)$ of G is a group under the composition of mappings. Let g be a fixed element of G and define an automorphism mapping I_g by $I_g(x) = gxg^{-1}$, for all $x \in G$. This map is called an **inner automorphism** of G, and the set of all inner automorphisms $Inn(G)$ of G forms a normal subgroup of $Aut(G)$.

Proposition 1.1.8. *Let G be any group. The quotient group $G/Z(G)$ is isomorphic to the group of inner automorphisms $Inn(G)$ of G where $Z(G)$ is the centre of G.*

Definition 1.1.7 (Lattice)**.** *A partially ordered set $(L, \leq)$ is called a* ***lattice*** *if any pair of elements in L has the least upper bound and the greatest lower bound. That is, L is said to be a lattice if for each x and y, there exist both the least upper bound called* ***join*** *and the greatest lower bound called* ***meet****, written as $x \vee y$ and $x \wedge y$, respectively.*

A lattice L is called **distributive** if for all $x, y, z \in L$

$$x \wedge (y \vee z) = (x \wedge y) \vee z.$$

Example 1.1.4. The lattice L of subgroups of the additive group $\mathbb{Z}_{30}$ of integers modulo 30 is

$$L = \{\langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle, \langle 5 \rangle, \langle 6 \rangle, \langle 10 \rangle, \langle 15 \rangle\}.$$

Let L be a lattice. We have the following.

Definition 1.1.8 (An Ideal of a Lattice)**.** *A subset $\mathfrak{a}$ of L is called an* ***ideal*** *of L if it satisfies the following properties.*

(i) if $x \in \mathfrak{a}$ and $y \leq x$, then $y \in \mathfrak{a}$,

(ii) if $x, y \in \mathfrak{a}$, then $x \vee y \in \mathfrak{a}$.

We say that an ideal $\mathfrak{a}$ is **proper** if $\mathfrak{a} \neq L$.

Definition 1.1.9 (A Filter of a Lattice)**.** *Let $\mathfrak{f}$ be a subset of L. $\mathfrak{f}$ is said to be a **filter** of L if the following conditions hold.*

(i) if $x \in \mathfrak{f}$ and $x \leq y$, then $y \in \mathfrak{f}$,

(ii) if $x, y \in \mathfrak{f}$, then $x \wedge y \in \mathfrak{f}$.

We say that a filter $\mathfrak{f}$ is **proper** if $\mathfrak{f} \neq L$.

Remark 1.1.5. Let $(L, \leq)$ be a lattice. Consider the opposite order $\leq_{op}$ of $\leq$, i.e.,

$$y \leq x \Rightarrow x \leq_{op} y, \qquad x, y \in L.$$

Then $(L, \leq_{op})$ is clearly a lattice. In addition, an ideal of $(L, \leq)$ is a filter of $(L, \leq_{op})$ and a filter of $(L, \leq)$ is an ideal of $(L, \leq_{op})$.

Definition 1.1.10 (Maximal Ideal of a Lattice)**.** *Let $\mathfrak{m}$ be an ideal of a lattice L. We say that $\mathfrak{m}$ is a **maximal ideal** if it is a proper ideal that is not properly included in any other proper ideal, i.e., whenever $\mathfrak{m} \neq L$ and $\mathfrak{a}$ is an ideal of L such that $\mathfrak{m} \subseteq \mathfrak{a} \subset L$, then either $\mathfrak{m} = \mathfrak{a}$ or $\mathfrak{a} = L$.*

In order to ensure the existence of maximal ideals, we first give some essential facts as follows.

Since a lattice L is a partially ordered set, a subset E of L is called a **chain** if every pair in E is comparable, i.e., if $x, y \in E$, then either $x \leq y$, $x \geq y$ or equal. Further, an element m in L is said to be **maximal** if there is no $l \in L$ for which $m < l$.

Proposition 1.1.9 (Zorn's Lemma)**.** *Let P be a partially ordered set in which every chain in P has an upper bound in P, then P has at least one maximal element (upper bound).*

Proposition 1.1.10. *if $\mathcal{A}$ is a chain of proper ideals of a lattice L with a maximal element 1, then $\bigcup \mathcal{A}$ is a proper ideal.*

Proof. Let $x \in \bigcup \mathcal{A}$ with $y \leq x$, then $x \in \mathfrak{a}$ for some $\mathfrak{a} \in \mathcal{A}$. Since $\mathfrak{a}$ is an ideal in L, it follows that $y \in \mathfrak{a} \subseteq \bigcup \mathcal{A}$. Also, if $x, y \in \bigcup \mathcal{A}$ such that $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ for some $\mathfrak{a}, \mathfrak{b} \in \mathcal{A}$, and without loss of generality $\mathfrak{b} \subseteq \mathfrak{a}$, then $x \vee y \in \mathfrak{a} \subseteq \bigcup \mathcal{A}$ as $\mathfrak{a}$ is an ideal with $x, y \in \mathfrak{a}$. Thus $\bigcup \mathcal{A}$ is an ideal. Finally, as every ideal of $\mathcal{A}$ is proper, we have that $1 \notin \mathfrak{a}$ for each $\mathfrak{a} \in \mathcal{A}$ so that $1 \notin \bigcup \mathcal{A}$. Hence, $\bigcup \mathcal{A}$ is a proper ideal. $\square$

According to Prop.1.1.9, we deduce that every proper ideal is contained in a maximal ideal. We can assert that the existence of maximal filters (usually, they are called **ultrafilters**) in lattices with a minimal element as well.

Definition 1.1.11 (Prime Ideal of a Lattice)**.** *An ideal $\mathfrak{p}$ of a lattice L is called **prime** if it is proper and whenever the meet $x \wedge y$ is in $\mathfrak{p}$, then either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.*

Similarly, one can define a prime filter $\mathfrak{pf}$ associated with the join of a pair in L and then one of them is in $\mathfrak{pf}$. Moreover, regarding to the Remark 1.1.5, prime ideals correspond to prime filters and vice versa.

Proposition 1.1.11. *Let $\mathfrak{m}$ and $\mathfrak{f}$ be a maximal ideal and ultrafilter of a distributive lattice L, respectively. Then*

(i) if L has an upper bound, $\mathfrak{m}$ is prime,

(ii) if L has a lower bound, $\mathfrak{f}$ is prime.

Proof. Let $(L, \leq)$ be a distributive lattice.

(i) Suppose that L has an upper bound 1 and let $\mathfrak{m}$ be a maximal ideal. Assume that $a \wedge b \in \mathfrak{m}$ with $a \notin \mathfrak{m}$. Our goal is to show that $b \in \mathfrak{m}$. Firstly, define

$$\mathfrak{a} := \{x \in L : \exists m \in \mathfrak{m} \text{ such that } x \leq a \vee m\}$$

which is an ideal in L. Indeed,

- if $x_1, x_2 \in \mathfrak{a}$ then there are $m_1, m_2 \in \mathfrak{m}$ in which $x_1 \leq a \vee m_1$ and $x_2 \leq a \vee m_2$. Thus,

  $$x_1 \vee x_2 \leq (a \vee m_1) \vee (a \vee m_2) = a \vee (m_1 \vee m_2),$$

  where $m_1 \vee m_2 \in \mathfrak{m}$ as $\mathfrak{m}$ is an ideal. It follows that $x_1 \vee x_2 \in \mathfrak{a}$,

- if $x \in \mathfrak{a}$ and $y \in L$ with $y \leq x$, then there exists $m \in \mathfrak{m}$ such that $x \leq a \vee m$. Therefore, $y \leq a \vee m$ and we get that $y \in \mathfrak{a}$.

Additionally, as $a \leq a \vee m$ for all $m \in \mathfrak{m}$, so we have $a \in \mathfrak{a}$. Likewise, as $m \leq a \vee m$, $m \in \mathfrak{a}$ for all $m \in \mathfrak{m}$. Hence, $\mathfrak{m}$ is strictly contained in $\mathfrak{a}$ and it follows that $\mathfrak{a}$ is an ideal. Since $a \in \mathfrak{a} \setminus \mathfrak{m}$ and $\mathfrak{m}$ is a maximal ideal, we deduce that $\mathfrak{a} = L$. In particular, $1 \in \mathfrak{a}$, so there is $m \in \mathfrak{m}$ for which $1 = a \vee m$. Therefore,

$$(a \wedge b) \vee m = (a \vee m) \wedge (b \vee m) = 1 \wedge (b \vee m) = b \vee m \geq b,$$

where both $a \wedge b$ and m are in $\mathfrak{m}$. So $(a \wedge b) \vee m \in \mathfrak{m}$ as $\mathfrak{m}$ is an ideal. Finally, since $b \leq (a \wedge b) \vee m$, we have $b \in \mathfrak{m}$. Thus, $\mathfrak{m}$ is prime as desired.

(ii) Suppose that L has a lower bound and let $\mathfrak{f}$ be an ultrafilter. We have $\mathfrak{f}$ is a maximal ideal in $(L, \leq_{op})$. By assumption, $(L, \leq)$ has a minimal element, so $(L, \leq_{op})$ has a maximal element. Thus, applying (i) gives that $\mathfrak{f}$ is a prime ideal of $(L, \leq_{op})$. Hence, it is a prime filter of $(L, \leq)$ according to the correspondence. □

1.2 Topological Spaces and Continuous Functions

In this section, we review the elementary definitions and theorems from topology which we need throughout this project. We first have topological spaces and followed by related notions. Then, we give the meaningful of being a function continuous on any topological space with surely associated ideas such as homeomorphisms.

Definition 1.2.1 (Topological Spaces). *Let X be any set and let τ be a collection of subsets of X. A family τ is called a **topology** on X if the following axioms hold.*

(i) $\emptyset$ and X are in τ,

(ii) the union of any collection $(O_i)_{i \in I}$ of elements of τ is in τ, and

(iii) the intersection of any finite collection $(O_i)_{i=1}^{n}$ of elements of τ is in τ.

A set X equipped with a topology τ is called a **topological spae** and we write (X, τ). Each element in τ is called an **open set** and any set whose complement is open called a **closed set**.

Example 1.2.1. Let X be any set.

1. The collection τ_d of all subsets of X, i.e., $\mathcal{P}(X)$, is a topology on X called the **discrete topology**.

2. The collection τ_i containing only $\emptyset$ and X is a topology on X called the **trivial topology** or **indiscrete topology**.

Cleary for any topology τ on a set X, we have $\tau_i \subseteq \tau \subseteq \tau_d$. In this case, we say that τ_d is **finer** than τ and τ_i is **coarser** than τ. Further, for any topology on X, τ_d is the **finest** topology on X.

3. The topology $\tau_f = \{O \subseteq X : X \setminus O \text{ is finite } \} \cup \{\emptyset\}$ defined on X is called the **finite complement topology**.

4. The two collections

$$\tau_u = \{O \subseteq \mathbb{R} : \forall x \in O, \exists (a,b) \text{ such that } x \in (a,b) \subseteq O\} \cup \{\emptyset\},$$
$$\tau_l = \{O \subseteq \mathbb{R} : \forall x \in O, \exists [a,b) \text{ such that } x \in [a,b) \subseteq O\} \cup \{\emptyset\}$$

define topologies on the set of real numbers $\mathbb{R}$ and are called the **usual topology** and the **lower limit topology**, respectively.

From the preceding definition, we can deduce the following properties for closed sets as well.

Proposition 1.2.1. *Let X be a topological space. Then*

(i) X and $\emptyset$ are closed in X.

(ii) The union of any finite collection of closed sets in X is closed in X.

(iii) The intersection of any family of closed sets is closed in X.

Definition 1.2.2 (Neighborhood of x)**.** *Let X be a topological space and x any element in X. Let O be an open set in X such that $x \in O$, then O is called a **neighborhood of x**.*

Given a topological space X, we denote the collection of neighborhoods of an element x by $\mathcal{N}(x)$. Now by the meaning of neighborhood, the identification of an open set becomes as below.

Definition 1.2.3 (Characterization of an Open Set)**.** *Let X be a topological space and let O be a non-empty subset of X. Then O is said to be an **open set** in X if for each $x \in O$, there is a neighborhood U_x of x such that $U_x \subseteq O$.*

Proposition 1.2.2. *Let X be a topological space and $x \in X$. Let $\mathcal{N}(x)$ be the collection of neighborhoods of x. Then the following properties are called the **neighborhood axioms**.*

(i) $\mathcal{N}(x)$ is non-empty and every element contains x,

(ii) any subset Y of X containing a neighborhood of x is an element in $\mathcal{N}(x)$,

(iii) any finite intersection of neighborhoods is an element in $\mathcal{N}(x)$,

(iv) if $y \in X$ such that $\mathcal{N}(y)$ is a family of neighborhoods of y and $O \in \mathcal{N}(y)$, then for any $U \in \mathcal{N}(x)$ such that $O \subset U$, $U \in \mathcal{N}(y)$ for all $y \in O$.

Definition 1.2.4 (Fundamental System of Neighborhoods)**.** *Let X be a topological space and let $x \in X$. Let $\mathscr{V}_x$ be a subset of the collection $\mathscr{U}_x$ of all neighborhoods of x. Then we say that $\mathscr{V}_x$ is a **fundamental system of neighborhoods of x** whenever for all U_x in $\mathscr{U}_x$ there is V_x in $\mathscr{V}_x$ such that $V_x \subset U_x$. Moreover, we call $\mathscr{V}_x$ a base for $\mathscr{U}_x$.*

Definition 1.2.5 (Interior, Derived, Closure and Boundary Sets)**.** *Let x be an element in a topological space X and A a subset of X. Then* *(i) x is an* ***interior point of A*** *if there is a neighborhood of x contained in A. The set of all interior points of A is denoted by $\mathring{A}$ or $Int(A)$.*

(ii) x is called a ***limit point*** *(or* ***accumulation point****) of A if for every neighbourhood O_x of x for which $O_x \cap A \setminus \{x\} \neq \emptyset$. The set containing all limit points of A is called the* ***derived set*** *of A and denoted by A'.*

(iii) The set containing points of A and its limit points is called the ***closure of*** *A and denoted as $\bar{A}$. Alternatively, whenever A meets every neighbourhood O_x of x, then $x \in \bar{A}$.*

(iv) The ***boundary of A*** *is the intersection of the closure of A and the closure of its complement and denoted by ∂A.*

Proposition 1.2.3. *Let A and B be two subsets of a topological space X. Then*

(i) $\mathring{A}$ is the largest open set contained in A.

(ii) A is open if and only if $A = \mathring{A}$.

(iii) $\bar{A}$ is the smallest closed set containing A.

(iv) A is closed if and only if $A = \bar{A}$.

(v) $\bar{\bar{A}} = \bar{A}$, $\overline{A \cup B} = \bar{A} \cup \bar{B}$ and $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.

(vi) $\overline{\bigcap_\alpha A_\alpha} \subset \bigcap_\alpha \bar{A}_\alpha$ and $\bigcup_\alpha \bar{A}_\alpha \subset \overline{\bigcup_\alpha A_\alpha}$.

(vii) $\partial A = \partial(X \setminus A) = \bar{A} \cap \overline{X \setminus A}$.

(viii) $\bar{A}$, ∂A and $\overline{X \setminus A}$ form a partition of X.

(ix) $\bar{A} = \mathring{A} \cup \partial A$, $\overline{X \setminus A} = Int(X \setminus A) \cup \partial A$, $\overline{X \setminus A} = Int(X \setminus A)$ and $X \setminus \bar{A} = Int(X \setminus A)$.

Definition 1.2.6 (Dense Set and Separable Space)**.** *Let X be a topological space and A a subset of X.*

(i) A is said to be ***dense*** *in X if its closure set is the entire space X.*

(ii) X is called ***separable*** *space if it has a countable dense subset.*

Next, in preceding examples, for instance, 3 and 4, there are some collections of subsets of X which are generating the concerned topologies. These families are called **bases** for these topologies which we define below.

Definition 1.2.7 (Basis for a Topology)**.** *Let X be a topological space and let $\mathcal{B}$ be a collection of subsets of X whose elements β are called* ***basis elements****. $\mathcal{B}$ is a* ***basis*** *for a topology on X if the following properties hold.*

(i) for each $x \in X$, there is $\beta \in \mathcal{B}$ such that $x \in \beta$,

(ii) if $x \in X$ and for any β_1 and β_2 in $\mathcal{B}$ such that $x \in \beta_1 \cap \beta_2$, there is $\beta_3 \in \mathcal{B}$ such that $x \in \beta_3 \subseteq \beta_1 \cap \beta_2$.

Note that if $\mathcal{B}$ satisfies the preceding definition, then we can define the **topology** $\tau_{\mathcal{B}}$ **generated by** $\mathcal{B}$ as follows. A subset O of X is said to be open in X, if for each $x \in O$, there is a basis element $\beta_x \in \mathcal{B}$ such that $x \in \beta_x \subseteq O$. So it is clear that each open set is a union of basis elements.

Example 1.2.2.

1. The collection of all singleton subsets of a set X is a basis for discrete topology τ_d on X.
2. Consider $X = \mathbb{R}$. The finite complement topology on $\mathbb{R}$ is a basis for itself.
3. The collections
$$\mathcal{B}_u = \{(a,b) : a, b \in \mathbb{R}, a < b\},$$
$$\mathcal{B}_l = \{[a,b) : a, b \in \mathbb{R}, a < b\}$$
are bases for the usual (standrad) and the lower limit topologies on $\mathbb{R}$, respectively.

Proposition 1.2.4. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bases for topologies τ_1 and τ_2 on a set X, respectively. Then the following statments are equivalent.*

(i) τ_1 *is finer than* τ_2,

(ii) for each $x \in X$ and each basis element $\beta_2 \in \mathcal{B}_2$ such that $x \in \beta_2$, there is a basis element $\beta_1 \in \mathcal{B}_1$ such that $x \in \beta_1 \subset \beta_2$.

Definition 1.2.8 (Neighborhood Base)**.** *We say that the collection $\mathcal{B}$ of subsets of X is a **neighborhood base of x** in X if for each open set U of X containing x, there is a basis element $\beta \in \mathcal{B}$ such that $x \in \beta \subset U$.*

Definition 1.2.9 (Sub-basis)**.** *Let X be any set. A **sub-basis** $\mathcal{S}$ for a topology τ on X is a collection of subsets of X whose union equals X. The topology τ of all union of finite intersections of elements of $\mathcal{S}$ is called the **topology generated by the sub-basis** $\mathcal{S}$.*

Another important subject that is of interest to the study of topological spaces is the product topology. We first define a projection map.

Definition 1.2.10 (Projection Map)**.** *Let $\{X_i\}_{i \in I}$ be a family of topological spaces and let X be the product of these spaces. For each $x \in X$, the mappings*

$$\rho_i \colon X \to X_i$$
$$x \mapsto x_i$$

*are called the **projections** of X onto X_i for each $i \in I$.*

Now, we express the product topology in term of its sub-basis. Given a family of topological spaces $\{(X_i, \tau_i)\}_{i \in I}$, the union of the collection

$$s = \{\rho_i^{-1}(O_i) : O_i \in \tau_i, i \in I\}$$

defines a sub-basis for the product of topological spaces.

Definition 1.2.11 (Product Topology)**.** *Let $\{X_i\}_{i \in I}$ be a collection of topological spaces and consider X the product of given spaces. The **product topology** of X is the family of all unions of all finite intersections of elements of the sub-basis s.*

The product set X endowed with the product topology is called a **product space**.

Definition 1.2.12 (Basis for the Product Topology)**.** *Let $\{X_i\}_{i \in I}$ be a collection of topological spaces and consider $X = \prod_{i \in I} X_i$. The collection*

$$\mathcal{B}_{prod} = \{\prod_{i \in I} O_i : O_i = X_i \text{ except for finitely many values of } i\}$$

*defines a **basis for the product topology** on X where each O_i is an open set in each X_i.*

Remark 1.2.1. Let $\{X_i\}_{i \in I}$ be a family of topological spaces and X be the product space. We say that $U = \prod_i U_i$ is **open in** X provided that for each $x \in U$, there is a basis element $\beta \in \mathcal{B}_{prod}$ of the form $\beta = \prod_{i \in I} \beta_i$ where $\beta_i = X_i$ except a finite number of indices with β_i is open in X_i such that $x \in \beta \subseteq U$.

Example 1.2.3. The product of the usual topology on $\mathbb{R}$ n-times is a topology on $\mathbb{R}^n$.

Definition 1.2.13 (Subspace Topology)**.** *Let (X, τ) be a topological space and let Y be a subset of X. Then the collection*

$$\tau_Y = \{Y \cap O : O \in \tau\}$$

*is a topology on Y called the **subspace topology** and we say that Y, equipped with τ_Y, is a **subspace of** X.*

Proposition 1.2.5. *Let X be a topological space. The basis for the subspace topology on Y is defined as*

$$\mathcal{B}_Y = \{\beta \cap Y : \beta \in \mathcal{B}\}$$

where $\mathcal{B}$ is a basis for a topology on X.

Remark 1.2.2. O is **open in** Y if there is an open set U in X such that $O = Y \cap U$.

Proposition 1.2.6. *Let X be a topological space and Y be a subspace of X.*

(i) If U is open in Y and Y is open in X, then U is open in X.

(ii) If C is closed in Y and Y is closed in X, then C is closed in X.

Definition 1.2.14 (Hausdorff Space)**.** *Let X be a topological space. X is called a **Hausdorff space** if for each distinct elements x_1 and x_2 of X, there are disjoint neighborhoods O_{x_1} and O_{x_2} of x_1 and x_2, respectively.*

Example 1.2.4.

1. Any set with discrete topology is a Hausdorff space.
2. The set of real numbers $\mathbb{R}$ equipped with the usual or with the lower limit topologies is a Hausdorff space.

Proposition 1.2.7.

(i) Any finite set in a Hausdorff space is closed.

(ii) Any subspace of a Hausdorff spaces is Hausdorff.

The converse of (i) in the preceding proposition is not true. For instance, the set of real numbers $\mathbb{R}$ together with the finite complement topology is not a Hausdorff space, but each finite set is closed. However, this property holds T_1 axiom which will be given in the next section.

Proposition 1.2.8. *If $\{X_i\}_{i \in I}$ is a family of topological spaces and X is the product space. Then*

(i) X is Hausdorff if and only if each X_i is Hausdorff,

(ii) a subset $Y = \prod_{i \in I} Y_i$ of X is a subspace if and only if each Y_i is a subspace of X_i.

Next, we move on to the meaning of continuity of functions on topological spaces which give many various properties.

Definition 1.2.15 (Continuity on a Topological Space)**.** *If X and Y are two topological spaces and f is a function from X into Y, then f is said be* ***continuous at a point x*** *if for each open subset O of Y containing $f(x)$ there is an open subset U of X containing x such that $f(U)$ is contained in O.*

We say that f is **continuous** if it is continuous at each x.

Proposition 1.2.9. *Let X and Y be two topological spaces and $f\colon X \to Y$ be a function. Then the following statments are equivalent.*

(i) f is continuous,

(ii) for any subset A of X, $f(\bar{A}) \subseteq \overline{f(A)}$,

(iii) for any closed set C in Y, $f^{-1}(C)$ is closed in X,

(iv) for any open set O in Y, $f^{-1}(O)$ is open in X.

Example 1.2.5. Let X, Y and Z be topological spaces.

1. If the two functions $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, the composition $g \circ f\colon X \to Z$ is continuous.

2. If A is a subspace of X and $i\colon A \to X$ is the inclusion map, then i is continuous.

3. The constant function (i.e., $f(x) = y_0$ for any $x \in X$ and $y_0 \in Y$) is continuous.

4. The projection map ρ_i defined in Def.1.2.10 is continuous.

5. If A is a subspace of X and a function $f\colon X \to Y$ is continuous, then the restriction $h = f|_A\colon A \to Y$ is also continuous.

Proposition 1.2.10. *If X is a Hausdorff space and $f\colon Y \to X$ a continuous injection, then Y is Hausdorff.*

Proposition 1.2.11. *Let $\{X_i\}_{i\in I}$ be a family of topological spaces and X the product space endowed with the product topology. Assume that Y is a topological space and $f\colon Y \to X$ is a map defined by $f(x) = (f_i(x))_{i\in I}$ where $f_i\colon Y \to X_i$ for all $i \in I$. Then f is continuous if and only if f_i is continuous for each $i \in I$.*

Proposition 1.2.12. *Let $\{X_i\}_{i\in I}$ be a family of topological spaces, and let X be the product space equipped with the product topology and with the projections $\rho_i\colon X \to X_i$. Suppose that Y be another topological space. Then a function $f\colon Y \to X$ is continuous if and only if the composition $\rho_i \circ f\colon Y \to X_i$ is continuous for each $i \in I$.*

Before giving the next result, we define the following.

Definition 1.2.16 (Open Map)**.** *Let X and Y be two topological spaces. A map $f\colon X \to Y$ is said to be* ***open*** *if for each open subset of X, its image under f is open in Y.*

Likewise for the identification of a closed map.

Proposition 1.2.13. *Let $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ be two families of topological spaces and let X and Y be two product spaces of two given families, respectively. If $f_i\colon X_i \to Y_i$ is an open surjection for each $i \in I$, then the function*

$$\begin{aligned} f\colon X &\to Y \\ x &\mapsto (f_i(x)) \end{aligned}$$

is an open surjection.

Definition 1.2.17 (Homeomorphism Function)**.** *Let X and Y be two topological spaces and $f\colon X \to Y$ a bijective function. Then f is called* ***homeomorphism*** *if both f and its inverse are continuous.*

Note that any homeomorphism map is open and closed at once. When two spaces are homeomorphic, this means that they have the same topological properties such as being Hausdorff or the connectedness property (see Sec.1.5). This is the topological analog of an isomorphism of groups.

Definition 1.2.18 (Homogeneous Space)**.** *A topological space X is said to be* ***homogeneous*** *if for any $x, y \in X$, there is a homeomorphism $\pi\colon X \to X$ such that $\pi(x) = y$.*

We are going to see soon how much the homogeneity of groups plays the crucial role in construction of many properties and important results on topological groups. So far, the following is another type of mappings which leads us to build the quotient spaces called **quotient map**.

Definition 1.2.19 (Quotient Map)**.** *Let X and Y be two topological spaces. A surjection $q\colon X \to Y$ is called* ***quotient map*** *provided a subset O of Y is open in Y if and only if $q^{-1}(O)$ is open in X.*

Thus, q is a continuous map on X by Prop.1.2.9.

Proposition 1.2.14. *Let X, Y and Z be topological spaces and let $q\colon X \to Y$ be the quotient map. Then a map $f\colon Y \to Z$ is continuous if and only if $f \circ q\colon X \to Z$ is continuous.*

Definition 1.2.20 (Quotient Topology)**.** *Let X be a space and Y be any set, and consider the quotient map $q\colon X \to Y$. If there is exactly one topology τ on Y relative to q, then this topology is called the* ***quotient topology*** *induced by q.*

The above definition has given generally the meaning of quotient topology. However, we are looking for the special situation in which the quotient topology occurs particularly frequently. So the following will be considered in this project.

Definition 1.2.21 (Quotient Space)**.** *Let (X,τ) be a topological space and consider the equivalence relation $\sim$ defined on X. In this case, the quotient map becomes*

$$q\colon X \to X/\sim \ \text{ such that } x \mapsto \bar{x}.$$

The topology induced by q defined by

$$\tau_q = \{U \subset X/\sim \colon q^{-1}(U) \text{ open in } X\}$$

is the quotient topology on $X/\sim$. The set which is endowed with the quotient topology is said to be a ***quotient space****.*

Note that the quotient topology is the finest topology making q continuous. Nevertheless, the quotient map is not always open as shown in the following example.

Example 1.2.6. Take the closed interval $X = [0, 1]$ and the equivalence relation $\sim$ identifying the points 0 and 1, equipped with the quotient map $q\colon X \to X/\sim$. We show that the image of an open subset $O = \left[0, \frac{1}{2}\right)$ under q is not open in $X/\sim$. Indeed,

$$q^{-1}(q(O)) = q^{-1}\left(q\left(0, \frac{1}{2}\right)\right) \cup q^{-1}\left(q\left\{0\right\}\right) = \left(0, \frac{1}{2}\right) \cup \{0, 1\}$$

and $\left[0, \frac{1}{2}\right) \cup \{1\}$ is not open in X. Thus, q is not an open map.

Proposition 1.2.15. *Let $\mathcal{B}$ be a basis for a topology defined on a space X. Then the family*

$$\mathscr{B} = \{q(U) : U \in \mathcal{B}\}$$

is a basis for the quotient topology if and only if q is an open map.

Proof. Suppose that $\mathscr{B}$ is a basis for the quotient topology, we show that q is open. Indeed, let O be an open set in X, then $O = \cup_{j\in J} U_j$. Thus, $q(O) = \cup_{j\in J} q(U_j)$ which is a union of open sets. Hence, q is open. Conversely, assume that q is an open map. Then

- let $\bar{x} \in X/\sim$, so $\bar{x} \in q(X)$ as q a surjection. Because of the openness of q and X, there is a neighborhood O of $\bar{x}$ such that $\bar{x} \in O \subset q(X)$. Note that $q^{-1}(O)$ is a neighborhood of x. By the continuity of q, there exists $U \in \mathcal{B}$ such that $x \in U \subset q^{-1}(O)$. That is, $\bar{x} \in q(U)$ and $q(U) \in \mathscr{B}$.

- Let $q(U_1)$ and $q(U_2)$ in $\mathscr{B}$ such that $\bar{x} \in q(U_1) \cap q(U_2)$. Thus $q(U_1) \cap q(U_2)$ is open in $X/\sim$ since q, U_1 and U_2 are open. It follows that $q^{-1}(q(U_1) \cap q(U_2))$ is open in X so that there is $U \in \mathcal{B}$ containing x such that $x \in U \subset q^{-1}(q(U_1) \cap q(U_2))$. In other words, $\bar{x} \in q(U) \subset q(U_1) \cap q(U_2)$ and $q(U) \in \mathscr{B}$. Hence, $\mathscr{B}$ is a basis for the generating topology $\tau_{\mathscr{B}}$.

To show that $\tau_q = \tau_{\mathscr{B}}$. Indeed, we have

$$\begin{aligned} O \in \tau_q &\iff q^{-1}(O) \text{ is open in } X \\ &\iff q^{-1}(O) = \cup_{j\in J} U_j \text{ for some } U_j \in \mathcal{B} \\ &\iff O = \cup_{j\in J} q(U_j) \\ &\iff O \in \tau_{\mathscr{B}}. \end{aligned}$$

□

1.3 Separation Axioms

In this section, we state other stronger proporties than the Hausdorff property so-called the **separation axioms**.

Definition 1.3.1 (Separation Axioms). *Let X be a topological space. We say that X is T_0 (respectively, T_1, T_2, T_3, T_4 and T_5) if*

(T_0) *for any x different to y in X, there is an open subset O of X such that either x in O or y in O,*

(T_1) *for each x different to y in X, there are two neighborhoods O_1 and O_2 of x and y, respectively, in which $x \notin O_2$ and $y \notin O_1$.*

(T_2) *it is a Hausdorff space.*

(T_3) *it is a T_1 space and for any closed subset C with $x \notin C$, there are two disjoint open subsets O_1 and O_2 such that $C \subseteq O_1$ and $x \in O_2$.*

(T_4) *it is a T_1 space and for any $x \in X$ and any closed set C with $x \notin C$, there exists a continuous function f from X into $[0, 1]$ such that $f(x) = 0$ and $f(C) = \{1\}$.*

(T_5) *it is a T_1 space and for all disjoint closed sets C_1 and C_2 in X, there are disjoint neighborhoods O_1 and O_2 of C_1 and C_2, respectively.*

We say that a topological space is **regular**, **completely regular** or **normal** if it is a T_3, T_4 or T_5 space, respectively.

Remark 1.3.1.

- We can see from the previous definition that for any topological space X,

$$X \text{ is } T_4 \Rightarrow X \text{ is } T_3 \Rightarrow X \text{ is } T_2 \Rightarrow X \text{ is } T_1 \Rightarrow X \text{ is } T_0.$$

  Later, we will see in the next chapter that these imblications are equivalent if X is a topological group.

- It is also clear that a normal space is completely regular and a completely regular space is regular.

Proposition 1.3.1. *Let X be a T_1 space.*

(i) X is regular if and only if given a point $x \in X$ and a neighborhood U of x, there is a neighborhood O of x such that $\overline{O} \subseteq U$.

(ii) X is normal if and only if given a closed set C of X and an open set U containing C, there is an open set O containing C such that $\overline{O} \subseteq U$.

Proposition 1.3.2. *Let X be a topological space.*

(i) X is T_1 if and only if every finite subset of X is closed.

(ii) If X is T_3, then every pair of points of X have neighborhoods whose closures are disjoint.

(iii) If X is regular with a countable neighborhood basis, then it is normal.

Proof. Let X be a topological space.

(i) Suppose that every finite set in X is closed. Let x and y be two different points in X. Then, the two one point sets $\{x\}$ and $\{y\}$ are closed so that their complements $X \setminus \{x\}$ and $X \setminus \{y\}$ are open. Note that $X \setminus \{x\} \in \mathcal{N}(y)$ with $x \notin X \setminus \{x\}$ and $X \setminus \{y\} \in \mathcal{N}(x)$ with $y \notin X \setminus \{y\}$. This means that X is T_1. Now suppose that X is a T_1 space and let $C = \{x_1, x_2, \cdots, x_n\} \subset X$. We show that C is closed. Indeed, take $x \in X \setminus C$ so that for all $i = 1, 2, \cdots, n$, $x \neq x_i$. By the assumption we get, there are two neighborhoods U_i and O_i of x and x_i, respectively, in which $x_i \notin U_i$ and $x \notin O_i$. Note that $U = \cap_{i=1}^n U_i \in \mathcal{N}(x)$. Hence,

$$x \in U = \cap_{i=1}^n U_i \subset \cap_{i=1}^n (X \setminus \{x_i\}) = X \setminus \cup_{i=1}^n \{x_i\} = X \setminus C.$$

We see that $X \setminus C$ is open and consequently, C is closed.

(ii) As X is regular, then for any pair of points x and y, there is disjoint neighborhoods U_x and U_y of x and y, respectively. By Prop.1.3.1 (i), both of them x and y have another neighborhoods O_x and O_y such that their closures contained in U_x and U_y, respectively. Hence,

$$\overline{O_x} \cap \overline{O_y} \subseteq U_x \cap U_y = \emptyset.$$

That is, O_x and O_y are disjoint.

(iii) See e.g. [13], p. 200-201. □

Proposition 1.3.3.

(i) A subspace of a regular space is regular and a product of regular spaces is regular as well.

(ii) A subspace of a completely regular space is completely regular and a product of completely regular spaces is completely regular as well.

(iii) A closed subspace of a normal space is normal.

1.4 Metrization of Spaces

In this section, we define and talk briefly about the pseudometrizability. We first recall the first and second countability axioms of topological spaces.

Definition 1.4.1 (First and Second Countabilities). *A topological space X is called* ***first countable*** *if every $x \in X$ has a countable neighbourhood base, and it is said to be* ***second countable*** *if it is has a countable base for its topology.*

Proposition 1.4.1.

(i) A subspace of a first countable space is first countable.

(ii) A product of first countable spaces is first countable.

Likewise for a second countability axiom.

Example 1.4.1. The space $\mathbb{R}$ equipped with τ_l is the first countable but not the second.

Definition 1.4.2 (Pseudometric). *A* ***pseudometric*** *on a set X is a function $d\colon X \times X \to [0, \infty)$ satisfying the following properties.*

(i) $d(x, x) = 0$ for all $x \in X$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry),

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (triangle inequality).

Remark 1.4.1.

- If d satisfies $d(x, y) = 0$ if and only if $x = y$ for any $x, y \in X$, we say that d is a **metric** and a set equipped with d is called a **metric space**, and clearly, d is a pseudometric.
- If d is a pseudometric on X, we define an open ball with radius $r > 0$ and centered at x by
$$B(x, r) = \{y \in X : d(x, y) < r\}$$
and the topology generated by d as
$$\tau_d = \{O \subseteq X : \forall x \in O, \exists r > 0 \text{ such that } B(x, r) \subseteq O\}.$$

One can note that a pseudometric d can generate the topology τ_d on a set X and can be either finer, coarser than some topologies of X, equivalent or neither.

Definition 1.4.3 (Pseudo-Metrizable Space). *A topological space X is said to be* ***pseudo-metrizable*** *if there exists a pseudometric on X generating its topology.*

Proposition 1.4.2. *If d is a pseudometric on a T_0 space, then d is a metric.*

Proof. Let d be a pseudometric on a T_0 space X. It is enough to see that for any $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$. On the contrary, suppose that $x \neq y$. As X is a T_0 space, there is an open ball with radius $r > 0$ in which either $x \notin B(y, r)$ or $y \notin B(x, r)$. In both cases we have $d(x, y) > r > 0$. It follows that d is metric. □

Theorem 1.4.1. *A metrizable space is second-countable if and only if it is separable.*

Proof. Let X be a metrizable space with metric d generating its topology. Suppose that X is a second countable, then it has a countable base $\mathcal{B}$. For each $\beta \in \mathcal{B}$, take $x \in \beta$ and define $D = \{x_\beta : \beta \in \mathcal{B}\}$. Clearly, D is dense and countable in X, so X is separable.

On the other hand, assume that X is separable. Let D be a countable dense subset of X and define $\mathcal{B} = \{B(x, \frac{1}{n}) : x \in D, n \in \mathbb{N}\}$. We show that $\mathcal{B}$ is a base for a topology on X. Indeed, let O be an open subset of X and choose $x \in O$. As $\{B(x, \frac{1}{n}) : n \in \mathbb{N}\}$ is a neighborhood base of x, there is $n_0 \in \mathbb{N}$ such that $B(x, \frac{1}{n_0}) \subseteq O$. Since D is dense, so every open subset has a point on it. In particular, there exists $y \in D \cap B(x, \frac{1}{2n_0})$. Thus, for $z \in B(y, \frac{1}{2n_0})$ and by triangular inequality, we have

$$d(x, z) \leq d(x, y) + d(y, z) < \frac{1}{2n_0} + \frac{1}{2n_0} = \frac{1}{n_0},$$

which implies that $z \in B(x, \frac{1}{n_0})$. Hence,

$$y \in B(y, \frac{1}{2n_0}) \subseteq B(x, \frac{1}{n_0}) \subseteq O.$$

Now since $B(y, \frac{1}{2n_0}) \in \mathcal{B}$, the claim follows. □

Definition 1.4.4 (Bounded Set). *A subset A of a metric space (X,d) is said to be* ***bounded*** *if there is some positive number M such that $d(x, y) \leq M$ for all x, y in A.*

1.5 Connected, Compact and Locally Compact Spaces

We divide this section into two subsections. The first one involves the concept of connectedness and the second one concerns the concept of compactness and locally compactness.

1.5.1 Connected Space

Definition 1.5.1 (Disconnected Space). *Let X be a topological space. X is said to be* ***disconnected*** *if there exist two non-empty disjoint open subsets O_1 and O_2 of X such that their union is X. In this case, we say that (O_1, O_2) is a* ***separation*** *of X.*

A topological space is **connected** if it is not disconnected. Below we give alternative and different identifications of connectedness.

Proposition 1.5.1. *Let X be a topological space. Then the following axioms are equivalent*

(i) X is a connected space,

(ii) if X is a union of two disjoint open sets, then one of them is empty,

(iii) if X is a union of two disjoint closed sets, then one of them sets is empty,

(iv) if any continuous function maps X into $\{0, 1\}$, then it is constant,

(v) the only clopen (open and closed) sets are the empty set and X itself.

Definition 1.5.2 (Connected Subspace). *Let Y be a subspace of a topological space X. Y is connected if it has no separation in the subspase topology τ_Y.*

Example 1.5.1. The set of rationals is not connected in $\mathbb{R}$, because we find that $\mathbb{Q} \cap (\sqrt{2}, \infty)$ and $\mathbb{Q} \cap (-\infty, \sqrt{2})$ is a separation in $\mathbb{Q}$.

Proposition 1.5.2. *Let X be a connected space and Y a topological space. If f is a continuous function from X into Y, then $f(X)$ is connected.*

That is, the image of any connected space under a continuous map is connected.

Proposition 1.5.3. *Let $\{C_i\}_{i\in I}$ be a family of connected subspaces of a topological space X in which their intersection is non-empty for each i. Then their union is a connected subspace of X.*

Proposition 1.5.4. *Let C be a connected subspace and A a subset of a topological space X. If $C \subset A \subset \overline{C}$, then A is connected.*

As consequence, the closure of a connected subspace is connected.

Corollary 1.5.1. *Let X be a topological space and let Y be a dense connected subset of X, then X is connected.*

Proposition 1.5.5. *If $\{X_i\}_{i\in I}$ is a collection of topological spaces, then the product space is connected in the product topology if and only if X_i is connected for each $i \in I$.*

Definition 1.5.3 (Path-connected Space)**.** *A topological space X is said to be **path-connected** if for all $x, y \in X$ there is a continuous function $f\colon [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$. This function is called **path** from x to y.*

Proposition 1.5.6.

(i) If a space is path-connected, then it is connected.

(ii) The image of a path connected space under a continuous map is path connected.

Definition 1.5.4 (Connected Component)**.** *Let X be a topological space and $x \in X$. The union of all connected subspaces of X containing x is called the **connected component** of X at x (or shortly the **component** of x).*

Proposition 1.5.7. *Let X be a topological space.*

(i) If $x \in X$, then the component of x is closed and connected.

(ii) The connected components of X form a partition of X.

One can deduce from the definition of a component of a point x and the preceding proposition that the component of x is the maximal connected subspace containing x in X.

Definition 1.5.5 (Totally Disconnected)**.** *We say that a topological space X is **totally disconnected** if the only connected subsets are the empty set and the one point set.*

1.5.2 Compact and Locally Compact Spaces

Here, in addition to what we give, we state and prove the **Tychonoff's theorem** which ensures whatever how many compact spaces are there, their product is necessarily compact. In the proof, we employ the lattice idea. Note that there is another proceduce to prove it, we refer to [11].

Definition 1.5.6 (Open Cover)**.** *Let X be a topological space. An **open cover** of X is a collection $\mathcal{O} = \{O_i\}_{i\in I}$ of open subsets of X such that $\bigcup_{i\in I} O_i = X$.*

In addition, a **subcover** $\mathcal{U}$ of $\mathcal{O}$ is a cover of X for which $\mathcal{U}$ is contained in $\mathcal{O}$.

Definition 1.5.7 (Compact Space). *A topological space X is called* ***compact*** *if every open cover of X has a finite subcover.*

Example 1.5.2.

1. Every finite space is compact.
2. Any space equipped with discrete topology is compact if and only if it is finite.
3. The set of real numbers $\mathbb{R}$ together with the usual topology is not compact.

Definition 1.5.8 (Compact Subspace). *A subspace Y of a topological space X is* ***compact*** *if every open cover in Y has a finite subcover. Alternatively, Y is compact if every covering of Y by open sets in X has a finite open covring of Y.*

Proposition 1.5.8.

(i) Any closed subset of a compact space is compact.

(ii) Any compact subspace of a Hausdorff space is closed.

(iii) A finite union of compact spaces is compact.

(iv) Any intersection of compact subspaces of a Hausdorff space is compact.

(v) If X is a compact space and $f\colon X \to Y$ a continuous surjection, then Y is compact.

The following is a well-known characterization for compactness in the Euclidean space $\mathbb{R}^n$.

Theorem 1.5.1 (Heine-Borel Theorem). *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Example 1.5.3.

1. Any interval $[a, b] \subseteq \mathbb{R}$ is compact as it is closed and bounded.
2. Let $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit sphere. Define a continuous surjection $f\colon [0,1] \to \mathbb{S}^1$ by $f(x) = e^{2inx}$. Then by Prop.1.5.8 (v), the 1-dimensional sphere $\mathbb{S}^1$ is compact as $[0, 1]$ is closed and bounded. By the same argument, we can show that the n-dimensional sphere $\mathbb{S}^n$ is compact for any $n \in \mathbb{N}$.

Definition 1.5.9 (Finite Intersection Property). *Given a topological space X. We say that X has the* ***finite intersection property*** *provided for every family $\{C_i\}_{i\in I}$ of closed subsets of X in which every finite subcollection has a non-empty intersection, then $\bigcap_{i\in I} C_i \neq \emptyset$.*

Proposition 1.5.9. *Let X be a topological space. Then X is compact if and only if X has the finite intersection property.*

Proof. Let X be a topological space. Suppose that X is a compact space and let $\{C_i\}_{i\in I}$ be a family of closed subsets of X such that every finite subcollection has a non-empty intersection. On the contrary, assume that $\bigcap_{i\in I} C_i = \emptyset$. Then, set $O_i = X \setminus C_i$ and note that O_i is open for each i. Thus we have

$$X = X \setminus \bigcap_{i\in I} C_i = \bigcup_{i\in I} O_i.$$

As X is compact, there exists a finite subset J of I for which $\bigcup_{j\in J} O_j = X$, and we get $\bigcap_{j\in J} C_j = \emptyset$ which is a contradiction.

On the other hand, let $\{O_i\}_{i\in I}$ be an open cover of X. Then for each i, set $C_i = X \setminus O_i$ so we have

$$\bigcap_{i\in I} C_i = \bigcap_{i\in I}(X \setminus O_i) = X \setminus \bigcup_{i\in I} O_i = \emptyset.$$

Thus $\{C_i\}_{i\in I}$ is a family of closed subsets of X such that $\bigcap_{i\in I} C_i = \emptyset$. By hypothesis, there is a finite subset J of I in which $\bigcap_{j\in J} C_j = \emptyset$. Hence, $\bigcup_{j\in J} O_j = X$. In other words, X is compact. □

Regarding to the lattice notion which had been discussed in Sec.1.1, if X is a topological space, OX and CX are the families of open sets and closed sets, respectively, then both are lattices with respect to the inclusion order.

Lemma 1.5.1. *Let X be a topological space. Then the following statements are equivalent:*

(i) X is a compact space,

(ii) if $\mathfrak{a}$ is a proper ideal in OX, then $\bigcup \mathfrak{a} \neq X$,

(iii) if $\mathfrak{f}$ is a proper filter, then $\bigcap \mathfrak{f} \neq \emptyset$.

Proof.
(ii)⇔(iii) First note that (ii) and (iii) are clearly equivalent by taking complements. Indeed, assume that (ii) holds and suppose that $\mathfrak{f}$ is a proper filter in CX. Then $\mathfrak{a} = \{X \setminus A : A \in \mathfrak{f}\}$ is a proper ideal in OX, so that $\bigcup \mathfrak{a} \neq X$. Thus $\bigcap \mathfrak{f} = X \setminus \bigcup \mathfrak{a} \neq X \setminus X = \emptyset$. In the same pattern, we can show that (iii) implies (ii).
(i)⇒(ii) Suppose that X is a compact space and let $\mathfrak{a}$ be a proper ideal in OX. On the contrary, if $\bigcup \mathfrak{a} = X$, then $\mathfrak{a}$ is an open cover of X. Thus X has a finite subcover as X is compact, i.e., there is $\{U_i\}_{i=1}^n$ in which $\bigcup_{i=1}^n U_i = X$. Since ideals are closed under the finite union, so $X \in \mathfrak{a}$ and therefore any open subset contained in X is in $\mathfrak{a}$. It follows that $\mathfrak{a} = OX$ and it is not proper, a contradiction.
(ii)⇒(i) Assume that $\mathfrak{a}$ is a proper ideal in OX and let $\{U_i\}_{i\in I}$ be an open cover of X. Define $\mathfrak{a}$ to be the family of all open subsets A of X for which A can be covered by finitely many U_i. In other words,

$$\mathfrak{a} = \{A \in OX : \exists J_A \subseteq I \text{ finite such that } A \subseteq \bigcup_{j\in J_A} U_j\}.$$

Now we want to check that $\mathfrak{a}$ is an ideal of OX. Take $A, B \in \mathfrak{a}$, there are finite subsets $J_A, J_B \subseteq I$ in which $A \subseteq \bigcup_{i\in J_A} U_i$ and $B \subseteq \bigcup_{i\in J_B} U_i$. Thus $J_A \cup J_B$ is finite and

$$A \cup B \subseteq \bigcup_{j\in J_A\cup J_B} U_j.$$

Therefore, $A \cup B \in \mathfrak{a}$. Further, if $E \in OX$ with $E \subseteq A$, then we have $E \in \mathfrak{a}$ as $E \subseteq \bigcup_{j\in J_A} U_j$. Hence $\mathfrak{a}$ is an ideal in OX. Finally, from the definition of $\mathfrak{a}$, $U_i \in \mathfrak{a}$ for each $i \in I$ and since $\{U_i\}_{i\in I}$ is a cover of X, it follows that $\bigcup \mathfrak{a} = X$. Hence, by assumption $\mathfrak{a}$ can not be a proper ideal, i.e., $\mathfrak{a} = OX$. Particularly, $X \in \mathfrak{a}$ and so there is $J_X \subseteq I$ for which $X = \bigcup_{j\in J_X} U_j$ as desired. □

Theorem 1.5.2 (Tychonoff's Theorem)**.** *Any product of compact spaces is compact.*

Proof. Let $\{X_i\}_{i\in I}$ be a family of compact sets. Let $X = \prod_{i\in I} X_i$ and assume that $\mathfrak{a}$ is a proper ideal of OX. We employ Lemma 1.5.1 by showing that $\bigcup \mathfrak{a} \neq X$. Indeed, let $\mathfrak{m}$ be a maximal ideal of OX containing $\mathfrak{a}$. It suffices to show that $\bigcup \mathfrak{m} \neq X$ since $\bigcup \mathfrak{a} \subseteq \bigcup \mathfrak{m}$. For each $i \in I$, define

$$\mathfrak{m}_i = \{U \in OX : \rho_i^{-1}(U) \in \mathfrak{m}\}$$

where $\rho_i \colon X \to X_i$ is the i-th projection. $\mathfrak{m}_i$ is a proper ideal of OX_i for all $i \in I$. Indeed,

- if U and W in $\mathfrak{m}_i$, then both $\rho_i^{-1}(U)$ and $\rho_i^{-1}(W)$ are in $\mathfrak{m}$. Thus, $\rho_i^{-1}(U \cup W) \in \mathfrak{m}$ as $\mathfrak{m}$ is closed under union, so that U and W are in $\mathfrak{m}_i$,
- if $U \in \mathfrak{m}_i$ and $W \in OX_i$ such that $W \subseteq A$, then $\rho_i^{-1}(W) \subseteq \rho_i^{-1}(U) \in \mathfrak{m}$. Thus, $\rho_i^{-1}(W) \in \mathfrak{m}$ so that $W \in \mathfrak{m}_i$.

On the other hand, since $\rho_i^{-1}(X_i) \neq X$, it follows that $\mathfrak{m}_i$ is a proper ideal of OX_i.

Now, as each X_i is compact, from Lemma 1.5.1, $\bigcup \mathfrak{m}_i \neq X_i$. Take $x = (x_i)_{i\in I} \in X$ such that each x_i is in X_i, but not in $\bigcup \mathfrak{m}_i$. We prove that $x \notin \bigcup \mathfrak{m}$. On the contrary, if $x \in \bigcup \mathfrak{m}$, then there is $U \in \mathfrak{m}$ containing x. As U open is in X, there exists $W \in \mathcal{N}(x)$ of the form $W = \bigcap_{j\in J} \rho_j^{-1}(W_j)$ such that $x \in W \subseteq U$ where J is a finite subset of I and $W_j \in OX_j$ for all $j \in J$. In particular, for each $j \in J$, we have $x \in W \subseteq \rho_j^{-1}(W_j)$, so that by taking the image of ρ_j both sides we get

$$x_j = \rho_j(x) \in \rho_j(\rho_j^{-1}(W_j)) \subseteq W_j.$$

Thus, $x_j \in W_j$. Set $U_j = \rho_j^{-1}(W_j)$ and note that as $U \in \mathfrak{m}$ and $W \subseteq U$, then $W \in \mathfrak{m}$. Further, as OX has an upper bound of X and according to Prop.1.1.11, $\mathfrak{m}$ is a prime ideal. Since $\bigcap_{j\in J} \in U_j \in \mathfrak{m}$, by the definition of a prime ideal, there exists $k \in J$ in which $U_k \in \mathfrak{m}_k$. Therefore, $W_k \in \mathfrak{m}_k$. Thus, $x_k \in W_k$ and so $x_k \in \bigcup \mathfrak{m}_k$. This contradicts the choice of x_k so that $x \notin \bigcup \mathfrak{m}$. The claim follows. □

Remark 1.5.1. The converse of the above theorem is also true. Indeed, if $X = \prod_{i\in I} X_i$ is compact, then the image of X under the projection mapping ρ_i is also compact. That is, $\rho_i(X) = X_i$ is compact and so is each X_i.

Definition 1.5.10 (Locally Compact Space)**.** *We say that a topological space X is* ***locally compact*** *if for each point x in X, there is a compact subspace containing a neighborhood of x.*

From definition we can see that a compact space is locally compact.

Proposition 1.5.10. *Let X be a Hausdorff space. Then X is locally compact if and only if for each $x \in X$ and each U a neighborhood of x, there is a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.*

Proposition 1.5.11. *Let X be a Hausdorff locally compact space, Y be a compact component. Then for any open set U such that $Y \subseteq U$, there is a compact open subset O such that $Y \subseteq O \subseteq U$.*

Proposition 1.5.12. *Let X be a Hausdorff space and K and C be disjoint compact subsets of X. Then there are disjoint open subsets U and V of X such that $K \subseteq U$ and $C \subseteq V$.*

Proposition 1.5.13. *If X be a locally compact space and $(X_i)_{i=0}^n$ is a finite descending sequence of topological spaces such that $X_0 = X$ and X_i either open or closed in X_{i-1}, then X_n is locally compact.*

Lemma 1.5.2. *Let X be a locally compact space such that*

$$X = \bigcup_{n\geq 1} X_n$$

where X_n is closed in X for each n. Then one of X_n contains an open set.

Proof. Suppose by contradiction for all $n \geq 1$, there is no an open set contained in X_n. Since X is locally compact, there exists a compact open set U_0 in X so that $U_0 \cap (X \setminus X_1)$ is a non-empty open set. It follows that we can find an open set U_1 contained in $U_0 \cap (X \setminus X_1)$ such that $\overline{U_1}$ is compact and

$$\overline{U_1} \cap X_1 = \emptyset.$$

Again, $U_1 \cap (X \setminus X_2)$ is a non-empty open set, it follows that we can find an open set U_2 contained in $U_1 \cap (X \setminus X_2)$ such that $\overline{U_2}$ is compact and

$$\overline{U_2} \cap X_2 = \emptyset.$$

Continue in the same fashion, we construct a sequence of open sets $\{U_n\}_{n\geq 1}$ such that

$$\overline{U_n} \subset U_{n-1} \quad \text{and} \quad \overline{U_n} \cap X_n = \emptyset. \tag{1.1}$$

Note that in $\overline{U_0}$, from (1.1), we can see that the sequence $\{\overline{U_n}\}_{n\geq 1}$ is nested whose intersection is non-empty. Now, if $U = \cap_{n\geq 1}\overline{U_n}$, then U does not meet any of X_n since $\overline{U_n} \cap X_n = \emptyset$ for all n. This contradicts to the fact that X is a union of X_n so that at least one of X_n must contain an open set. □

Proposition 1.5.14. *A finite product of locally compact spaces is locally compact.*

Proposition 1.5.15. *Let X be a Hausdorff locally compact space, K be a compact subset of X, U_1 and U_2 be open subsets of X such that $K \subseteq U_1 \cup U_2$. Then there are compact sets K_1 and K_2 such that $K = K_1 \cup K_2$ with $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$.*

Proof. Let $C_1 = K \setminus U_1$ and $C_2 = K \setminus U_2$. Then both C_1 and C_2 are disjoint compact sets. Thus, by Prop.1.5.12, there are V_1 and V_2 disjoint open sets such that $C_1 \subseteq V_1$ and $C_2 \subseteq V_2$. If we consider $K_1 = K \setminus V_1$ and $K_2 = K \setminus V_2$, then K_1 and K_2 are both compact and contained in U_1 and U_2, respectively, and satisfy $K = K_1 \cup K_2$. □

Remark 1.5.2. One can generalize the above the proposition for all $n \in \mathbb{N}$ inductively. Indeed, assume that it holds for $n-1$, i.e., if K is a compact subset in X such that $K \subseteq \bigcup_{i=1}^{n-1} U_i$, there is $\{K_i\}_{i=1}^{n-1}$ compact subsets in X such that

$$K = \bigcup_{i=1}^{n-1} K_i \quad \text{and} \quad K_i \subseteq U_i \quad \text{for all } i = 1, 2, \cdots, n-1.$$

We want to show that it is also true for n. By our assumption and the proposition above, if K is a compact subset of X such that $K \subseteq \bigcup_{i=1}^{n-1} U_i \cup U_n \subseteq \bigcup_{i=1}^{n} U_i$, then there are $\{K_i\}_{i=1}^{n-1}$ and K_n compact subsets of X such that

$$K = \bigcup_{i=1}^{n} K_i \quad \text{and} \quad K_i \subseteq U_i \quad \text{for all } i = 1, 2, \cdots, n.$$

Hence, the proposition holds for all n.

Proposition 1.5.16. *Let X be a locally compact space, K be a compact subset of X and U be an open subset of X such that $K \subseteq U$. Then there is an open subset V of X with a compact closure $\overline{V}$ such that $K \subseteq V \subseteq \overline{V} \subseteq U$.*

Proof. Prop.1.5.10 implies that each point $x \in K$ has a neighborhood V_x whose closure is compact and contained in U. K being compact, then there is a finite subcovering $\{V_{x_i}\}_{i=1}^{n}$ of K. Let $V = \bigcup_{i=1}^{n} V_{x_i}$. Since $\overline{V}_{x_i} \subseteq U$ for all i and by Prop.1.2.3 (vi), the claim follows. □

Chapter 2

General Theory of Topological Groups

The study of topological groups is a nice theory that relates the algebraic properties of the group to the analysis properties resulting from the topology. The group operations are, however, not independent, but they are connected by the condition of continuity: every group operation must be continuous in a topological group. The fundamental relations holding for abstract group and topological space are more or less bodily carried over into topological groups. In this chapter, we discuss the concept of topological group and introduce the meaning of topological subgroups, quotient groups and product groups. Further, we represent the separation axioms and the metrization. The homogeneity of topological groups makes it possible to have results proved locally, for example on the identity element, valid successfully on the whole group.

2.1 Main Definitions and Properties

Definition 2.1.1 (Topological Group)**.** *A **topological group** $\mathcal{G}$ is a group and topological space such that the mappings*

$$\begin{aligned} \psi\colon \mathcal{G}\times\mathcal{G} &\to \mathcal{G} & \quad \textit{and} \quad \phi\colon \mathcal{G} &\to \mathcal{G} \\ (x,y) &\mapsto xy & x &\mapsto x^{-1} \end{aligned}$$

are both continuous.

Remark 2.1.1. It is sufficient to consider the application $\mu\colon \mathcal{G}\times\mathcal{G} \to \mathcal{G}$ defined as $(x,y)\mapsto xy^{-1}$ to be continuous instead of the multiplication and inversion functions in the above definition.

Example 2.1.1.

1. The additive groups $\mathbb{C}$ and $\mathbb{R}$ together with the usual topology are topological groups since the map $(x,y)\mapsto x-y$ is continuous. Likewise for the multiplicative groups $\mathbb{C}^*$ and $\mathbb{R}^*$.

2. Any group equipped either with discrete or indiscrete topology is a topological group.

3. The one-dimension sphere $\mathbb{S}^1 = \{z\in\mathbb{C} : |z| = 1\}$ together with the subspace topology induced from the usual topology in $\mathbb{C}$ is a topological group.

4. The general linear group $GL_n(\mathbb{R})$ endowed with the matrix multiplication is a topological group. Indeed, if we identify each matrix of $GL_n(\mathbb{R})$ with enteries in $\mathbb{R}^{n^2}$ as follows

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \longleftrightarrow \left(x_{11},\cdots,x_{1n},\cdots,x_{n1},\cdots,x_{nn}\right),$$

then we can consider on $GL_n(\mathbb{R})$ the induced topology from $\mathbb{R}^{n^2}$. In this case, the multiplication is a polynomial defined by $\psi\colon \mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ such that $(A, B) \mapsto AB$ where

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj},$$

and the inversion is a rational function defined by $\phi\colon \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ such that $A \mapsto A^{-1}$ where

$$A^{-1} = \frac{1}{\det A} adj(A)$$

and $adj(A)$ is the transpose of the cofactors matrix. Remark that the determinant map given by $\det\colon GL_n(\mathbb{R}) \to \mathbb{R}^*$ is clearly continuous. Thus, both are continuous functions since ψ is a polynomial of continuous coeficients and ϕ is a product of a continuous matrix with the determinant's inverse. It follows that $GL_n(\mathbb{R})$ is a topological group. Similarly, we can show that $GL_n(\mathbb{C})$ is a topological group.

Given a topological group $\mathcal{G}$ and let a be a fixed point. The applications R_a and L_a are called the **right translation of** $\mathcal{G}$ by a and the **left translation** of $\mathcal{G}$ by a and defined as

$$x \mapsto xa \qquad x \mapsto ax,$$

respectively. Now the following shows that both are homeomorphisms.

Proposition 2.1.1. *Let $\mathcal{G}$ be a topological group and $x \in \mathcal{G}$. Then the following mappings are homeomorphisms.*

(i) R_x and L_x, and

(ii) the inversion function ϕ.

Proof.
(i) Let ψ and ϕ be the multiplication and inversion functions defined on $\mathcal{G}$, respectively. We know that the identity and the constant ($g \mapsto x$ for any $g \in \mathcal{G}$) functions are continuous, then the function

$$\begin{aligned} \gamma_x\colon \mathcal{G} &\to \mathcal{G} \times \mathcal{G} \\ g &\mapsto (g, x) \end{aligned}$$

is also continuous by Prop.1.2.11. The composition application $R_x = \psi \circ \gamma_x$ ($g \mapsto gx$) is the right translation map which is a homeomorphism. Indeed, $R_{x^{-1}}$ is the inverse of R_x and it is clear that both are continuous. Also, R_x is bijective since for any $g_1 \in \mathcal{G}$, there is $g_2 := g_1 x^{-1} \in \mathcal{G}$ such that $R_x(g_2) = g_1$, and R_x satisfies the definition of injective functions. Hence, R_x is a homeomorphism. Similarly, we can show that L_x is also a homeomorphism.
(ii) Clearly ϕ is bijective by Remark 1.1.1 and it is continuous. Likewise, ϕ^{-1} is continuous. Indeed, remark first thar for all $x \in \mathcal{G}$

$$\phi^{-1}(x) = (x^{-1})^{-1} = x = \phi^{-1}(x^{-1}) = ((x^{-1})^{-1})^{-1} = x^{-1} = \phi(x).$$

since $\mathcal{G}$ has a group structure and ϕ is bijective. Now for any open set O in $\mathcal{G}$, $\phi(O)$ is open in $\mathcal{G}$ as $\phi^{-1}(O)$ is open. Hence, ϕ is a homeomorphism. □

In the next, we give some relations which connect the group morphisms with elements in topology.

Definition 2.1.2 (Topological Group Homomorphism and Isomorphism)**.** *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two topological groups and f an application from $\mathcal{G}_1$ into $\mathcal{G}_2$. We say that f is a **topological group homomorphism** or a **topological homomorphism** if it is a continuous group homomorphism, and it is said to be a **topological group isomorphism** or a **topological isomorphism** if it is a homeomorphism group homomorphism.*

The following result shows that every topological group is homogeneous. This is the main difference between topological groups and ordinary topological spaces. In this case, it will give us many results and advantages, for instance, the fact that topological groups are homogeneous, is useful when describing a topology on a group by neighborhood bases (see the next section).

Proposition 2.1.2. *Every topological group is a homogeneous space.*

Proof. Let $x, y \in \mathcal{G}$. $R_{x^{-1}y}$ is a homeomorphism and $R_{x^{-1}y}(x) = xx^{-1}y = y$ for any $x, y \in \mathcal{G}$. We conclude that $\mathcal{G}$ is a homogeneous space. □

Remark 2.1.2. The converse of the above proposition is not true. Indeed, the Sorgenfrey line S (the set of real numbers equipped with the lower limit topology) cannot be a topological group although it is a homogeneous space. Indeed, in Sec.2.7, we will give the reason by using the metrization of groups.

Let A and B be two subsets of a topological group $\mathcal{G}$, we denote

$$\begin{aligned} A^{-1} &= \{a^{-1} : a \in A\}, \\ AB &= \{ab : a \in A, b \in B\}, \\ aB &= \{ab : b \in B\}, \text{ and} \\ Ab &= \{ab : a \in A\}. \end{aligned}$$

The following ensures how the right and left translations are applicable and helpful to identify many properties, specially in study the openness and closeness concept in groups.

Proposition 2.1.3. *Let $\mathcal{G}$ be a topological group, H_1, H_2 are two subsets of $\mathcal{G}$ and let x in $\mathcal{G}$. Then*

(i) if H_1 is open (respectively, closed), then so are H_1x and xH_1,

(ii) if H_1 is open, then so are H_1H_2 and H_2H_1,

(iii) if H_1 is open (respectively, closed), then so is H_1^{-1},

(iv) if H_1 is closed and H_2 is finite, then H_1H_2 and H_2H_1 are closed.

Proof.
(i) It is clear that xH_1 and H_1x are the (direct) image of the set H_1 under the right R_x and the left L_x translations, respectively. Therefore, if H_1 is open (or closed), then so are xH_1 and H_1x.
(ii) Write $H_1H_2 = \bigcup_{h \in H_2} H_1h$ and $H_2H_1 = \bigcup_{h \in H_2} hH_1$. So when H_1 is open then H_1H_2 and H_2H_1 are the unions of open subsets, hence both are open.
(iii) Since the inversion map ϕ is homeomorphism, then H_1^{-1} is open (or closed) whenever H_1 is.
(iv) Again, write $H_1H_2 = \bigcup_{h \in H_2} H_1h$ and $H_2H_1 = \bigcup_{h \in H_2} hH_1$. So when H_1 is closed and H_2 is finite, it follows that H_1H_2 and H_2H_1 are the unions of finitely closed subsets, hence both are closed. □

Proposition 2.1.4. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two topological groups with e_1 is the identity element of $\mathcal{G}_1$ and $f\colon \mathcal{G}_1 \to \mathcal{G}_2$ be a group homomorphism. Then*

(i) f is continuous if and only if f is continuous at e_1,

(ii) f is open if and only if f is open at e_1,

(iii) if f is bijective, then its inverse is continuous if and only if f is open,

(iv) if f is bijective, then f is topologically isomorphism if and only if f is open and continuous.

Proof.
(i) The first imblication is obvious. Conversely, assume that f is continuous at e_1 and let $x \in \mathcal{G}_1$. Let U be an open set containing $f(x)$. Then $Uf(x)^{-1}$ is a neighborhood of $e_2 = f(e_1)$ (see Prop.2.2.1 in Sec.2.2). Thus, by continuity of f at e_1, there is O an open set containing e_1 such that $f(O) \subset Uf(x)^{-1}$. f being homomorphism, we have

$$f(Ox) = f(O)f(x) \subset U$$

(see Prop.1.1.4). That is, we find an open set Ox (see Prop.2.1.3) containing x such that its image under f contained in U. Therefore, f is continuous.
(ii) It is clear that if f is open then particularly it is open at e_1.

Conversely, suppose that f is open at e_1 and let O be an open set containing an element x. Then Ox^{-1} is a neighborhood of e_1. It follows that $f(Ox^{-1})$ is an open set in $\mathcal{G}_2$ containing e_2. Since f is a homomorphism, we get

$$e_2 \in f(Ox^{-1}) = f(O)f(x^{-1}) = f(O)f(x)^{-1}$$

(see Prop.1.1.4). Hence, $f(O)$ is open in $\mathcal{G}_2$ containing $f(x)$. Consequently, f is open.
(iii) The equivalence comes from the fact that f is open and its inverse is continuous.
(iv) The equivalence follows the above definition with result of preceding. □

Example 2.1.2. Let $\mathcal{G}$ be a topological group.

1. For a fixed point $a \in \mathcal{G}$, recall that the inner automorphism I_a defined as $x \mapsto axa^{-1}$ is a topological isomorphism on $\mathcal{G}$. Indeed, it is a composition of right $R_{a^{-1}}$ and left L_a translations which both are homeomorphisms. Also it is clearly a group homomorphism. Indeed, for any $x, y \in \mathcal{G}$,

$$I_a(xy) = axya^{-1} = ax(a^{-1}a)ya^{-1} = (axa^{-1})(aya^{-1}) = I_a(x)I_a(y).$$

2. If $\mathcal{G}$ is abelian, then the inversion map ϕ is a topological isomorphism. Indeed, by Prop.2.1.1, ϕ is a homeomorphism. Thus we only show that it is a homomorphism. For any $x, y \in \mathcal{G}$

$$\phi(xy) = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = \phi(x)\phi(y).$$

The claim follows.

Definition 2.1.3 (Transformation Group)**.** *Let X be a topological space and $\mathcal{G}$ a topological group. A **left-action** of $\mathcal{G}$ on X is a continuous map $\psi\colon \mathcal{G} \times X \to X$ satisfying the conditions below.*

(i) $\psi(st, x) = \psi(s, \psi(t, x))$ for all s and t in $\mathcal{G}$ and all x in X,

(ii) $\psi(e,x) = x$ *for all* x *in* X*, where* e *is the idntity element* $\mathcal{G}$*.*

A ***left-transformation group*** *(or a* ***left*** $\mathcal{G}$***-space****) is a pair* (X,ψ) *consisting of a space* X *equipped with a left-action* ψ *of* $\mathcal{G}$ *on* X*.*

Remark 2.1.3.

- We shall usually denote $\psi(s,x)$ by sx for any $s \in \mathcal{G}$ and $x \in X$.
- Similarly, we can define a **right-action** of $\mathcal{G}$ on X as a continuous map $(x,s) \mapsto xs$, $x \in X$ and $s \in \mathcal{G}$ satisfying the conditions below.

  (i) $(xt)s = x(ts)$ for all s and t in $\mathcal{G}$ and all x in X,

  (ii) $xe = x$ for all x in X, where e is the idntity element $\mathcal{G}$.

  A **right** $\mathcal{G}$**-space** is a space X equipped with a right-action of $\mathcal{G}$ on X.
- We say that $\mathcal{G}$ **effectively operates** in X if $ex = x$ for all $x \in X$.

Example 2.1.3. Let a be a fixed point in a topological group $\mathcal{G}$, then both translations R_a and L_a are a right-action and a left-action of $\mathcal{G}$ on $\mathcal{G}$, respectively. Hence, $(\mathcal{G}, R_a)$ and $(\mathcal{G}, L_a)$ are a right $\mathcal{G}$-space and a left $\mathcal{G}$-space, respectively.

Proposition 2.1.5. *Every topological group is a transformation group.*

Proof. See the above example. □

Definition 2.1.4 (Isotropy Group)**.** *Let* x *be in a topological space* X *and let* H *be a set of elements* s *of a topological group* $\mathcal{G}$ *such that* $sx = x$*.* H *is called* ***isotropy (or stability)*** *subgroup of* $\mathcal{G}$ *at* x*.*

2.2 Neighborhood Bases

We will see that the homogeneity of a topological group plays a role to represent a topology on a group by neighbourhood bases which we examine its properties.

Proposition 2.2.1. *Let* $\mathcal{G}$ *be a topological group and* $\mathcal{B}$ *a neighborhood basis of the identity element* e*. Then for each* x *in* $\mathcal{G}$*, the collections* $x\mathscr{U} = \{x\beta : \beta \in \mathcal{B}\}$ *and* $\mathscr{U}x = \{\beta x : \beta \in \mathcal{B}\}$ *form neighborhood bases of* x*.*

Proof. It is sufficient to notice that $x\beta = L_x(\beta)$ where L_x is a homeomorphism. Similarly, $\beta x = R_x(\beta)$ where R_x is a homeomorphism. □

As consequence, if $\mathscr{U}_x$ is a neighborhood base of x, then $\mathcal{B} = \{g^{-1}U : U \in \mathscr{U}_x\}$ is a neighborhood base of e. Indeed, take $x \in \mathcal{G}$ and let U be an open subset of $\mathcal{G}$ containing e. Hence, we can find $O \in \mathscr{U}_x$ such that $e = g^{-1}g \in g^{-1}O \subseteq U$.

Now we give the fundamental properties of the neighbourhood base of the identity element e.

Proposition 2.2.2. *Let* $\mathcal{B}$ *be a neighborhood base of* e *in a topological group* $\mathcal{G}$*. Then, the following properties are satisfied.*

(P_1) $\mathcal{B}$ *is non-empty,*

(P_2) *for each* $\beta_1, \beta_2 \in \mathcal{B}$ *there exists* $\beta_3 \in \mathcal{B}$ *such that* $\beta_3 \subseteq \beta_1 \cap \beta_2$*,*

(P_3) *for each* $\beta_1 \in \mathcal{B}$ *there exists* $\beta_2 \in \mathcal{B}$ *such that* $\beta_2\beta_2 \subseteq \beta_1$*,*

(P_4) *for each* $\beta_1 \in \mathcal{B}$ *there exists* $\beta_2 \in \mathcal{B}$ *such that* $\beta_2^{-1} \subseteq \beta_1$*,*

(P_5) *for each* $\beta_1 \in \mathcal{B}$ *and* $x \in \mathcal{G}$ *there exists* $\beta_2 \in \mathcal{B}$ *such that* $x^{-1}\beta_2 x \subseteq \beta_1$*,*

(P_6) *for each* $\beta_1 \in \mathcal{B}$ *and* $x \in \beta_1$ *there exists* $\beta_2 \in \mathcal{B}$ *such that* $x\beta_2 = \beta_1$*,*

(P_7) *for each* $\beta_1 \in \mathcal{B}$ *and* $x \in \mathcal{G}$ *there exists* $\beta_2 \in \mathcal{B}$ *such that* $x\beta_2^{-1}x \subseteq \beta_1$*.*

Proof. Let $\mathcal{G}$ be a topological group with multiplication ψ and inversion ϕ functions and let $\mathcal{B}$ be a neighbourhood base of e.
(P_1) Because $\mathcal{G} \in \mathcal{B}$.
(P_2) Every topological space satisfies this property, in particular, topological groups.
(P_3) Let $\beta_1 \in \mathcal{B}$. As ψ is continuous, $\psi^{-1}(\beta_1)$ is a neighborhood of (e, e) and so there exist $\beta_2, \beta_3 \in \mathcal{B}$ such that $\beta_2 \times \beta_3 \subseteq \psi^{-1}(\beta_1)$. By ($P_2$) we find $\beta^* \in \mathcal{B}$ such that $\beta^* \subseteq \beta_2 \cap \beta_3$. Then, $\beta^* \times \beta^* \subseteq \psi^{-1}(\beta_1)$, and by applying ψ we have $\beta^*\beta^* \subseteq \psi(\psi^{-1}(\beta_1)) \subseteq \beta_1$.
(P_4) Let $\beta_1 \in \mathcal{B}$. Since $\phi^{-1}(\beta_1)$ is a neighborhood of e, there exists $\beta_2 \in \mathcal{B}$ such that $\beta_2 \subseteq \phi^{-1}(\beta_1)$. Taking the image by ϕ in both sides, we get $\beta_2^{-1} = \phi(\beta_2) \subseteq \beta_1$.
(P_5) Let $x \in \mathcal{G}$ and $\beta_1 \in \mathcal{B}$. The composition $\gamma_x = L_{x^{-1}} \circ R_x$ is given by $\gamma_x \colon \mathcal{G} \to \mathcal{G}$ such that $\gamma_x(g) = x^{-1}gx$. Clearly, γ_x is continuous as it is the composition of two continuous functions. Hence, $\gamma_x(\beta_1)$ is a neighborhood of e. Thus, if we take $\beta_2 \in \mathcal{B}$ such that $\beta_2 \subseteq \gamma_x^{-1}(\beta_1)$, and the image γ_x in both sides, we get $\gamma_x(\beta_2) = x^{-1}\beta_2 x \subseteq \beta_1$.
(P_6) Let $\beta_1 \in \mathcal{B}$ and $x \in \beta_1$. Set $\beta_2 = x^{-1}\beta_1$, then β_2 is a neighborhood of e as $x^{-1}x = e$ for each $x \in \beta_1$. It follows that $\beta_2 \in \mathcal{B}$ and hence $x\beta_2 = \beta_1$ as desired.
(P_7) Let $\beta_1 \in \mathcal{B}$. The map $L_x \circ R_x \circ \phi$ is continuous, so $(L_x \circ R_x \circ \phi)^{-1}(\beta_1)$ is a neighborhood of e. Thus, there is $\beta_2 \in \mathcal{B}$ such that $\beta_2 \subset (L_x \circ R_x \circ \phi)^{-1}(\beta_1)$, i.e.,

$$x\beta_2^{-1}x = (L_x \circ R_x \circ \phi)(\beta_2) \subseteq \beta_1.$$

□

Remark 2.2.1.

- In the preceding proposition, the properties (P_3) and (P_4) are equivalent to say:
  (P_8) for all $\beta_1 \in \mathcal{B}$ there exists $\beta_2 \in \mathcal{B}$ such that $\beta_2^{-1}\beta_2 \subseteq \beta_1$. Indeed, suppose that ($P_2$) and ($P_3$) hold. By ($P_2$), let $\beta_1 \in \mathcal{B}$, there exists $\beta_2 \in \mathcal{B}$ such that $\beta_2\beta_2 \subseteq \beta_1$. Also by ($P_3$) with taking $\beta_2 \in \mathcal{B}$, there exists $\beta_3 \in \mathcal{B}$ such that $\beta_3^{-1} \subseteq \beta_2$, and by ($P_1$), there exists $\beta^* \in \mathcal{B}$ such that $\beta^* \subseteq \beta_2 \cap \beta_3$. Therefore,

  $$\beta^{*^{-1}}\beta^* \subseteq \beta_3^{-1}\beta_2 \subseteq \beta_2\beta_2 \subseteq \beta_1$$

  as required. Conversely, suppose that (P_8) holds. Let $\beta_1 \in \mathcal{B}$, there exists $\beta_2 \in \mathcal{B}$ such that $\beta_2^{-1}\beta_2 \subseteq \beta_1$. Thus, $\beta_2^{-1} \subseteq \beta_2^{-1}\beta_2 \subseteq \beta_1$, so that ($P_3$) is satisfied. By ($P_2$), take a $\beta^* \in \mathcal{B}$ such that $\beta^* \subseteq \beta_2^{-1} \cap \beta_2$. Then $\beta^* \subseteq \beta_2^{-1}$ and $\beta^* \subseteq \beta_2$, so we get $\beta^*\beta^* \subseteq \beta_2^{-1}\beta_2 \subseteq \beta_1$.

- Similarly, we can deduce the equivalence of (P_3) and (P_4) with (P_9) which says that for all $\beta_1 \in \mathcal{B}$ there exists $\beta_2 \in \mathcal{B}$ such that $\beta_2\beta_2^{-1} \subseteq \beta_1$.

- The converse of this proposition holds: a non-empty collection $\mathcal{B}$ of subsets of a group $\mathcal{G}$ satisfying these properties generates a **group topology** on $\mathcal{G}$, i.e., a topology on $\mathcal{G}$ making $\mathcal{G}$ a topological group.

Example 2.2.1.

1. The collection $\{(-\epsilon, \epsilon) : \epsilon > 0\}$ generates a group topology (the usual topology) on the additive group $\mathbb{R}$.

2. For a fixed prime p, we define $\mathcal{B} = \{p^n\mathbb{Z} : n \in \mathbb{N}\}$ which is the family of subsets of $\mathbb{Z}$. This family satisfies the properties in Prop. 2.2.2, so it generates a group topology on $\mathbb{Z}$. This topology is called **p-adic topology**.

3. Let $\mathcal{G}$ be any group and $\mathcal{B}$ the family of all subgroups of finite index of $\mathcal{G}$, i.e.,

$$\mathcal{B} = \{H \leq \mathcal{G} : [\mathcal{G} : H] < \infty\}.$$

   We can show that $\mathcal{B}$ satisfies the properties in Prop. 2.2.2 and thus it generates a group topology on $\mathcal{G}$. This topology is called **profinite topology**.

The following proposition let us carrying on about the last point in the remark above.

Proposition 2.2.3. *Let $\mathcal{B}$ be a non-empty collection of subsets of a group $\mathcal{G}$ containing e. If $\mathcal{B}$ satisfies the properties (P_2, P_3, P_4, P_5), then there is a unique group topology such that $\mathcal{B}$ is a neighbourhood base of e in $\mathcal{G}$.*

Proof. Let $\mathcal{G}$ be any group and $\mathcal{B}$ be a non-empty collection of subsets containing e and satisfying the properties P_2, P_3, P_4 and P_5. Define

$$\tau = \{H \subseteq \mathcal{G} : \forall x \in H, \exists \beta \in \mathcal{B} \text{ such that } x\beta \subseteq H\}.$$

Our goal is to prove that τ is a group topology on $\mathcal{G}$.

Firstly, we show that τ forms a topology on $\mathcal{G}$. Clearly, $\emptyset$ and $\mathcal{G}$ are in τ. Let $H_1, H_2 \in \tau$ with $H_1 \cap H_2 \neq \emptyset$ and let $x \in H_1 \cap H_2$. By definition of τ, there exist $\beta_1, \beta_2 \in \mathcal{B}$ such that $x\beta_1 \subseteq H_1$ and $x\beta_2 \subseteq H_2$, and by (P_2) we can take $\beta_3 \in \mathcal{B}$ such that $\beta_3 \subseteq \beta_1 \cap \beta_2$. Then

$$x\beta_3 \subseteq x(\beta_1 \cap \beta_2) \subseteq x\beta_1 \cap x\beta_2 \subseteq H_1 \cap H_2.$$

Thus $H_1 \cap H_2 \in \tau$. Let $H_i \in \tau$ for all $i \in I$ and let $x \in \bigcup_{i\in I} H_i$. Then $x \in H_{i_0}$ for some $i_0 \in I$ and so there exists $\beta \in \mathcal{B}$ such that $x\beta \subseteq H_{i_0} \subseteq \bigcup_{i\in I} H_i$. Thus, $\bigcup_{i\in I} H_i \in \tau$ and it follows that τ is a topology on $\mathcal{G}$ such that $x\mathscr{U} = \{x\beta : \beta \in \mathcal{B}\}$ is a neighborhood base of x for any $x \in \mathcal{G}$.

Secondly, we show that the multiplication ψ function is continuous. Let $\beta_1 \in \mathcal{B}$ and $(x, y) \in \mathcal{G} \times \mathcal{G}$. Since $xy\beta_1$ is a neighborhood of $\psi(x, y) = xy$, it suffices to find a neighbourhood of (x, y) in $\mathcal{G} \times \mathcal{G}$ contained in $\psi^{-1}(xy\beta_1)$. By (P_3), there exists $\beta_2 \in \mathcal{B}$ such that $\beta_2\beta_2 \subseteq \beta_1$ and then $xy\beta_2\beta_2 \subseteq xy\beta_1$. By ($P_5$), we take $\beta_3 \in \mathcal{B}$ such that $y^{-1}\beta_3 y \subseteq \beta_2$ and if we let $\beta_3^* = \beta_3 \cap \beta_2$, then β_3^* is a neighborhood of e such that $y^{-1}\beta_3^* y \subseteq \beta_2$ since $\beta_3^* \subseteq \beta_2$. Thus,

$$\psi(x\beta_3^* \times y\beta_3^*) = x\beta_3^* y\beta_3^* = xy(y^{-1}\beta_3^* y)\beta_3^* \subseteq xy\beta_2\beta_2 \subseteq xy\beta_1.$$

Taking the preimage in both sides, we get $x\beta_3^* \times y\beta_3^* \subseteq \psi^{-1}(xy\beta_1)$. Therefore, ψ is continuous since $x\beta_3^* \times y\beta_3^*$ is a neighbourhood of (x, y).

Finally, we show that the inversion ϕ function is continuous. Indeed, let $\beta_1 \in \mathcal{B}$ and $x \in \mathcal{G}$. It is sufficient to find a neighborhood of x^{-1} contained in $\phi^{-1}(x\beta_1)$. By (P_4), there is $\beta_2 \in \mathcal{B}$ such that $\beta_2^{-1} \subseteq \beta_1$. Thus, $\phi(\beta_2 x^{-1}) = x\beta_2^{-1} \subseteq x\beta_1$, and with taking the inverse image in both sides, we get $\beta_2 x^{-1} \subseteq \phi^{-1}(x\beta_1)$. Now by ($P_5$), there exists $\beta_3 \in \mathcal{B}$ such that $x^{-1}\beta_3 x \subseteq \beta_2$. Hence,

$$x^{-1}\beta_3 = (x^{-1}\beta_3 x)x^{-1} \subseteq \beta_2 x^{-1} \subseteq \phi^{-1}(x\beta_1),$$

and it follows that ϕ is continuous. □

Let us mention now that by the homogeneity of a topological group and Prop.2.2.2, we can deduce some features of neighborhood base at any point analogously.

Proposition 2.2.4. *Let $\mathscr{U}_x$ be a neighborhood base of x in a topological group $\mathcal{G}$. Then*

(i) for each $U \in \mathscr{U}_x$ there exists $O \in \mathscr{U}_x$ such that $OO \subseteq U$,

(ii) for each $U \in \mathscr{U}_x$ there exists $O \in \mathscr{U}_x$ such that $O^{-1} \subseteq U$,

(iii) for each $U \in \mathscr{U}_x$ there exists $O \in \mathscr{U}_x$ such that $O^{-1}O \subseteq U$,

(iv) for each $U \in \mathscr{U}_x$ there exists $O \in \mathscr{U}_x$ such that $OO^{-1} \subseteq U$.

Proof.
(i) Let U be a neighborhood of x. We have ψ is continuous at (x, e) and $\psi(x, e) = x \in U$ which is open. Then $\psi^{-1}(U)$ is open in $\mathcal{G} \times \mathcal{G}$ and containing (x, e). Thus there exists O neighborhood of x such that $O \times O \subseteq \psi^{-1}(U)$. Hence, $\psi(O \times O) = OO \subseteq U$.
(ii) Let U be a neighborhood of x. Then we can write $U = x\beta_1$ or $U = \beta_1 x$ for some β_1 a neighborhood of e. Hence, in view of (P_7), there is β_2 containing e such that $x^{-1}\beta_2^{-1}x^{-1} \subseteq \beta_1$. This implies that $\beta_2^{-1}x^{-1} \subseteq x\beta_1$, i.e., $(x\beta_2)^{-1} \subseteq U$. Set $O = x\beta_2$, so we have $O^{-1} \subseteq U$ with $O \in \mathscr{U}_x$. The case of $U = \beta_1 x$ is smilar.
(iii) Let U be a neighborhood of x, then by using (i) and (ii), there are O and V neighborhoods of x such that

$$OO \subseteq U \quad \text{and} \quad V^{-1} \subseteq O.$$

Let $W = V \cap O$ which is a neighboorhood of x. We get

$$W^{-1}W \subseteq V^{-1}O \subseteq OO \subseteq U.$$

The claim follows.
(iv) Similar to the proof of (iii). □

Proposition 2.2.5. *Let $\mathcal{G}$ be a topological group and H_1 and H_2 be two subsets. Then*

(i) $\overline{H_1}\,\overline{H_2} \subseteq \overline{H_1 H_2}$.

(ii) $\overline{H_1}^{-1} = \overline{H_1^{-1}}$.

(iii) $x\overline{H_1}y = \overline{xH_1y}$.

Proof.
(i) Let $x \in \overline{H_1}$, $y \in \overline{H_2}$ and let β_1 be a neighborhood of e. Then there is a neighborhood β_2 of e such that $x\beta_2 y\beta_2 \subseteq xy\beta_1$. It follows that

$$\emptyset \neq (x\beta_2 \cap H_1)(y\beta_2 \cap H_2) \subseteq x\beta_2 y\beta_2 \cap H_1H_2 \subseteq xy\beta_1 \cap H_1H_2$$

meaning that $xy \in \overline{H_1H_2}$. Thus, $\overline{H_1}\,\overline{H_2} \subseteq \overline{H_1H_2}$.
(ii) The inversion map ϕ is a homeomorphism, so it is a closed map, i.e., $\phi(\overline{H_1}) = \overline{H_1}^{-1}$ is closed. Thus, $\overline{H_1}^{-1} = \overline{\overline{H_1}^{-1}}$. It follows that

$$\overline{H_1^{-1}} \subseteq \overline{\overline{H_1}^{-1}} = \overline{H_1}^{-1}.$$

On the other hand, as ϕ is continuous, $\overline{H_1}^{-1} \subseteq \overline{H_1^{-1}}$ by Prop.1.2.9. Hence the equality holds.
(iii) The composition map $\gamma_{xy} = L_x \circ R_y$ is a homeomorphism, so it is a closed map, i.e., $\gamma_{xy}(\overline{H_1}) = x\overline{H_1}y$ is closed. Thus, $x\overline{H_1}y = \overline{x\overline{H_1}y}$. It follows that

$$\overline{xH_1y} \subseteq \overline{x\overline{H_1}y} = x\overline{H_1}y.$$

On the other hand, as γ_{xy} is continuous, then $x\overline{H_1}y \subseteq \overline{xH_1y}$ by Prop.1.2.9. Hence the equality holds. □

Proposition 2.2.6. *Let $\mathcal{G}$ be a topological group and D be a dense subset in $\mathcal{G}$. Assume that $\mathcal{B}$ is a fundamental system of neighborhoods of e, then the collection*

$$\mathscr{U} = \{Ux : x \in D, U \in \mathcal{B}\}$$

is a basis for a topology on $\mathcal{G}$.

Proof. Let O be a neighborhood of x, so Ox^{-1} is a neighborhood of e. Thus there is $U \in \mathcal{B}$ such that $UU^{-1} \subseteq Ox^{-1}$ by (P_9). As D is dense, then so is xD^{-1} because

$$X = \gamma_x(\overline{D}) \subseteq \overline{\gamma_x(D)}$$

where the map $\gamma_x = L_x \circ \phi$ is continuous. So, there is $y \in U \cap xD^{-1}$ and $z \in D$ such that $y = xz^{-1}$, i.e., $z = y^{-1}x$. We get that $Uy^{-1}x \in \mathscr{B}$. Further as $y \in U$ and for all $x \in O$, we have

$$Uy^{-1}x \subseteq UU^{-1}x \subseteq Ox^{-1}x = O.$$

Therefore, $\bigcup_{x\in O} Uy^{-1}x \subseteq O$. Likewise, since $e \in Uy^{-1}$, then for all $x \in O$, we have $x \in Uy^{-1}x$. Hence, $O \subseteq \bigcup_{x\in O} Uy^{-1}x$. Consequently, $O = \bigcup_{x\in O} Uy^{-1}x$. □

Definition 2.2.1 (Symmetric Neighborhood)**.** *Let $\mathcal{G}$ be a topological group. A neighborhood β is said to be* ***symmetric*** *if $\beta = \beta^{-1}$.*

Proposition 2.2.7. *Let $\mathcal{G}$ be a topological group, then there exists a base of symmetric open sets. In particular, there is a fundamental system of symmetric neighborhoods of each point of $\mathcal{G}$.*

Proof. Let U be an open subset of $\mathcal{G}$. Then U^{-1} is an open subset of $\mathcal{G}$ as well. Put $O = U \cap U^{-1}$, so O is an open and $O = O^{-1}$ since

$$O = U \cap U^{-1} = U^{-1} \cap (U^{-1})^{-1} = (U \cap U^{-1})^{-1} = O^{-1}.$$

Further, $O \subseteq U$, so that there is a symmetric of open sets. □

2.3 Subgroups

Definition 2.3.1 (Topological Subgroup)**.** *Let $\mathcal{G}$ be a topological group and H a subgroup of $\mathcal{G}$. Consider H equipped with the topology induced from $\mathcal{G}$, then H is called a* ***topological subgroup*** *of $\mathcal{G}$.*

For illustrating the idea of the definition, we give the proposition below.

Proposition 2.3.1. *A subgroup of a topological group is a topological group.*

Proof. Let H be a subgroup of a topological group $\mathcal{G}$. It is enough to discuss the continuity of the inversion ϕ and multiplication ψ functions on H. Indeed, since both are also restrictions on H, it follows that they are continuous on H (see Ex.1.2.5(5)). And the claim follows. □

Proposition 2.3.2. *Let $\mathcal{G}$ be a topological group and H a subgroup of $\mathcal{G}$. Then*

- *(i) if H is open, H is closed,*
- *(ii) if H is closed and of finite index, H is open,*
- *(iii) if H contains a non-empty open subset, H is open.*

Proof. Let $\mathcal{R}$ be a set of representatives of the cosets xH except H. Then

$$\mathcal{G} \setminus H = \bigcup_{x \in \mathcal{R}} xH. \tag{2.1}$$

(i) If H is open, then according to the Prop.2.1.3, its complement in (2.1) is a union of open subsets so that it is open. Therefore, H is closed.
(ii) If H is closed and having a finite index, then $\mathcal{R}$ is finite and the complement of H in (2.1) is a finite union of closed subsets, so H is open.
(iii) If H contains a non-empty open subset, say O, then $H = OH$ and by Prop.2.1.3, H is open. □

Proposition 2.3.3. *Let H be a subset of a topological group $\mathcal{G}$. Then*

(i) if H is a subgroup, then so is its closure $\overline{H}$,

(ii) if H is a normal subgroup, then so is its closure $\overline{H}$.

Proof. Let $\mathcal{G}$ be a topological group.
(i) $\overline{H}$ is clearly non-empty since $e \in H \subset \overline{H}$. Let $x, y \in \overline{H}$ and let U be a neighborhood of xy^{-1}. As the application μ is continuous (see Remark 2.1.1), then there are a neighborhood O of x and a neighborhood V of y such that $O \times V \subseteq \mu^{-1}(U)$, i.e., $OV^{-1} \subseteq U$. Also, as x and y are in $\bar{H}$, then there are $h_1 \in O \cap H$ and $h_2 \in V \cap H$. Hence, $h_1 h_2^{-1} \in OV^{-1} \cap H$ since H is a subgroup. Thus, $xy^{-1} \in \overline{H}$. It follows that $\overline{H}$ is a subgroup.
(ii) For all $x \in \mathcal{G}$, by Prop.2.2.5(iii), we get

$$x\overline{H}x^{-1} = \overline{xHx^{-1}} = \overline{H}$$

as H is normal. Therefore, $\bar{H}$ is normal. □

Example 2.3.1. Recall that the general linear group $GL_n(\mathbb{R})$ is a topological group.

1. The subset $GL_{n,+}(\mathbb{R})$ is the set of all matrices having positive determinant, i.e.,

   $$GL_{n,+}(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det(A) > 0\}.$$

   Then it is an open subgroup of $GL_n(\mathbb{R})$. Indeed, $GL_{n,+}(\mathbb{R})$ is non-empty since the determinant of the identity matrix is 1. If A and B are two elements of $GL_{n,+}(\mathbb{R})$, then

   $$\det(AB^{-1}) = \det(A)\det(B^{-1}) = \frac{\det(A)}{\det(B)} > 0.$$

   On the other hand, since the determinant map over $GL_n(\mathbb{R})$ is continuous, it follows that $GL_{n,+}(\mathbb{R})$ is open as $\det^{-1}(0, \infty) = GL_{n,+}(\mathbb{R})$. Consequently, it is closed by Prop.2.3.2(i).

2. Let us recall that the special linear group $SL_n(\mathbb{R})$ is defined as

   $$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : det(A) = 1\}.$$

   Then it is a closed subgroup of $GL_n(\mathbb{R})$. Indeed, clearly it is a subgroup of $GL_n(\mathbb{R})$. On the other hand, since the determinant map over $GL_n(\mathbb{R})$ is continuous, it follows that $SL_n(\mathbb{R})$ is closed as $\det^{-1}(\{1\}) = SL_n(\mathbb{R})$.

3. The orthogonal matrices group $\mathcal{O}_n(\mathbb{R})$ over $GL_n(\mathbb{R})$ is defined as

$$\mathcal{O}_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : A^t A = I\}.$$

Then it is a closed subgroup of $GL_n(\mathbb{R})$. Indeed, it is clear that $\mathcal{O}_n(\mathbb{R})$ is a subgroup of $GL_n(\mathbb{R})$. Note that if $A \in \mathcal{O}_n(\mathbb{R})$, then $A^t A = I$ which is equivalent to

$$\sum_{k=1}^{n} a_{ki}a_{kj} = \delta_{ij} \qquad \forall i, j = 1, 2, \cdots, n,$$

where δ_{ij} is the Kronecker delta which is equal to 1 if $i = j$ and 0 otherwise. Now, define the continuous function $f_{ij}\colon GL_n(\mathbb{R}) \to \mathbb{R}$ as $f_{ij}(A) = \sum_{k=1}^{n} a_{ki}a_{kj}$, $i, j = 1, 2, \cdots, n$. Thus, $\mathcal{O}_n(\mathbb{R})$ is closed since

$$\mathcal{O}_n(\mathbb{R}) = f_{ij}^{-1}(\{\delta_{ij}\}) = \begin{cases} f_{ij}^{-1}(\{0\}) & \text{if } i \neq j \\ f_{ij}^{-1}(\{1\}) & \text{if } i = j. \end{cases}$$

Lemma 2.3.1. *Let U be a symmetric neighborhood of e in a topological group $\mathcal{G}$. Then $H = \bigcup_{n\geq 1} U^n$ is a clopen subgroup of $\mathcal{G}$.*

Proof. Let $x, y \in H$. Then there exist positive integers n and m such that $x \in U^n$ and $y \in U^m$. Hence,

$$x^{-1}y \in (U^n)^{-1}U^m = (U^{-1})^n U^m = U^n U^m = U^{n+m} \subseteq H$$

since U is symmetric. Hence, H is a subgroup. On the other hand, we show that H is open. Indeed, if U is a neighborhood of e, then for each $x \in H$, $xU \in xH = H$, so H is open. Consequently, H is closed by Prop.2.3.2. □

Proposition 2.3.4. *A subgroup H of a topological group $\mathcal{G}$ is closed if and only if for some closed neighborhood U of e, $H \cap U$ is closed in $\mathcal{G}$.*

The proof follows easily.

Definition 2.3.2 (Discrete Subgroup)**.** *A subgroup H of a toplogical group is called a* ***discrete subgroup*** *if for each $x \in H$, there exists a neighborhood U of x such that $U \cap H = \{x\}$.*

2.4 Quotient Groups

Let $\mathcal{G}$ be a topological group endowed with a topology τ and H a subgroup of $\mathcal{G}$ (not necessarily normal). Consider the equivalence relation in $\mathcal{G}$ given by

$$x \sim y \quad \text{if and only if} \quad xH = yH.$$

For any $x \in \mathcal{G}$, the equivalence class $[x]$ is exactly the coset xH, because

$$[x] = \{y \in \mathcal{G} : x \sim y\} = \{y \in \mathcal{G} : xH = yH\} = \{y \in \mathcal{G} : x^{-1}y \in H\} = \{y \in \mathcal{G} : y \in xH\} = xH.$$

We shall denote the set whose elements are the cosets xH by $\mathcal{G}/H$. Further, define the **canonical projection** κ by

$$\begin{aligned} \kappa\colon \mathcal{G} &\to \mathcal{G}/H \\ x &\mapsto xH. \end{aligned} \tag{2.2}$$

It is easy to check that κ is a surjection. Now we construct the collection τ_κ of subsets of $\mathcal{G}/H$ induced by κ as follows

$$\tau_\kappa = \{O \subseteq \mathcal{G}/H : \kappa^{-1}(O) \in \tau\},$$

then it is a topology on $\mathcal{G}/H$. In fact, τ_κ is the quotient topology on $\mathcal{G}/H$ since it is the finest topology making κ continuous. In fact, κ is the quotient map and $\mathcal{G}/H$ is the quotient space.

Remark 2.4.1.

- If H is a normal subgroup of $\mathcal{G}$, then $\mathcal{G}/H$ has a natural group structure.
- If $\mathcal{G}/H$ is a group, the canonical projection κ is obviously an epimorphism. Indeed, for any x and y in $\mathcal{G}$, we get

$$\kappa(xy) = xyH = xHyH = \kappa(x)\kappa(y).$$

- When we say O open in $\mathcal{G}/H$, this actually means that there is an open set U in $\mathcal{G}$ such that $U = \kappa^{-1}(O)$, i.e., $\kappa(U) = O$. Besides, we have

$$O = \kappa^{-1}(\kappa(O))$$

as κ is surjective.

Now suppose that the group $\mathcal{G}$ has the multiplication ψ and the inversion ϕ mappings, and H be a normal subgroup of $\mathcal{G}$. Further, assume that $\mathcal{G}/H$ has the product ψ^* and the inversion ϕ^* maps. If we consider the map $\kappa \times \kappa$ given by $(\kappa \times \kappa)(x, y) = (\kappa(x), \kappa(y))$, then we get the following diagrams which commute.

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\phi} & \mathcal{G} \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\kappa} \\ \mathcal{G}/H & \xrightarrow[\phi^*]{} & \mathcal{G}/H \end{array} \qquad \begin{array}{ccc} \mathcal{G}\times\mathcal{G} & \xrightarrow{\psi} & \mathcal{G} \\ {\scriptstyle\kappa\times\kappa}\downarrow & & \downarrow{\scriptstyle\kappa} \\ \mathcal{G}/H\times\mathcal{G}/H & \xrightarrow[\psi^*]{} & \mathcal{G}/H \end{array} \tag{2.3}$$

In our case, ϕ^* is continuous if and only if the composition $\phi^* \circ \kappa$ is also continuous. By commutativity of (2.3), $\phi^* \circ \kappa = \kappa \circ \phi$ and as the latter is a composition of continuous functions, then so is $\phi^* \circ \kappa$. Thus, the inversion ϕ^* is continuous. The map κ is open and surjective (see Prop.2.4.1), then so is $\kappa \times \kappa$ by Prop.1.2.13. Therefore, $\kappa \times \kappa$ is a quotient map which is continuous, open and surjective. By commutativity of (2.3), $\psi^* \circ (\kappa \times \kappa)$ is continuous, then by Prop.1.2.14 so is ψ^* in which $\mathcal{G}/H$ is a topological group.

Definition 2.4.1 (Topological Quotient Group)**.** *The quotient space $\mathcal{G}/H$ defined above is a **topological quotient group** provided that $\mathcal{G}/H$ has a natural group structure and both ψ^* and ϕ^* are continuous.*

On topological groups, the canonical projection κ which is also a quotient map, is always open as it is described in the following proposition.

Proposition 2.4.1. *Let H be a subgroup (not necessarily normal) of a topological group $\mathcal{G}$, and let $\kappa\colon \mathcal{G} \to \mathcal{G}/H$ be the canonical projection. Then κ is an open map.*

Proof. Let O be an open subset of $\mathcal{G}$. According to the definition of the quotient topology, $\kappa(O)$ is open if and only if $\kappa^{-1}(\kappa(O))$ is open in $\mathcal{G}$. Thus,

$$\kappa^{-1}(\kappa(O)) = \kappa^{-1}\left(\{xH : x \in O\}\right) = \{y \in \mathcal{G} : yH = xH \text{ for some } x \in O\},$$

and we know that $yH = xH$ if and only if $y \in xH$. Hence,

$$\kappa^{-1}(\kappa(O)) = \bigcup_{x\in O} \{y \in \mathcal{G} : y \in xH\} = \bigcup_{x\in O} xH = OH$$

which is open by Prop.2.1.3. □

Remark 2.4.2.

- We have argued that the canonical projection has many features on topological groups. It is a group isomorphism, open and continuous. Hence, by Prop.2.1.4, it is a topological isomorphism.

- In general, the canonical map κ is not closed. For instance, consider the additive group $\mathbb{R}$ and the subgroup $\mathbb{Z}$, then $\kappa\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is not closed since the closed set $\{n + \frac{1}{2^n} : n \in \mathbb{N}\}$ is mapped onto a set $\{\frac{1}{2^n} : n \in \mathbb{N}\}$ which is not closed. Nevertheless, we will see in Sec.3.2 that κ is closed whenever H is compact (see Prop.3.2.9).

Proposition 2.4.2. *Let $\mathcal{B}$ be a basis for a topology on $\mathcal{G}$. The collection*

$$\mathscr{B} = \{\{xH : x \in U\}, U \in \mathcal{B}\}$$

is a basis for the quotient topology on $\mathcal{G}/H$.

Proof.

- Let $xH \in \mathcal{G}/H$ with $x \in \mathcal{G}$. Thus, there is $U \in \mathcal{B}$ such that $x \in U$. Hence, $\{xH\} \in \mathscr{B}$.

- If $xH \in O \cap V$ and $O, V \in \mathscr{B}$, then by the continuity of κ, $\kappa^{-1}(O \cap V)$ is open in $\mathcal{G}$. Thus, there is $U \in \mathcal{B}$ containing x such that $U \subset \kappa^{-1}(O \cap V)$. In other words, $\kappa(U) \subset O \cap V$. It follows that $\kappa(U) = \{uH : u \in U\} \in \mathscr{B}$.

Therefore, $\mathscr{B}$ is a basis for the topology $\tau_{\mathscr{B}}$ generating by $\mathscr{B}$. On the other hand, it is clear that $\tau_{\mathscr{B}} = \tau_\kappa$ (see Prop.1.2.15). We conclude that $\mathscr{B}$ is a basis for the quotient topology defined on $\mathcal{G}/H$. □

The converse of Prop.2.4.1 is not true in general. Indeed, if the canonical projection $\kappa\colon \mathcal{G} \to \mathcal{G}/H$ is an open map, then in general $(\mathcal{G}, \tau)$ fails to be a topological group as we will see in the next example. For this, the following lemma is needed.

Lemma 2.4.1. *A subgroup of $\mathbb{R}$ which is not of the form $s\mathbb{Z}$ for some $s \in \mathbb{Z}$ is necessarily dense in $\mathbb{R}$.*

Proof. Let H be a subgroup of $\mathbb{R}$ not of the form $s\mathbb{Z}$. We prove that there is no least positive element in H. On the contrary, suppose that t is the least positive element of H, then $nt \in H$ for any $n \in \mathbb{Z}$ so that $t\mathbb{Z}$ is contained in H. Take x in H and not in $t\mathbb{Z}$ and let m be the integer part of $\frac{x}{t}$. Note that $|x - mt|$ is an element of H and $0 < |x - mt| < t$, a contradiction. As consequence, H has no least positive element, and so there is a strictly decreasing positive sequence in H,

$$x_1 > x_2 > \cdots > x_i > \cdots$$

converging to 0. Now, given any interval (a, b), we can take an element x_i of the sequence such that $0 < x_i < b - a$. So the element $nx_i \in H$ lies in (a, b) for some $n \in \mathbb{Z}$. Hence, H is dense in $\mathbb{R}$. □

Now we go through to the intended example.

Example 2.4.1. As we mentioned earlier, the Sorgenfrey line S is not a topological additive group. Indeed, the inverse image of an open subset $[a, b)$ (in the lower limit topology) under the inversion ϕ function $(x \mapsto -x)$ is $(-b, -a]$ which is not open in S. Hence, ϕ is not continuous. Now, let us see that for any subgroup H of S, the canonical projection κ is an open map. Indeed, by the preceding lemma, H may be either of the form $s\mathbb{Z}$ or dense in $\mathbb{R}$ (with the usual topology). We distinguish two cases.

- If H is of the form $s\mathbb{Z}$, then for an open subset $[a, b)$ of S,

$$\begin{aligned}\kappa^{-1}(\kappa[a,b)) &= \{x \in S : \kappa(x) \in \kappa[a,b)\}\\ &= \{x \in S : ks + x \in [a,b) \text{ for some } k \in \mathbb{Z}\}\\ &= \bigcup_{k\in\mathbb{Z}} [a+ks, b+ks)\end{aligned}$$

  which is open, because it is a union of open subsets. Thus κ is an open map.

- If H is dense in $\mathbb{R}$. We have to prove that $\kappa^{-1}(\kappa[a,b)) = S$ for any subset $[a,b)$ of S. For any $x \in S$, the subset $(a-x, b-x)$ is open in $\mathbb{R}$ and so there exists $h \in H$ such that $a - x < h < b - x$. Then $a \leq h + x \leq b$ and so there is $y \in (a,b)$ such that $h + x = y$. Thus, $y - x \in H$ and

$$\kappa(x) = \kappa(y) \in \kappa(a,b) \subseteq \kappa[a,b).$$

  This implies that $x \in \kappa^{-1}(\kappa[a,b))$. Hence, κ is an open map.

Proposition 2.4.3. *Let $\mathcal{G}/H$ be the quotient of a topological group $\mathcal{G}$ by a subgroup H. The application T_s on $\mathcal{G}/H$ defined as $xH \mapsto sxH$ is a homeomorphism.*

Proof. T_s is clearly a bijection. We show that T_s is continuous. Indeed, let O be an open set in $\mathcal{G}/H$, then according to the definition of the quotient topology, there is an open set U in $\mathcal{G}$ such that $O = \kappa(U)$. Now we have,

$$\begin{aligned}T_s^{-1}(O) &= \{xH \in \mathcal{G}/H : T_s(xH) \in O\}\\ &= \{xH \in \mathcal{G}/H : sxH \in O\}\\ &= \{xH \in \mathcal{G}/H : xH \in s^{-1}O\}\\ &= s^{-1}O.\end{aligned}$$

Hence,

$$T_s^{-1}(O) = s^{-1}O = L_{s^{-1}}(\kappa(U)) = (L_s^{-1} \circ \kappa)(U) \tag{2.4}$$

as $L_{s^{-1}} = L_s^{-1}$ so that T_s is continuous since form (2.4) gives us that $T_s^{-1}(O)$ is the image of U under the composition of two continuous mappings. On the other hand, as T_s is surjective, its inverse exists on $\mathcal{G}/H$ which is defined as $xH \mapsto s^{-1}xH$, i.e., $T_s^{-1} = T_{s^{-1}}$. Note that for any open set O in $\mathcal{G}/H$ and from (2.4), we obtain

$$T_s^{-1}(O) = (L_s \circ \kappa^{-1})(U)$$

for some open set U. Hence T_s^{-1} is continuous as well, meaning that T_s is a homeomorphism. □

Proposition 2.4.4. *Let $\mathcal{G}$ be a topological group and H be a subgroup, then $\mathcal{G}/H$ is homogeneous.*

Proof. Since the map $T_{yx^{-1}}$ is a homeomorphism on $\mathcal{G}/H$, it follows that for any xH and yH in $\mathcal{G}/H$, we obtain

$$T_{yx^{-1}}(xH) = yx^{-1}xH = yH.$$

Hence, $\mathcal{G}/H$ is homogeneous. □

Proposition 2.4.5 (First Isomorphism Theorem). *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two topological groups and $f\colon \mathcal{G}_1 \to \mathcal{G}_2$ a continuous open epimorphism. Then the map*

$$\begin{aligned}\varphi\colon \mathcal{G}_1/\ker f &\to \mathcal{G}_2\\ x(\ker f) &\mapsto f(x)\end{aligned}$$

is a topological isomorphism.

Proof. According to Prop.1.1.5, φ is a group isomorphism. It remains is to show that φ is open and continuous. Note that if $\kappa\colon \mathcal{G}_1 \to \mathcal{G}_1/\ker f$ is the canonical projection, then $\varphi \circ \kappa = f$. That is,

$$\varphi(\kappa(x)) = \varphi(x(\ker f)) = f(x) \quad \text{for all } x \in \mathcal{G}_1.$$

Now, since f is continuous, by Prop.1.2.14, φ is continuous. Finally, we show that φ is an open map. Indeed, for any open subset O of $\mathcal{G}_1/\ker f$, as κ is continuous and f is open, $f(\kappa^{-1}(O))$ is open in $\mathcal{G}_2$. However, κ is surjective, then

$$f(\kappa^{-1}(O)) = (\varphi \circ \kappa \circ \kappa^{-1})(O) = \varphi(O).$$

Hence φ is open. Thus, by Prop.2.1.4 (iv), the claim follows. □

Example 2.4.2.

1. Consider the topological groups $\mathbb{R}$ and $\mathbb{S}^1$ and the exponential function f from $\mathbb{R}$ into $\mathbb{S}^1$ defined as $f(x) = e^{2\pi i x}$ for all $x \in \mathbb{R}$. Clearly, f is a homomorphism. By considering $\mathbb{S}^1$ as a subset of $\mathbb{R}^2$, we may define by $f(x) = (\cos 2\pi x, \sin 2\pi x)$ for all $x \in \mathbb{R}$, and since both components are continuous, then so is f. Let (a, b) be an open interval of $\mathbb{R}$, the image of (a, b) can be either $\mathbb{S}^1$ (if $b - a > 1$), an open arc (if $b - a < 1$), or $\mathbb{S}^1 \setminus \{q\}$ for some $q \in \mathbb{S}^1$ (if $b - a = 1$). So the image of f is open in all cases. Thus f is an open map. Also, clear that $f(\mathbb{R}) = \mathbb{S}^1$ and $\ker f = \{x \in \mathbb{R} : e^{2\pi i x} = 1\} = \mathbb{Z}$. Therefore, by Prop.2.4.5, the topological group $\mathbb{R}/\mathbb{Z}$ is topologically isomorphic to $\mathbb{S}^1$.

2. In Ex.2.1.1, recall that $GL_n(\mathbb{R})$ is equipped with the induced topology from $\mathbb{R}^{n^2}$, and the determinant function given by $\det\colon GL_n(\mathbb{R}) \to \mathbb{R}^*$ is a topological homomorphism. det is an open map also. Indeed, let O be an open subset of $GL_n(\mathbb{R})$. As $\{0\}$ is closed in $\mathbb{R}$ and det is continuous, then $\det^{-1}(\{0\})$ is closed in $\mathbb{R}^{n^2}$, and so its complement $GL_n(\mathbb{R})$ is open in $\mathbb{R}^{n^2}$. Therefore, O is also open in $\mathbb{R}^{n^2}$. Now we show that $\det(O)$ is open. Fix $d \in \det(O)$ and take $x \in O$ with determinant d. As $x \in O$ and O is open, there is an open ball $B(x, r)$ with $r > 0$ such that $B(x, r) \subseteq O$. We ensure that there exists $\epsilon > 0$ such that $tx \in B(x, r)$ for all $t \in (1 - \epsilon, 1 + \epsilon)$. In fact, we have

$$\begin{aligned} tx \in B(x, r) &\iff \|x - tx\| < r \\ &\iff |1 - t|\|x\| < r \\ &\iff |1 - t| < \frac{r}{\|x\|}, \end{aligned}$$

we can take $\epsilon := \frac{r}{\|x\|} > 0$. Now by taking the image of tx under det, we have $\det(tx) = t^n d \in \det(O)$ for all $t \in (1 - \epsilon, 1 + \epsilon)$. We see that $((1 - \epsilon)^n d, (1 + \epsilon)^n d) \subseteq \det(O)$. Hence, det is open.
For each $d \in \mathbb{R}^*$, the matrix

$$\begin{pmatrix} d & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

has determinant d, thus ker is an epimorphism. Further, since $\ker \det = SL_n(\mathbb{R})$ (see Ex.1.1.2 part(1)), by Prop.2.4.5, $GL_n(\mathbb{R})/SL_n(\mathbb{R})$ is topologically isomorphic to $\mathbb{R}^*$.

Proposition 2.4.6 (Third Isomorphism Theorem)**.** *Let N and M be two normal subgroups of a topological group $\mathcal{G}$ with $N \leq M$. Then*

$$\frac{\mathcal{G}/N}{M/N} \cong \frac{\mathcal{G}}{M}$$

in the sense of being topologically isomorphic.

Proof. Since $\mathcal{G}$ has a group structure, then the two quotient groups are isomorphic, i.e., there is a group isomorphism f form $\frac{\mathcal{G}/N}{M/N}$ onto $\frac{\mathcal{G}}{M}$. On the other hand, the canonical projection $\kappa\colon \mathcal{G} \to \mathcal{G}/N$ is a topological isomorphism and also the map $\vartheta\colon \frac{\mathcal{G}}{M} \to \frac{\mathcal{G}/N}{M/N}$ is a homeomorphism. Thus, the composition $\vartheta \circ \kappa$ is a homeomorphism as well. Hence, by Prop.2.4.5,

$$\vartheta \circ \kappa\colon \mathcal{G}/\ker(\vartheta \circ \kappa) \to \frac{\mathcal{G}/N}{M/N}$$

is a homeomorphism such that $M = \ker(\vartheta \circ \kappa)$. In other words, we find a homeomorphism group homomorphism $f = \vartheta \circ \kappa$ between these quotients. Consequently, $\frac{\mathcal{G}/N}{M/N}$ is topologically isomorphic to $\frac{\mathcal{G}}{M}$. □

One can assume that the second isomorphism theorem is also applicable in topological groups as in groups (see Prop.1.1.6), but it is not true in general. The following example shows in fact that it does not hold for topological groups in general.

Example 2.4.3. Consider the additive group $\mathbb{R}$ and the normal subgroup $\mathbb{Z}$ of $\mathbb{R}$. Also consider $\alpha\mathbb{Z}$ where α is an irrational number. By Prop.2.4.6, the group $(\mathbb{Z} + \alpha\mathbb{Z})/\mathbb{Z}$ is isomorphic to $\alpha\mathbb{Z}/(\mathbb{Z} \cap \alpha\mathbb{Z})$. As α is irrational, $\alpha\mathbb{Z}/(\mathbb{Z} \cap \alpha\mathbb{Z}) = \alpha\mathbb{Z}/\{0\}$ is discrete. The subgroup $\mathbb{Z} + \alpha\mathbb{Z}$ of $\mathbb{R}$ can not be written in the form $s\mathbb{Z}$. If not, $\mathbb{Z} + \alpha\mathbb{Z} = s\mathbb{Z}$ implies $1 = ms$ and $\alpha = ns$ for some $m, n \in \mathbb{Z}$. So we obtain $\alpha = \frac{n}{m}$ which is a contradiction. Thus, $\mathbb{Z} + \alpha\mathbb{Z}$ would be dense in $\mathbb{R}$, by Lemma 2.4.1. Therefore, the quotient $(\mathbb{Z} + \alpha\mathbb{Z})/\mathbb{Z}$ is dense as a subspace of $\mathbb{S}^1$. Any open subset of this quotient contains infinitely many elements of it, so it is not discrete and thus not homeomorphic to $\alpha\mathbb{Z}/(\mathbb{Z} \cap \alpha\mathbb{Z})$.

Proposition 2.4.7. *Let $\mathcal{G}$ be a topological group and N, M be two normal subgroups of $\mathcal{G}$ with $N \leq M$. If τ_1 is a topology on M/N as a subspace of $\mathcal{G}/N$ and τ_2 a topology on M/N as a quotient space of M, then $\tau_1 = \tau_2$.*

Proof. Let $\kappa\colon \mathcal{G} \to \mathcal{G}/N$ be the canonical projection. Note that $\kappa(x) = xN$ is in M/N if and only if $x \in M$. Define

$$\begin{aligned} f\colon M &\to (M/N, \tau_1) \\ x &\mapsto \kappa(x) \end{aligned}$$

which is clearly a group epimorphism. We show that f is continuous. Let $O \in \tau_1$, there exists an open subset U of $\mathcal{G}/N$ such that $O = U \cap M/N$. Then

$$f^{-1}(O) = \kappa^{-1}(O) = \kappa^{-1}(U \cap M/N) = \kappa^{-1}(U) \cap \kappa^{-1}(M/N) = \kappa^{-1}(U) \cap M.$$

Since κ is continuous, $\kappa^{-1}(U)$ is open in $\mathcal{G}$. Hence, f is continuous. Now we prove that f is open. Let O be open in M, we shall write $O = M \cap U$ with U open in $\mathcal{G}$. As $M = \bigcup_{x \in M} xN$, we have

$$O = \bigcup_{x \in M} (xN \cap U)$$

and we get,

$$\begin{aligned} f(O) &= \kappa\left(\bigcup_{x \in M} (xN \cap U)\right) \\ &= \bigcup_{x \in M} \kappa(xN \cap U) \\ &= \bigcup_{x \in M} \{yN : y \in xN \cap U\} \\ &= \bigcup_{x \in M} (\kappa(x) \cap \{yN : y \in U\}). \end{aligned}$$

Therefore,

$$f(O) = \kappa(M) \cap \kappa(U) = M/N \cap \kappa(U).$$

By Prop.2.4.1, κ is open so that $\kappa(U)$ is open. Hence, f is an open map. Futhermore, $\ker f = \{x \in M : xN = N\} = N$. Thus according to the Prop.2.4.5, the map

$$\begin{aligned}(M/N, \tau_2) &\to (M/N, \tau_1)\\ xN &\mapsto xN\end{aligned}$$

is a homeomorphism. Hence, $\tau_1 = \tau_2$. □

Corollary 2.4.1. *If N is a normal subgroup of a topological group $\mathcal{G}$, then every subgroup of $\mathcal{G}/N$ is topologically isomorphic to a quotient group M/N where $N \leq M \trianglelefteq \mathcal{G}$.*

Proof. We know that if H is a subgroup of $\mathcal{G}/N$ then there exists a normal subgroup M of $\mathcal{G}$ containing N such that $H = M/N$. By the preceding proposition, we see that H and M/N define the same topology, so they are homeomorphic. □

Proposition 2.4.8. *Let $\mathcal{G}$ be a topological group and $\mathcal{G}/H$ be the quotient group of $\mathcal{G}$ by a normal subgroup H. The application $\varphi\colon \mathcal{G} \times \mathcal{G}/H \to \mathcal{G}/H$ is continuous.*

Proof. Let O be an open set in $\mathcal{G}/H$, then there is an open set U in $\mathcal{G}$ such that $O = \kappa(U)$. As $\mathcal{G}$ is a topological group, then the multiplication map ψ is continuous. This implies that for any $(x, y) \in \psi^{-1}(U)$, there are neighborhoods V_1 of x and V_2 of y such that $V_1 \times V_2 \subset \psi^{-1}(U)$, i.e., $\psi(V_1, V_2) \subset U$. Note that

$$\varphi(V_1, \kappa(V_2)) = \kappa(\psi(V_1, V_2))$$

by the diagram below. Thus,

$$\varphi(V_1, \kappa(V_2)) = \kappa(\psi(V_1, V_2)) \subset \kappa(U) = O.$$

Hence, we find a neighborhood $V_1 \times \kappa(V_2)$ of $(x, \kappa(y))$ such that $V_1 \times \kappa(V_2) \subset \varphi^{-1}(O)$. Consequently, φ is continuous.

We illustrate the relation between the concerning functions by the following commutative diagram

$$\begin{array}{ccc}\mathcal{G} \times \mathcal{G} & \xrightarrow{\psi} & \mathcal{G}\\ {\scriptstyle id\times\kappa}\downarrow & & \downarrow{\scriptstyle \kappa}\\ \mathcal{G} \times \mathcal{G}/H & \xrightarrow[\varphi]{} & \mathcal{G}/H\end{array}$$

□

Definition 2.4.2 (Homogeneous Space of a Topological Group)**.** *We say that a topological group $\mathcal{G}$ **operates transitively** in a space X if for all x,y in X, there is s in $\mathcal{G}$ such that sx=y. In this case, X is called a **homogeneous space** of $\mathcal{G}$.*

Example 2.4.4. Let H be a subgroup of a topological group $\mathcal{G}$. Then the quotient space $\mathcal{G}/H$ is a homogeneous space of $\mathcal{G}$. Since by Remark.2.1.3, the operation is defined as

$$\psi(s, tH) = stH \quad \text{for all } s \in \mathcal{G}, tH \in \mathcal{G}/H.$$

Proposition 2.4.9. *Let X be a homogeneous space of a topological group $\mathcal{G}$. Let x in X and let H be an isotropy group of $\mathcal{G}$ at x. Consider the canonical projection κ from $\mathcal{G}$ onto the quotient group $\mathcal{G}/H$. If s and t are two elements in $\mathcal{G}$ such that*

$$\kappa(s) = \kappa(t),$$

then sx=tx.

Proof. We have $s^{-1}t = h \in H$. So that $tx = shx = sx$. $\square$

Conversely, if $sx = tx$, then $s^{-1}tx = x$. That is, $s^{-1}t \in H$. So s and t are in the same class, i.e., $sH = tH$. This allows us to define an application $g\colon \mathcal{G}/H \to X$ by $g(\kappa(s)) = sx$.

Proposition 2.4.10. *Let X be a homogeneous space of a topological group $\mathcal{G}$ and let H be an isotropy group of $\mathcal{G}$ at a point $x \in X$. Then the map $g\colon \mathcal{G}/H \to X$ such that $g(\kappa(s)) = sx$ is bijective and continuous.*

Proof. Since X is a homogeneous space of $\mathcal{G}$, it follows that $\mathcal{G}$ operates transitively in X. Thus, g is clearly surjective. Given below is a commutative diagram showing the continuity of g

$$\begin{array}{ccc} \mathcal{G} & & \\ {\scriptstyle\kappa}\big\downarrow & \overset{f}{\searrow} & \\ \mathcal{G}/H & \xrightarrow[g]{} & X \end{array}$$

where f defined by $f(s) = sx$ for all $x \in X$ is continuous. Let O be an open set in X, then according to the diagram above, we obtain

$$\kappa^{-1}(g^{-1}(O)) = f^{-1}(O).$$

Therefore, $f^{-1}(O)$ is open and so is $g^{-1}(O)$ in $\mathcal{G}/H$. Hence, g is continuous. $\square$

2.5 Product Groups

Let us recall that the notion of product topology has been introduced in Sec.1.2. Now let $\{\mathcal{G}_i\}_{i\in I}$ be a family of topological groups and let $\mathcal{G} = \prod_{i\in I}\mathcal{G}_i$ with the projections defined as $\rho_i\colon \mathcal{G} \to \mathcal{G}_i$. $\mathcal{G}$ has a natural group structure obtained by multiplying elements of $\mathcal{G}$, i.e., if $x, y \in \mathcal{G}$, then the product of x and y is $xy = (x_i y_i)_{i\in I}$. In the following we show that the group $\mathcal{G}$ together with the product topology is a topological group. Whatever the group operation is defined in $\mathcal{G}$, the following diagrams are commutative for each $i \in I$.

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\phi} & \mathcal{G} \\ {\scriptstyle\rho_i}\big\downarrow & & \big\downarrow{\scriptstyle\rho_i} \\ \mathcal{G}_i & \xrightarrow[\phi_i]{} & \mathcal{G}_i \end{array} \qquad \begin{array}{ccc} \mathcal{G}\times\mathcal{G} & \xrightarrow{\psi} & \mathcal{G} \\ {\scriptstyle\rho_i\times\rho_i}\big\downarrow & & \big\downarrow{\scriptstyle\rho_i} \\ \mathcal{G}_i\times\mathcal{G}_i & \xrightarrow[\psi_i]{} & \mathcal{G}_i \end{array} \tag{2.5}$$

Since ρ_i, ϕ_i and ψ_i are continuous for each $i \in I$, then so are $\phi_i \circ \rho_i$ and $\psi_i \circ (\rho_i \times \rho_i)$. By commutativity of (2.5), $\rho_i \circ \phi$ and $\rho_i \circ \psi$ are continuous and by Prop.1.2.12, ϕ and ψ are continuous. Consequently, $\mathcal{G}$ is a topological group.

Definition 2.5.1 (Topological Product Group)**.** *The product space $\mathcal{G} = \prod_{i\in I}\mathcal{G}_i$ defined above is a **topological product group** provided that $\mathcal{G}$ has a natural group structure obtained by multiplying its elements and both ψ and ϕ are continuous.*

Next, as a result of the Prop.1.2.13 (i.e., the product of open surjections is again an open surjection), we introduce the following.

Corollary 2.5.1. *Let $\{\mathcal{G}_i\}_{i\in I}$ be a family of topological groups and let N_i be a normal subgroup of $\mathcal{G}_i$ for each i. Let $\mathcal{G} = \prod_{i\in I}\mathcal{G}_i$ and $N = \prod_{i\in I} N_i$. Then,*

$$\frac{\mathcal{G}}{N} \cong \prod_{i\in I}\frac{\mathcal{G}_i}{N_i}$$

in the sense of being topologically isomorphic.

Proof. Let $\kappa_i\colon \mathcal{G}_i \to \mathcal{G}_i/N_i$, $i \in I$, be the canonical projections. According to Prop.2.4.5, it suffices to show that the map

$$\begin{aligned} f\colon \mathcal{G} &\to \prod_{i\in I} \frac{\mathcal{G}_i}{N_i} \\ x &\mapsto (\kappa_i(x)) \end{aligned}$$

is an open continuous epimorphism with ker$f = N$. Indeed, as each κ_i is an open surjection, then by Prop.1.2.13 f is an open surjection. In addtion, f is a group homomorphism since for $x, y \in \mathcal{G}$,

$$f(xy) = (\kappa_i(xy)) = (\kappa_i(x)\kappa_i(y)) = (\kappa_i(x))(\kappa_i(y)) = f(x)f(y).$$

Consider now the diagram

$$\begin{array}{ccc} G & & \\ {\scriptstyle f}\downarrow & \searrow^{\rho_i \circ f} & \\ \prod_{i\in I} \frac{\mathcal{G}_i}{N_i} & \xrightarrow[\rho_i]{} & \frac{\mathcal{G}_i}{N_i} \end{array}$$

and note that $\rho_i \circ f = \kappa_i \circ \rho_i$. Since $\kappa_i \circ \rho_i$ is continuous, by Prop.1.2.13 so is f.
Finally, it is clear that ker$f = N$ since

$$(\kappa_i(x_i)) = N_i \text{ for each } i \iff x_i \in N_i \text{ for all } i \in I \iff x \in N$$

and the claim follows. □

Example 2.5.1. The n-torus $\mathbb{T}^n$, defined as $\mathbb{T}^n = \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{\text{n-times}}$ is a topological group, and by Cor.2.5.1, it is topologically isomorphic to $\mathbb{R}^n/\mathbb{Z}^n$.

2.6 Separation Axioms

In this section, we discuss separation axioms on topological groups and explore equivalences between them.

Proposition 2.6.1. *Every topological group is regular.*

Proof. Let $\mathcal{G}$ be a topological group. Firstly, we show that if C is a closed subset with $e \notin C$, then there are two open subsets O and U separating C and e. As C is closed, its complement is an open neighbourhood of e. By Prop.2.2.2 (P_8), there is a neighbourhood O of e such that $O^{-1}O$ is a subset of $\mathcal{G} \setminus C$. Moreover,

$$\begin{aligned} O^{-1}O \subseteq \mathcal{G} \setminus C &\iff x^{-1}y \notin C \quad \forall x, y \in O \\ &\iff y \notin xC \quad \forall x, y \in O \\ &\iff O \cap OC = \emptyset. \end{aligned}$$

Since O is open, so is OC by Prop.2.1.3, and we have $C \subseteq OC$ with $e \in O$ and $OC \cap O = \emptyset$. Therefore O and OC separate e and C.

Finally, if C is closed with $x \notin C$, then $x^{-1}C$ is a closed subset such that $e \notin x^{-1}C$, so there are open susets O and U separating $x^{-1}C$ and e. By Prop.2.1.3, xO and xU are two open subsets separating C and x. Hence, $\mathcal{G}$ is regular (i.e., a T_3 space). □

Proposition 2.6.2. *Every topological group is completely regular.*

Proposition 2.6.3. *For any topological groups, the separation axioms T_0, T_1, T_2 and T_3 are equivalent.*

Proof. Let $\mathcal{G}$ be any topological group. By Prop.2.6.1 and since a T_3 space is equivalent to a T_1 space, it suffices to show that a T_0 topological group is T_1. Suppose that $\mathcal{G}$ is a T_0 and let $x \neq y \in \mathcal{G}$. Without loss of generality, assume that O is an open subset in which $x \in O$ but $y \notin O$. We have $\mathcal{G} \setminus O$ is a closed subset with $x \notin \mathcal{G} \setminus O$ and by regularity there are two open subsets U_1 and U_2 separating $\mathcal{G} \setminus O$ and x. Thus we can say that $y \in \mathcal{G} \setminus O \subseteq U_1$ and $x \in U_2$ with $U_1 \cap U_2 = \emptyset$. Hence $\mathcal{G}$ is Hausdorff (or T_2) and consequently T_1. □

Proposition 2.6.4. *Let $\mathcal{G}$ be a topological group and $\mathcal{B}$ be a family of neighborhoods of e. Then $\mathcal{G}$ is a Hausdorff space if and only if $\cap_{\beta \in \mathcal{B}} \beta = \{e\}$.*

Proof. Suppose that $\mathcal{G}$ is a Hausdorff space. Then for any point $x \neq e$ in $\mathcal{G}$, there are disjoint neighborhoods of x and e. Hence, there is $\beta \in \mathcal{B}$ that does not contain x. Thus, $\cap_{\beta \in \mathcal{B}} \beta$ does not contain x also. Therefore, $\cap_{\beta \in \mathcal{B}} \beta = \{e\}$

Conversely, assume that $\cap_{\beta \in \mathcal{B}} \beta = \{e\}$ and let x and y be two distinct points in $\mathcal{G}$. Then $x^{-1}y$ is different from e, so there is $\beta \in \mathcal{B}$ such that $x^{-1}y \notin \beta$. Thus, using Prop.2.2.2 (P_9), there is $\beta_0 \in \mathcal{B}$ such that $\beta_0 \beta_0^{-1} \subset \beta$. Hence $x\beta_0 \cap y\beta_0 = \emptyset$. If not, pick a point z in common, then we get $x^{-1} \in \beta_0 z^{-1}$ and $y \in z\beta_0^{-1}$ such that

$$x^{-1}y \in \beta_0 \beta_0^{-1} \subset \beta,$$

which is a contradiction. Therefore, $\mathcal{G}$ is a Hausdorff space. □

The following proposition gives some equivalent characterizations of the Hausdorff property for topological groups. The statements (i) and (ii) are equivalent for any topological space while the statements (i), (iii), (iv) and (v) are equivalent for any topological group.

Proposition 2.6.5. *Let $\mathcal{G}_1$ be a topological group and $\mathcal{B}$ a neighbourhood base of e. Then the following statements are equivalent*

(i) *$\mathcal{G}_1$ is Hausdorff,*

(ii) *the diagonal map $d : \mathcal{G}_1 \to \mathcal{G}_1 \times \mathcal{G}_1$ given by $x \mapsto (x, x)$ is a closed map,*

(iii) *if $\mathcal{G}_2$ is a topological group and $f : \mathcal{G}_2 \to \mathcal{G}_1$ a topological homomorphism, then* $\ker f$ *is a closed subgroup of $\mathcal{G}_2$,*

(iv) *$\{e\}$ is a closed subset of $\mathcal{G}_1$,*

(v) $\bigcap_{\beta \in \mathcal{B}} \beta = \{e\}$.

Proof.
(i)⇒(ii) Suppose that $\mathcal{G}_1$ is a Hausdorff space and consider C a closed subset of $\mathcal{G}_1$. If $(x, y) \notin d(C)$, then either x is different from y or equal. When $x \neq y$, there exist disjoint open subsets O and U such that $x \in O$ and $y \in U$. Thus, $O \times U$ is an open neighborhood of (x, y) and we have

$$(O \times U) \cap d(C) = \{(z, z) : z \in C \text{ and } z \in O \cap U\} = \emptyset.$$

Otherwise, if $x = y$, since $(x, x) \notin d(C)$, we get $x \notin C$. C being closed, then there is an open subset O of $\mathcal{G}_1$ such that $x \in O$ with both O and C are disjoint. Further, $O \times O$ is an open neighborhood of (x, x) and with $(O \times O) \cap d(C) = \emptyset$, we get $d(C)$ closed in $\mathcal{G}_1 \times \mathcal{G}_1$.
(ii)⇒(iii) Assume that the diagonal map $d : \mathcal{G}_1 \to \mathcal{G}_1 \times \mathcal{G}_1$ given by $x \mapsto (x, x)$ is a closed map. Let $d_f : \mathcal{G}_2 \to \mathcal{G}_1 \times \mathcal{G}_1$ be defined by $x \mapsto (f(x), e)$ where f is continuous, so is d_f. Since $d(\mathcal{G}_1)$ is closed in $\mathcal{G}_1 \times \mathcal{G}_1$, then by the continuity of d_f

$$d_f^{-1}(d(\mathcal{G}_1)) = \{x \in \mathcal{G}_2 : f(x) = e\} = \ker f$$

is closed.
(iii)⇒(iv) Take in particular $\mathcal{G}_2 = \mathcal{G}_1$ and f the identity map $id\colon \mathcal{G}_1 \to \mathcal{G}_1$. It is clear that id is a topological homomorphism. Hence by (iii), $\ker id = \{e\}$ is closed.
(iv)⇒(v) Assume that $\{e\}$ is a closed subset of $\mathcal{G}_1$ and let $x \in \mathcal{G}_1 \setminus \{e\}$. By homogeneity of $\mathcal{G}_1$, $\{x\}$ is a closed and then there is $\beta \in \mathcal{B}$ in which $x \notin \beta$. Thus $x \notin \bigcap_{\beta\in\mathcal{B}} \beta$ and consequently $\bigcap_{\beta\in\mathcal{B}} \beta = \{e\}$.
(v)⇒(i) See the proposition above. □

Proposition 2.6.6. *Let $\{\mathcal{G}_i\}_{i\in I}$ be a family of topological groups. Then $\prod_{i\in I} \mathcal{G}_i$ is Hausdorff if and only if $\mathcal{G}_i$ is Hausdorff for each $i \in I$.*

Proof. The result follows immediately from Prop.1.2.8. □

Example 2.6.1.

1. As $\mathbb{R}$ is Hausdorff, so are $\mathbb{R}^n$ and all its subgroups for each $n \in \mathbb{N}$. Also, $\mathbb{C}^n$ is Hausdorff since it is homeomorphic to $\mathbb{R}^{2n}$.

2. Both $GL_n(\mathbb{R})$ and $SL_n(\mathbb{R})$ are Hausdorff as subsets of the Hausdorff space $\mathbb{R}^{n^2}$.

One can combine the preceding equivalences in one result as below.

Theorem 2.6.1. *Let $\mathcal{G}$ be a topological group. Then the following statements are equivalent.*

(i) $\mathcal{G}$ *is* T_0,

(ii) $\mathcal{G}$ *is* T_1,

(iii) $\mathcal{G}$ *is a Hausdorff space,*

(iv) $\mathcal{G}$ *is a regular space,*

(v) $\mathcal{G}$ *is a completely regular space,*

(vi) $\bigcap_{\beta\in\mathcal{B}} \beta = \{e\}$ *where* $\mathcal{B}$ *is a neighborhood base of* e.

Remark 2.6.1.

- Any topological group satisfies the theorem above, we may call it a **topological Hausdorff group** or **Hausdorff group**.

- One can wonder whether a Hausdorff group is normal (i.e., every toplogical group is a T_5 space), however, this is not true in general. Indeed, for any completely regular space, we enable to construct a Hausdorff group which defines a closed subgroup. But any closed subspace in a normal space is normal (see Prop.1.3.3(iii)). Thus, any Hausdorff group is normal and then it is completely regular which is impossible in general (see Remark 1.3.1).

Proposition 2.6.7. *Let $\mathcal{G}$ be a topological group. Then the following subgroups are closed.*

(i) The isotropy group of $\mathcal{G}$ at a point $x \in X$ whenever X is a Hausdorff space.

(ii) The centre $Z(\mathcal{G})$ and any discrete subgroup whenever $\mathcal{G}$ is a Hausdorff group.

Proof.
(i) Define a map $f\colon \mathcal{G} \to X$ by $s \mapsto sx$, then f is clearly continuous. Therefore, $H = f^{-1}(\{x\})$ is closed since $\{x\}$ is closed in the Hausdorff space X.
(ii) Let $a \in \overline{Z(\mathcal{G})}$. We show that $a \in Z(\mathcal{G})$. Otherwise, if $a \notin Z(\mathcal{G})$ then there is $x \in \mathcal{G}$ such that $a \neq x^{-1}ax$. Put $b = x^{-1}ax$ and since $\mathcal{G}$ is Hausdorff and so is regular, then there exist neighborhoods U and V of a and b, respectively, such that their closures are disjoint. Set $O = Z(\mathcal{G}) \cap U$ and as $a \in \overline{Z(\mathcal{G})}$, so it is clear that $a \in \overline{O}$. Therefore,

$$b = x^{-1}ax \in x^{-1}\overline{O}x = \overline{x^{-1}Ox} = \overline{O} \subset \overline{U}$$

as long as $Z(\mathcal{G})$ is a normal subgroup, a contradiction since $b \notin \overline{U}$. Hence, $a \in Z(\mathcal{G})$ and so $Z(\mathcal{G})$ is closed.

On the other hand, we show that a discrete subgroup H is closed. Indeed, as $\mathcal{G}$ is regular by Prop.2.6.1, there is a closed neighborhood U of e such that $U \cap H = \{e\}$. $\mathcal{G}$ being Hausdorff, then $\{e\}$ is closed. Hence, $U \cap H$ is closed and consequently by Prop.2.3.4, H is closed as well. □

In what follows, we discuss some results due to separation axioms on quotient groups.

Proposition 2.6.8. *Let $\mathcal{G}/H$ be the quotient group of a topological group $\mathcal{G}$ by a subgroup H. Then*

(i) if $\mathcal{G}$ is Hausdorff so is H,

(ii) $\mathcal{G}/H$ is Hausdorff if and only if H is closed,

(iii) if H and $\mathcal{G}/H$ are Hausdorff, then so is $\mathcal{G}$.

Proof.
(i) Let $f : H \to \mathcal{G}$ be the inclusion map which is injective. Hence the result follows Prop.1.2.10.
(ii) We know that the identity element of $\mathcal{G}/H$ is H. Thus by Prop.2.6.5,

$$\begin{aligned} \mathcal{G}/H \text{ Hausdorff } &\iff H \text{ closed in } \mathcal{G}/H \\ &\iff \kappa^{-1}(H) \text{ closed in } \mathcal{G}. \end{aligned}$$

Further, $\kappa^{-1}(H) = \{x \in \mathcal{G} : xH = H\} = H$. Thus H is closed.
(iii) If H is Hausdorff, then $\{e\}$ is closed in H. So there is a closed subset C of $\mathcal{G}$ such that $H \cap C = \{e\}$. Also if $\mathcal{G}/H$ is Hausdorff, then by (ii) H is closed in $\mathcal{G}$ and thus $H \cap C = \{e\}$ is closed in $\mathcal{G}$. Hence, $\mathcal{G}$ is Hausdorff. □

In consequence, if H is closed and Hausdorff, then by (ii) $\mathcal{G}/H$ is Hausdorff and thus by (iii) $\mathcal{G}$ is Hausdorff.

Example 2.6.2.

1. The unit circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is Hausdorff since $\mathbb{Z}$ is closed in $\mathbb{R}$. In other words, as $\mathbb{S}^1$ is a subgroup of the Hausdorff group $\mathbb{C}^*$, then $\mathbb{S}^1$ is Hausdorff.

2. As $\mathbb{Q}$ is not closed in $\mathbb{R}$, the quotient $\mathbb{R}/\mathbb{Q}$ is not a Hausdorff topological group.

Proposition 2.6.9. *Let $\mathcal{G}$ be a topological group and H be a subgroup of $\mathcal{G}$, then $\mathcal{G}/H$ is a T_3 space (i.e., regular space).*

Proof. Let U be a basis element containing H in $\mathcal{G}/H$. That is,

$$U = \{xH : x \in \beta_1\} = \kappa(\beta_1 H)$$

where β_1 is a basis element containing e in $\mathcal{G}$. Thus, by Prop.2.2.2 (P_8), there is β_2 containing e such that $\beta_2^{-1}\beta_2 \subseteq \beta_1$. We want to show that there is a neighborhood V of H such that $\overline{V} \subseteq U$. First we prove that $\overline{\beta_2 H} \subseteq \beta_1 H$. Indeed, let $x \in \overline{\beta_2 H}$, then $\beta_2 x \cap \beta_2 H \neq \emptyset$. It follows that there is $h \in H$ and $a, b \in \beta_2$ such that $ax = bh$. So, $x = a^{-1}bh \in \beta_2^{-1}\beta_2 H \subseteq \beta_1 H$. Set

$$V = \{yH : y \in \beta_2\} = \kappa(\beta_2 H).$$

Now, if $yH \in V$, then $yH \in \beta_2 H$ and so it is in $\overline{\beta_2 H}$. Let $yH \in \overline{V}$. We have two cases $y \in \beta_2$ or $y \notin \beta_2$. If $y \in \beta_2$, then $yH \in V$ and so

$$yH \in \beta_2 H \subseteq \overline{\beta_2 H} \subseteq \beta_1 H.$$

We also have

$$\{yH\} = \kappa(yH) \subseteq \kappa(\beta_1 H) = U.$$

This implies that $yH \in U$. Hence, $\overline{V} \subseteq U$. If $y \notin \beta_2$, then there is β_3 a basis element containing e such that $y \in \beta_3$. So $\{yH\}$ is an open subset in $\mathcal{G}/H$ containing yH. But

$$\{yH\} \cap V = \emptyset$$

since $y \notin \beta_2$. Hence, $yH \notin \overline{V}$, a contradition. Therefore, $\overline{V} \subseteq U$. According to Prop.1.3.1 (i), $\mathcal{G}/H$ is a T_3 space. □

Proposition 2.6.10. *A topological quotient group is a T_4 space (i.e., completely regular space).*

Proof. See the Theorem.2.6.1 with the preceding proposition. □

2.7 Metrization of Groups

In this section, our aim is to present a proof of the **Birkhoff-Kakutani theorem** which states that a topological group is metrizable if and only if it is a first countable T_0 space. Again, we are going to interpret why the Sorgenfrey line S cannot be a topological group while it is a homogeneous space. More precisely, we show that S does not admit any group stucture making it a topological group.

Remark 2.7.1. Note that by Prop.2.2.1, a topological group is first countable if and only if the identity element has a countable neighborhood base.

Lemma 2.7.1. *If $\mathcal{G}$ is a first countable topological group, then there is a neighborhood base $\{\beta_n\}_{n\in\mathbb{N}}$ of e such that each β_n is symmetric and $\beta_{n+1}\beta_{n+1}\beta_{n+1} \subseteq \beta_n$ for all $n \in \mathbb{N}$.*

Proof. Let $\{O_n\}_{n\in\mathbb{N}}$ be a neighborhood base of e. Set $U_n = O_n \cap O_n^{-1}$ and we can get a neighborhood base $\{U_n\}_{n\in\mathbb{N}}$ of e consisting of all symmetric neighborhoods.

Now take $i_1 = 1$. As $\mathcal{G}$ is a topological space, according to the Prop.2.2.2, there is $j > i_1$ for which $U_j U_j \subseteq U_{i_1}$. So let $i_2 > i_1$ for which $U_{i_2}U_{i_2} \subseteq U_j$, we have

$$U_{i_2}U_{i_2}U_{i_2} \subseteq U_{i_2}U_j \subseteq U_j U_j \subseteq U_{i_1}.$$

Again we can find an $i_3 > i_2$ such that $U_{i_3}U_{i_3}U_{i_3} \subseteq U_{i_2}$. Continuing in this manner, we construct a strictly increasing sequence $(i_n)_{n\in\mathbb{N}}$ for which $U_{i_{n+1}}U_{i_{n+1}}U_{i_{n+1}} \subseteq U_{i_n}$ for all $n \in \mathbb{N}$.

Set $\beta_n = U_{i_n}$ for all $n \in \mathbb{N}$, we obtain a neighborhood base $\{\beta_n\}_{n\in\mathbb{N}}$ of e (as i_n is strictly increasing) consisting of all symmetric neighborhoods such that $\beta_{n+1}\beta_{n+1}\beta_{n+1} \subseteq \beta_n$ for all $n \in \mathbb{N}$. □

Lemma 2.7.2. *A topological group is pseudometrizable if and only if it is first countable.*

Proof. Let $\mathcal{G}$ be a topological group together with a topology τ. Suppose that $\mathcal{G}$ is pseudometrizable. This implication is immediate. Indeed, if $\mathcal{G}$ is a pseudometrizable topological group, then there is a pseudometric d on $\mathcal{G}$ generating τ. For each $x \in \mathcal{G}$, $\{\beta(x, 1/n)\}_{n\in\mathbb{N}}$ is a countable neighborhood base of x and so $\mathcal{G}$ is first countable.

Conversely, suppose that $\mathcal{G}$ is first countable. By Lemma 2.7.1, there is a neighborhood base $\{\beta_n\}_{n\in\mathbb{N}}$ of e consisting of all symmetric neighborhoods such that $\beta_{n+1}\beta_{n+1}\beta_{n+1} \subseteq \beta_n$ for all $n \in \mathbb{N}$.

Step 1: (Defining a symmetric function f). Put $\beta_0 = \mathcal{G}$ and define $f\colon \mathcal{G} \times \mathcal{G} \to [0, \infty)$ by

$$f(x, y) = \begin{cases} 0, & \text{if } x^{-1}y \in \bigcap_{n\in\mathbb{N}} \beta_n \\ 2^{-n}, & \text{if } x^{-1}y \in \beta_n \setminus \beta_{n+1}. \end{cases}$$

In other words, $f(x, y) = 2^{-n}$ if n is the greatest non-negative integer in which $x^{-1}y \in \beta_n$ and $f(x, y) = 0$ if n does not exist. Further, $f(x, x) = 0$ and as each β_n is symmetric, $x^{-1}y \in \beta_n$ if and only if $(x^{-1}y)^{-1} = y^{-1}x \in \beta_n$. Hence, f is symmetric as $f(x, y) = f(y, x)$.

Step 2: (Defining a left-invariant pseudometric d generating τ). Set

$$\mathcal{F}_{x,y} = \{f(x_1, x_2) + \cdots + f(x_k, x_{k+1}) : k \in \mathbb{N}, x_1 = x, x_{k+1} = y\} \tag{2.6}$$

and define the function

$$\begin{aligned} d\colon \mathcal{G} \times \mathcal{G} &\to [0, \infty) \\ (x, y) &\mapsto \inf \mathcal{F}_{x,y}. \end{aligned}$$

d is a pseudometric generating τ. Indeed, for any $x, y, z \in \mathcal{G}$, the first property ($d(x, y) \geq 0$ and $d(x, x) = 0$) is clear according to the definition of d. Since f is symmetric, so is d. For the last condition, i.e., the triangle inquality, we can see that from (2.6), $\mathcal{F}_{x,y}+\mathcal{F}_{y,z} \subseteq \mathcal{F}_{x,z}$. Therefore, we get

$$d(x, z) = \inf \mathcal{F}_{x,z} \leq \inf(\mathcal{F}_{x,y} + \mathcal{F}_{y,z}) = \inf \mathcal{F}_{x,y} + \inf \mathcal{F}_{y,z} = d(x, y) + d(y, z).$$

Hence d is a pseudometric. Clearly, d is left-invariant (i.e., $d(x, y) = d(ax, ay)$). Because for any $a \in \mathcal{G}$, $x^{-1}y = (ax)^{-1}(ay)$, so that f is left-invariant and then so is d.

Step 3: (Showing that $\mathcal{G}$ is a pseudometrizable). We prove now that $\mathcal{G}$ is pseudometrizable, i.e., there is a topology τ_d generated by d such that $\tau = \tau_d$. Since d is left-invariant,

$$B(x, r) = \{y \in \mathcal{G} : d(x, y) < r\} = x\{x^{-1}y \in \mathcal{G} : d(e, x^{-1}y) < r\} = xB(e, r).$$

So we just have to check the neighborhoods at the identity e. We show that τ is finer than τ_d (i.e., $\tau \subseteq \tau_d$). Indeed, fix $r > 0$ and let n be a non-negative integer in which $2^{-n} < r$. Let $x \in \beta_{n+1}$, then $f(e, x) \leq 2^{-n-1}$ and by the definition of d, $d(e, x) \leq f(e, x) \leq 2^{-n-1} \leq 2^{-n}$. Thus $x \in B(e, 2^{-n})$ and so $\beta_{n+1} \subseteq B(e, 2^{-n}) \subseteq B(e, r)$. On the other hand, we show that τ_d is finer than τ. Indeed, for each $n \in \mathbb{N}$ there is $r > 0$ in which $B(e, r) \subseteq \beta_n$. Let $x \in B(e, 2^{-n})$ and as $d(e, x) < 2^{-n}$, we can find $k \in \mathbb{N}$ and $x_1, \cdots, x_{k+1} \in \mathcal{G}$ with $x_1 = e$ and $x_{k+1} = x$ for which

$$d(e, x) \leq f(x_1, x_2) + \cdots + f(x_k, x_{k+1}) < 2^{-n}.$$

Note that as $x_1^{-1}x_{k+1} = x$, it remains to show that

$$x_1^{-1}x_{k+1} \in \beta_n \tag{2.7}$$

by using induction on k.

- When $k = 1$, $f(x_1, x_2) < 2^{-n}$, so either $f(x_1, x_2) = 0$ or $f(x_1, x_2) = 2^{-j}$ for some $j \geq n$. However, $x_1^{-1}x_2 \in \beta_j \subseteq \beta_n$. So it holds for $k = 1$.
- For $k \geq 2$, assume that if

  $$f(y_1, y_2) + \cdots + f(y_p, y_{p+1}) < 2^{-n},$$

  then $y_1^{-1}y_{p+1} \in \beta_n$ for any $y_1, \cdots, y_{p+1} \in \mathcal{G}$ with $p < k$.
  Suppose that

  $$f(x_1, x_2) + \cdots + f(x_k, x_{k+1}) < 2^{-n}. \tag{2.8}$$

  Clearly, for any i, $f(x_i, x_{i+1}) < 2^{-n}$. So $f(x_i, x_{i+1}) \leq 2^{-n-1}$ and thus, $x_i^{-1}x_{i+1} \in \beta_{n+1}$. If $f(x_1, x_2) \geq 2^{-n-1}$, then $f(x_1, x_2) = 2^{-n-1}$ and $x_1^{-1}x_2 \in \beta_n$. Therefore, to hold (2.8), we have

  $$f(x_2, x_3) + \cdots + f(x_k, x_{k+1}) < 2^{-n-1}.$$

  It follows that by inductive hypothesis, $x_2^{-1}x_{k+1} \in \beta_{n+1}$. Thus,

  $$x_1^{-1}x_{k+1} = (x_1^{-1}x_2)(x_2^{-1}x_{k+1}) \in \beta_{n+1}\beta_{n+1} \subseteq \beta_n.$$

  Finally, suppose that $f(x_1, x_2) < 2^{-n-1}$ and let $1 \leq i \leq k$ be the greatest integer for which $f(x_1, x_2) + \cdots + f(x_i, x_{i+1}) < 2^{-n-1}$. We consider two cases.

  - When $i = k$ or $i = k - 1$, then

    $$f(x_1, x_2) + \cdots + f(x_{k-1}, x_k) < 2^{-n-1}$$

    and by inductive assumption, $x_1^{-1}x_k \in \beta_{n+1}$. Further, $x_k^{-1}x_{k+1} \in \beta_{n+1}$, so that

    $$x_1^{-1}x_{k+1} = (x_1^{-1}x_k)(x_k^{-1}x_{k+1}) \in \beta_{n+1}\beta_{n+1} \subseteq \beta_n.$$

  - When $i < k - 1$, by the choice of i, we have

    $$f(x_1, x_2) + \cdots + f(x_{i+1}, x_{i+2}) \geq 2^{-n-1}$$

    and from (2.8), we get

    $$f(x_{i+2}, x_{i+3}) + \cdots + f(x_k, x_{k+1}) < 2^{-n-1}.$$

    Thus,

    $$x_1^{-1}x_{i+1}, x_{i+1}^{-1}x_{i+2}, x_{i+2}^{-1}x_{k+1} \in \beta_{n+1}.$$

    Hence,

    $$x_1^{-1}x_{k+1} = (x_1^{-1}x_{i+1})(x_{i+1}^{-1}x_{i+2})(x_{i+2}^{-1}x_{k+1}) \in \beta_{n+1}\beta_{n+1}\beta_{n+1} \subseteq \beta_n.$$

Therefore the theorem is complete. □

Theorem 2.7.1 (Birkhoff-Kakutani Theorem)**.** *A topological group is metrizable if and only if it is a first countable T_0 space. In this case, any topological group admits a left-invariant metric generating its topology.*

Proof. Let $\mathcal{G}$ be a topological group. Suppose that $\mathcal{G}$ is metrizable. Since every metrizable topological space is a T_0 space and first countable, the first implication holds immediately.

Conversely, assume that $\mathcal{G}$ is a T_0 space and first countable. By Lemma 2.7.2, $\mathcal{G}$ is pseudometric and if it is also T_0, it must be metrizable by Prop.1.4.2. Finally, whenever $\mathcal{G}$ is metrizable, it admits a left-invariant metric as the pseudometric d ($\mathcal{G}$ is T_0, so d is a metric) is left-invariant and generates the topology of $\mathcal{G}$ (see the proof of Lemma 2.7.2). □

Next, as we said earlier, we are going to see that the Sorgenfrey line S does not form a topological group whereas it is a homogeneous space.

We may first recall the structure and properties of S. As we know the Sorgenfrey line is the real line $\mathbb{R}$ equipped with the lower limit topology τ_l. In Ex.2.4.1, we have shown that S together with addition is not a topological group since the inversion map is discontinuous. Our goal now is to prove that for any operation defines on S, S is not a topological group. In the following remark we argue that it is separable and first countable, but not second countable.

Remark 2.7.2. The subset $\mathbb{Q}$ of S is countable and dense since for any interval $[a, b)$, by density theorem, we can find a rational lying on it. Thus, S is separable. Further, it is first countable as $\{[x, x + 1/n) : n \in \mathbb{N}\}$ is a neighborhood base for each $x \in S$. On the other hand, the second countability axiom fails for S. Indeed, suppose by contradiction that $\mathcal{B}$ is a countable base for the topology on S. For any $x \in S$, the interval $[x, x+1)$ is open and contains x. Choose $\beta_x \in \mathcal{B}$ in which $x \in \beta_x \subseteq [x, x + 1)$. Now for $x \neq y$ and without loss generality we consider $x < y$. The subsets β_x and β_y are distinct as $x \notin [y, y + 1) \supseteq \beta_y$. Therefore, the map

$$\begin{aligned} S &\to \mathcal{B} \\ x &\mapsto \beta_x \end{aligned}$$

is injective, a contradiction since $card(S) > card(\mathcal{B})$. Hence, S is separable and first countable, but not second countable.

Theorem 2.7.2. *The Sorgenfrey line S does not admit any group structure making it a topological group.*

Proof. On the contrary, assume that S is a topological group. As S is T_0 and first countable (see the preceding remark), by Theorem 2.7.1, S is metrizable. On the other hand, as S is separable, by Theorem 1.4.1, S is second countable, a contradiction. Hence S cannot be a topological group. □

Chapter 3

Connectedness and Compactness of Topological Groups

In this chapter, we discuss many ideas concerning the connectedness and the compactness on groups. The first section is devoted to connected groups while the second concerns compact and locally compact groups.

3.1 Connected Groups

Proposition 3.1.1. *A connected topological group has neither proper open subgroups nor proper closed subgroups of finite index.*

Proof. According to Prop.2.3.2, note that every open subgroup is closed and every closed subgroup of finite index is open. □

Proposition 3.1.2. *Let $\mathcal{C}$ be a connected group and choose O a non-empty open subset of $\mathcal{C}$. Then $\mathcal{C}$ is the group generated by O.*

Proof. Let $\mathcal{C}$ be a connected group and O a non-empty open subset of $\mathcal{C}$. As $\langle O\rangle$ is a subgroup of $\mathcal{C}$ containing a non-empty open subset, by Prop.2.3.2, it is an open subgroup. By Prop.3.1.1, $\langle O\rangle$ is not a proper subgroup of $\mathcal{C}$. Thus, $\langle O\rangle = \mathcal{C}$. □

In fact, any connected topological group can be also generated by a neighborhood of its identity.

Proposition 3.1.3. *For any neighborhood U of e in a connected group $\mathcal{C}$, we have $\mathcal{C} = \bigcup_{n\geq 1} U^n$.*

Proof. In view of Prop.2.2.7, we may assume that U is a symmetric neighborhood of e. By Lemma 2.3.1, $\bigcup_{n\geq 1} U^n$ is clopen. $\mathcal{C}$ being connected, then the only clopen sets are empty set and the entire set. Hence, $\mathcal{C} = \bigcup_{n\geq 1} U^n$. □

Remark 3.1.1. The converse of above the proposition does not hold. Indeed, consider the additive group of rationals $\mathbb{Q}$ endowed with the subspace topology from the usual topology in $\mathbb{R}$. $\mathbb{Q}$ is not connected, but every symmetric neighborhood U of 0 in $\mathbb{Q}$ being the trace of a symmetric neighborhood of 0 in $\mathbb{R}$ that generates $\mathbb{R}$, generates $\mathbb{Q}$.

Example 3.1.1. The subgroup of all positive reals $\mathbb{R}^+$ of the multiplicative group $\mathbb{R}^*$ is connected. Thus, any open interval (a,b) of $\mathbb{R}^+$ generates $\mathbb{R}^+$, i.e., given any $x\in\mathbb{R}^+$, we can write x as a product of finitely many elements of (a,b) and its inverses.

The following result is an application of Prop.1.5.5 in topological groups.

Proposition 3.1.4. *Let $\{\mathcal{C}_i\}_{i\in I}$ be a family of topological groups and let N be a normal subgroup of a topological group $\mathcal{C}$. Then*

(i) if $\mathcal{C}$ is connected, then so is $\mathcal{C}/N$,

(ii) the product space of $\{\mathcal{C}_i\}_{i\in I}$ is connected if and only if $\mathcal{C}_i$ is connected for all $i \in I$,

(iii) if N and $\mathcal{C}/N$ are connected then so is $\mathcal{C}$.

Proof.
(i) holds since the canonical projection is a continuous surject and (ii) follows immediately form Prop.1.5.5.
(iii) Suppose on the contrary that $\mathcal{C}$ is disconnected, i.e., there is a separation (O_1, O_2) of $\mathcal{C}$. Assume that N and $\mathcal{C}/N$ are connected. Then without loss of generality, let $e \in O_1$. If there is $x \in \mathcal{C}$ the coset xN is not contained in O_1 nor in O_2, then $xN \cap O_1 \neq \emptyset$ and $xN \cap O_2 \neq \emptyset$ and they are disjoint. Thus $(xN \cap O_1, xN \cap O_2)$ forms a separation of xN which must be connected as it is homeomorphic to N. Therefore for all $x \in \mathcal{C}$, xN is contained either in O_1 or in O_2, so we can write

$$O_1 = \bigcup\{xN : x \in O_1\} \text{ and } O_2 = \bigcup\{xN : x \in O_2\}.$$

Since κ is an open map (see Prop.2.4.1), both $\kappa(O_1)$ and $\kappa(O_2)$ are open in $\mathcal{C}/N$. Moreover, $\kappa(O_1) = \{xN : x \in O_1\}$ and $\kappa(O_2) = \{xN : x \in O_2\}$, so that they are disjoint and $\kappa(O_1) \cup \kappa(O_2) = \kappa(O_1 \cup O_2) = \mathcal{C}/N$. It follows that $(\kappa(O_1), \kappa(O_2))$ is a separation of $\mathcal{C}/N$, a contradiction. Thus $\mathcal{C}$ is connected. □

Theorem 3.1.1. *If $\mathcal{G}$ is a topological group and N the component of e, then N is a closed and connected normal subgroup of $\mathcal{G}$ and for any $x \in \mathcal{G}$, xN is the component of x.*

Proof. Let $\mathcal{G}$ be a topological group and N the component of e. By Prop.1.5.7, it follows immediately that N is closed and connected. We show that N is a normal subgroup of $\mathcal{G}$. Let $n \in N$ and $x \in \mathcal{G}$, then both $n^{-1}N$ and $x^{-1}Nx$ are homeomorphic to N, so they are connected. Since $e \in n^{-1}N$, $n^{-1}N \subseteq N$ and hence N is a subgroup of $\mathcal{G}$. Likewise, $e \in x^{-1}Nx$, so $x^{-1}Nx \subseteq N$. Thus, N is a normal subgroup of $\mathcal{G}$.

Finally, as the left translation $L_x \colon \mathcal{G} \to \mathcal{G}$ is a homeomorphism, xN is a connected component of $\mathcal{G}$ of x for any $x \in \mathcal{G}$. □

Example 3.1.2.

1. Any interval of reals is connected.

2. As $\mathbb{R} = \bigcup_{n\in\mathbb{N}}(-n, n)$, by Prop.1.5.3 and 1., $\mathbb{R}$ is a connected group.

3. The additive group $\mathbb{R}$ is connected and its subgroup $\mathbb{Z}$ is normal, so $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ is connected. Hence, the n-torus $\mathbb{T}^n$ is also connected for any $n \in \mathbb{N}$.

4. The multiplicative group $\mathbb{R}^*$ is not connected since $O_1 = (-\infty, 0)$ and $O_2 = (0, \infty)$ form a separation of $\mathbb{R}^*$. In addition O_1 and O_2 are clearly connected, so they are the connected component of $\mathbb{R}^*$.

5. The general linear group $GL_n(\mathbb{C})$ of $n \times n$ non-singular matrices with enteries in $\mathbb{C}$ is connected, but $GL_n(\mathbb{R})$ is not. Indeed, $GL_n(\mathbb{R})$ is homeomorphic to $\mathbb{R}^{n^2} \setminus \ker(det)$ where $det \colon \mathbb{R}^{n^2} \to \mathbb{R}$. Clearly, the inverse image of two open subsets of $\mathbb{R}$, which was given in 4., under det are open subsets of $\mathbb{R}^{n^2}$. We also have $O_1 \cup O_2 = \mathbb{R}^{n^2} \setminus \ker(det)$ and they are disjoint, thus $GL_n(\mathbb{R})$ is disconnected. However, if $n = 1$, $\mathbb{R} \setminus \ker(det) = \mathbb{R}^*$, so the result follows from 4.

Proposition 3.1.5. *Let $\mathcal{G}$ be a topological group. Then*

(i) If $\mathcal{G}$ is connected, then the connected component of e is the entire group $\mathcal{G}$.

(ii) If the connected component of e is $\{e\}$, then $\mathcal{G}$ is totally disconnected.

Proof.
(i) Since the connected component of e is the largest connected subspace containing e, so it is obviously the whole $\mathcal{G}$.
(ii) By definition of a totally disconnected space and Theorem 3.1.1. □

Proposition 3.1.6. *Let N be normal discrete subgroup of a connected group $\mathcal{C}$, then N is a subgroup of the center of $\mathcal{C}$.*

Proof. By Prop.2.6.7, N is closed. Let $x \in N$, then there is a neighborhood U of x such that $U \cap N = \{x\}$. Since we can write $e^{-1}xe = x$, then there exists a neighborhood V of e such that $V^{-1}xV \subset U$. Let $v \in V$, then $v^{-1}xv \in U$ and $v^{-1}xv \in N$ because $x \in N$. Thus, $v^{-1}xv \in U \cap N$, i.e., $v^{-1}xv = x$. Hence, $x \in Z_{\mathcal{C}}(V)$. Let $y \in \mathcal{C}$, then by Theorem 3.1.3, $\mathcal{C} = \bigcup_{n\geq 1} V^n$ so that $y = v_1 \cdots v_k$, $v_i \in V$. Thus,

$$y^{-1}xy = v_k^{-1} \cdots v_1^{-1} x v_1 \cdots v_k = x$$

since $x \in Z_{\mathcal{C}}(V)$. Therefore, $x \in Z(\mathcal{C})$. □

Remark 3.1.2. We can see easily that N is closed in the centre of $\mathcal{C}$ whenever $\mathcal{C}$ is a Hausdorff group. Indeed, by Prop.2.6.7, N is closed in $\mathcal{C}$ and hence closed in $Z(\mathcal{C})$ since $N = N \cap Z(\mathcal{C})$.

Proposition 3.1.7. *Let $\mathcal{G}$ be a Hausdorff group such that for all neighborhoods U of e, there is an open subgroup H of $\mathcal{G}$ contained in U. Then $\mathcal{G}$ is totally disconnected.*

Proof. Since H is a non-empty ($e \in H$) clopen and $\mathcal{G} = H \cup (\mathcal{G} \setminus H)$, then the component of e is in H since $e \in H$. This compopent is in the intersection of all neighborhoods of e which is $\{e\}$ since $\mathcal{G}$ is Hausdorff according to the Theorem 2.6.1. So the compopent of e is just $\{e\}$. Therefore, $\mathcal{G}$ is totally disconnected. □

3.2 Compact and Locally Compact Groups

Proposition 3.2.1. *If K and C are compact subsets of a topological group $\mathcal{G}$ and $x \in \mathcal{G}$, then the sets xK, Kx, KC and K^{-1} are compact.*

Proof. Since the multiplication ψ and inversion ϕ are continuous, it follows that KC and K^{-1}. Further, as R_x and L_x are homeomorphism, we obtain xK and Kx are compact. □

Proposition 3.2.2. *Let $\mathcal{G}$ be a topological group. Let C be a compact subset of $\mathcal{G}$ and U an open subset of $\mathcal{G}$ containing C. Then there exists an open neighborhood N of e such that $NC \subseteq U$.*

Proof. Since any point $x \in C$ is an interior point of U, by Prop.2.2.1, there is a neighborhood M_x base of e such that $M_x x \subseteq U$ and from Prop.2.2.2 (P_3), there exists a neighborhood N_x of e such that $N_x N_x \subseteq M_x$. Because of $x \in N_x x$, the family $\{N_x x\}_{x\in C}$ is an open cover of C. C being compact, then there is a finite subcover of C, i.e., $C \subseteq \bigcup_{i=1}^{n}(N_i x_i)$ where $N_1, N_2, \cdots, N_n$ are corresponding to $x_1, x_2, \cdots, x_n$, respectively. Set $N = \bigcap_{i=1}^{n} N_i$, then N is a neighborhood of e. Thus,

$$NC \subseteq N\bigcup_{i=1}^{n}(N_i x_i) = \bigcup_{i=1}^{n}(NN_i x_i).$$

Also, as

$$NN_ix_i \subseteq N_iN_ix_i \subseteq M_ix_i \subseteq U$$

for all $i = 1, 2, \cdots, n$, it follows that $NC \subseteq U$ where $M_1, M_2, \cdots, M_n$ are corresponding to $x_1, x_2, \cdots, x_n$, respectively. □

Proposition 3.2.3. *Every open subgroup of a compact group has a finite index.*

Proof. Let H be an open subgroup of a compact topological group $\mathcal{G}$. Then $\{xH\}_{x\in\mathcal{G}}$ is an open cover of $\mathcal{G}$, and as cosets form a partition of $\mathcal{G}/H$ so any pair is either equal or disjoint. Thus, $\mathcal{G}$ has no proper subcovers. Hence, $\{xH\}_{x\in\mathcal{G}}$ must be finite and so H has a finite index. □

Proposition 3.2.4. *A topological group is locally compact if and only if there exists a compact neighborhood of the identity.*

Proof. Assume that $\mathcal{G}$ is a locally compact group, then there is a compact neighborhood of e. Conversely, assume that U is a compact neighborhood of e. Then as $\mathcal{G}$ is regular by Prop.2.6.1, there is a neighborhood V of e such that $\overline{V} \subset U$. Thus, $\overline{V}$ is compact since $\overline{V}$ is closed in U (see Prop.1.5.8). Now, for any $x \in \mathcal{G}$, since the right translation map R_x is a homeomorphism, then $x\overline{V}$ is a compact neighborhood of x and in view of Prop.2.2.5, $x\overline{V} = \overline{xV}$ is a compact neighborhood of x. As consequence, $\mathcal{G}$ is locally compact. □

Proposition 3.2.5. *Let $\{\mathcal{G}_i\}_{i\in I}$ be a family of topological groups and let $\mathcal{G} = \prod_{i\in I}\mathcal{G}_i$. Then*

(i) $\mathcal{G}$ is compact if and only if each $\mathcal{G}_i$ is compact.

(ii) If all but a finite number of $\mathcal{G}_i$ are compact and all $\mathcal{G}_i$ are locally compact, then $\mathcal{G}$ is locally compact.

Proof.
(i) Clearly from Theorem 1.5.2, the claim follows.
(ii) Assume that each $\mathcal{G}_j$ is locally compact and there are only finite number of $\mathcal{G}_{j_i}$ which are not compact, i.e.,

$$\mathcal{G}_{j_i} \text{ is } \begin{cases} \text{not compact}, & \text{for } 1 \leq i \leq n \\ \text{compact}, & \text{otherwise.} \end{cases}$$

By Prop.1.5.14, we get that $\prod_{1\leq i\leq n}\mathcal{G}_{j_i}$ is locally compact and by (i), $\prod_{i\neq k}\mathcal{G}_{j_i}$ is compact for $k = 1, 2, \cdots, n$. Since

$$\mathcal{G} = \prod_{1\leq i\leq n}\mathcal{G}_{j_i} \times \prod_{i\neq k}\mathcal{G}_{j_i}$$

is a product of two locally compact groups with $k = 1, 2, \cdots, n$, then $\mathcal{G}$ is locally compact. □

Example 3.2.1.

1. The topological group $\mathbb{S}^1$ is compact. Hence, the n-torus $\mathbb{T}^n$ is compact too according to Prop.3.2.5.

2. Let $\mathcal{O}_n(\mathbb{R})$ be the orthogonal group defined in Ex.2.3.1. We have shown that $\mathcal{O}_n$ is closed in $GL_n(\mathbb{R})$. On the other hand, whenever $i = j$, we have $\sum_{k=1}^{n} a_{ki}^2 = 1$, so that $|a_{ki}| \leq 1$ for all $k, i = 1, 2, \cdots, n$. It follows that $\mathcal{O}_n(\mathbb{R})$ is bounded in $\mathbb{R}^{n^2}$ and thus it is compact.

Proposition 3.2.6. *Let H be a subgroup (not necessarily normal) of a topological group $\mathcal{G}$. Then*

(i) if $\mathcal{G}$ is compact, then so is $\mathcal{G}/H$,

(ii) if $\mathcal{G}$ is locally compact, then so is $\mathcal{G}/H$,

(iii) if both H and $\mathcal{G}/H$ are compact, then so is $\mathcal{G}$,

(iv) if both H and $\mathcal{G}/H$ are locally compact, then so is $\mathcal{G}$.

Proof.
(i) As the canonical projection $\kappa\colon \mathcal{G} \to \mathcal{G}/H$ is continuous and surjective, the result follows Prop.1.5.8 (ii).
(ii) Let U be a compact neighborhood of e since $\mathcal{G}$ is locally compact. As $\mathcal{G}$ is regular, there is a neighborhood V of e such that $\overline{V} \subseteq U$. We have $\overline{V}$ is compact since it is closed in U. As the canonical projection κ is a homeomorphism, it follows that $\kappa(V)$ is a neighborhood of $H = \kappa(e)$. Further, $\kappa(\overline{V})$ is compact in $\mathcal{G}/H$ which is regular. Thus, $\kappa(\overline{V})$ is closed. On the other hand,

$$\begin{aligned} V \subseteq \overline{V} &\Rightarrow \kappa(V) \subseteq \kappa(\overline{V}) \\ &\Rightarrow \overline{\kappa(V)} \subseteq \kappa(\overline{V}). \end{aligned}$$

We also have $\kappa(\overline{V}) \subseteq \overline{\kappa(V)}$ as κ is continuous. Therefore, $\overline{\kappa(V)} = \kappa(\overline{V})$ is a compact neighborhood of H. Hence, $\mathcal{G}/H$ is locally compact. In addition, employing the map T_s defined in Prop.2.4.3 gives for any arbitrary element in $\mathcal{G}/H$ has a compact neighborhood of it.
(iii) In this case, we assume that H is a normal subgroup of the topological group $\mathcal{G}$ such that H and $\mathcal{G}/H$ are both compact. Let $\{U_i\}_{i\in I}$ be an open cover of $\mathcal{G}$. For any $x \in \mathcal{G}$, the coset xH is compact and it is covered by $\{U_i\}_{i\in I}$. Thus, there is a finite subcovering of xH, i.e., $xH \subseteq \bigcup_{i\in J_x} U_i$ where $J_x \subseteq I$. Since $\bigcup_{i\in J_x} U_i$ is open, then by Prop.3.2.2, there exists a neighborhood N_x of e such that

$$N_x xH \subseteq \bigcup_{i\in J_x} U_i.$$

On the other hand, as the canonical projection κ is open (see Prop.2.4.1) and since $x \in N_x xH$, the collection $\{\kappa(N_x xH)\}_{x\in\mathcal{G}}$ is an open cover of $\mathcal{G}/H$. Therefore, there are $x_1, x_2, \cdots, x_n \in \mathcal{G}$ such that

$$\mathcal{G}/H = \bigcup_{i=1}^{n} \kappa(N_{x_i} x_i H).$$

Remark that $N_x xH$ is a union of cosets. Hence,

$$\kappa^{-1}(\kappa(N_x xH)) = \kappa^{-1}(\kappa(\bigcup_{y\in N_x x} yH)) = \bigcup_{y\in N_x x} \kappa^{-1}(\kappa(yH)) = \bigcup_{y\in N_x x} yH = N_x xH.$$

Thus,

$$\mathcal{G} = \kappa^{-1}(\mathcal{G}/H) = \kappa^{-1}(\bigcup_{y\in N_x x} \kappa(N_{x_i} x_i H)) = \bigcup_{y\in N_x x} \kappa^{-1}\kappa(N_{x_i} x_i H) = \bigcup_{y\in N_x x} (N_{x_i} x_i H) \subseteq \bigcup_{i=1}^{n} \bigcup_{j\in J_{x_i}} U_j.$$

We deduce that $\mathcal{G}$ is compact.
(iv) One can find a compact neighborhood of the identity element of $\mathcal{G}$, and then the locally compactness of $\mathcal{G}$ follows. □

Proposition 3.2.7. *Let H be a closed subgroup of a topological group $\mathcal{G}$. Then*

(i) H is compact if $\mathcal{G}$ is,

(ii) H is locally compact if $\mathcal{G}$ is.

Proof.
(i) As $\mathcal{G}$ is compact, by Prop.1.5.8 (i), the claim follows immediately.
(ii) Let $x \in H$, so $x \in \mathcal{G}$. As $\mathcal{G}$ is locally compact, then there is a neighborhood U of x such that $\overline{U}$ is compact. Thus $V = U \cap H$ is a neighborhood of x in H. Further, we have $V \subset U$, and so $\overline{V} \subset \overline{U}$. Hence, $\overline{V}$ is compact in $\overline{U}$ and then in $\mathcal{G}$ since it is closed in $\overline{U}$ which is compact. This implies that $\overline{V}$ is compact in $\mathcal{G}$. As $V \subset H$, then $\overline{V} \subset H$ because H is closed. Therefore, $\overline{V}$ is compact in H, and then H is locally compact. □

Lemma 3.2.1. *Let H be a closed subset and C a compact subset of a topological group $\mathcal{G}$ such that $H \cap C = \emptyset$. Then there exists a neighborhoods β of e such that*

(i) $H\beta \cap C\beta = \emptyset$, and

(ii) $\beta H \cap \beta C = \emptyset$.

Proof.
(i) It sufficies to show that there exists a neighborhood O of e such that $HOO^{-1} \cap C = \emptyset$. Indeed, for each neighborhood U of e, set

$$H(U) = \overline{HUU^{-1}}$$

which is clearly closed. Hence,

$$H(U) = \overline{H(U)} = \bigcap_{V \in \mathscr{U}(e)} HUU^{-1}V.$$

Indeed, pick $x \in \bigcap_{V \in \mathscr{U}(e)} HUU^{-1}V$. Then $x \in HUU^{-1}V$ for all $V \in \mathscr{U}(e)$. This implies that $xV^{-1} \subset HUU^{-1}$ and so that $xV^{-1} \cap HUU^{-1} \neq \emptyset$ where $xV^{-1} \in \mathscr{U}(x)$. Thus, $x \in H(U)$. On the other hand, let $x \in H(U)$ and $V \in \mathscr{U}(e)$. Then $V^{-1} \in \mathscr{U}(e)$ with

$$xV^{-1} \cap HUU^{-1} \neq \emptyset.$$

Therefore, $x \in HUU^{-1}V$ and hence, $x \in \bigcap_{V \in \mathscr{U}(e)} HUU^{-1}V$ since V is arbitrary. So the equality holds. However,

$$H(U) = \bigcap_{V \in \mathscr{U}(e)} HUU^{-1}V = \bigcup_{W \in \mathscr{U}(e)} HW = \overline{H} = H$$

since H is closed and where $W = UU^{-1}V$. It follows that

$$H(U) \cap C = H \cap C = \emptyset.$$

In other words, $C \subset \mathcal{G} \setminus H(U)$ for each U and $\mathcal{G} \setminus H(U)$ is open. C being compact, so $\{\mathcal{G} \setminus H(U)\}_{U \in \mathscr{U}(e)}$ is an open covering of C which has a finite subcovering $\{\mathcal{G}/H(U_i)\}_{i=1}^{n}$ of C such that $H(U_i) \cap C = \emptyset$, for each $i = 1, 2, \cdots, n$. Thus,

$$\bigcap_{i=1}^{n} H(U_i) \cap C = \emptyset. \tag{3.1}$$

Put $O = \bigcap_{i=1}^{n} U_i$. We have

$$HOO^{-1} = H\bigcap_{i=1}^{n} U_i \left(\bigcap_{i=1}^{n} U_i\right)^{-1} = H\bigcap_{i=1}^{n} U_i \bigcap_{i=1}^{n} U_i^{-1} = \bigcap_{i=1}^{n} HU_iU_i^{-1}.$$

By taking the closure in both sides in the equation above, we get

$$HOO^{-1} \subset \overline{HOO^{-1}} = H(O) = \overline{\left(\bigcap_{i=1}^{n} HU_iU_i^{-1}\right)} \subset \bigcap_{i=1}^{n} \overline{HU_iU_i^{-1}} = \bigcap_{i=1}^{n} H(U_i),$$

that is,

$$HOO^{-1} \subset \bigcap_{i=1}^{n} H(U_i). \tag{3.2}$$

It follows that $HOO^{-1} \cap C = \emptyset$ by (3.1) and (3.2). Consequently, $HO \cap CO = \emptyset$.
(ii) Likewise, we prove the existence of some neighborhood V of e satisfying $VH \cap VC = \emptyset$.
Now let $\beta = O \cap V$, then for this β, (i) and (ii) follow. □

Proposition 3.2.8. *Let H be a closed subset and C a compact subset of a topological group $\mathcal{G}$, then CH and HC are closed.*

Proof. Let $x \in \mathcal{G} \setminus HC$. By Prop.2.1.3, $H^{-1}x$ is closed as H is. Note that $H^{-1}x \cap C = \emptyset$ since $x \notin HC$. Therefore, since C is compact, by Lemma 3.2.1, there is a neighborhood β of e such that

$$H^{-1}x\beta \cap C\beta = \emptyset.$$

That is, $x\beta\beta^{-1} \cap HC = \emptyset$. Hence, $x\beta\beta^{-1}$ is a neighborhood of x such that $x\beta\beta^{-1} \subseteq \mathcal{G} \setminus HC$. Hence, $\mathcal{G} \setminus HC$ is open and so HC is closed. Analogously, CH is closed. □

Remark 3.2.1. If C is not compact, then the above proposition is no longer valid. Indeed, for instance, in the additive group $\mathbb{R}$ are equipped with the usual metric, $\mathbb{Z}$ and $k\mathbb{Z}$, k irrational, are both closed subgroups of $\mathbb{R}$. However, $\mathbb{Z} + k\mathbb{Z}$ is dense in $\mathbb{R}$, so that it is not closed.

Proposition 3.2.9. *Let N be a compact normal subgroup of a topological group $\mathcal{G}$, then $\kappa\colon \mathcal{G} \to \mathcal{G}/N$ is closed.*

Proof. Let C be a closed subset of $\mathcal{G}$. Then we show that the set

$$H = (\mathcal{G}/N) \setminus \kappa(C) = \{xN : x \notin CN\}$$

is open. Indeed, let $xN \in H$, then $x \notin CN$. Since N is a compact subgroup, by Prop.3.2.8, CN closed. Thus, there exists a neighborhood U of x such that $U \cap CN = \emptyset$. As κ is an open map, then $\kappa(U)$ is a neighborhood of xN. It follows that

$$\kappa(U) = \{yN : y \in U\} = \{yN : y \notin CN\} \subseteq H.$$

That is, we find an open neighborhood of xN contained in H. Since xN is arbitrary, we deduce that H is open and consequently, $\kappa(C)$ is closed. Thus, κ is a closed map, as desired. □

Theorem 3.2.1. *Let X be a homogeneous space of a topological group $\mathcal{G}$. Suppose that X and $\mathcal{G}$ are locally compact and have a countable base. Let H be the isotropy group of $\mathcal{G}$ at a point x, then the map*

$$g\colon \mathcal{G}/H \to X$$

which is defined in Prop.2.4.9, is a homeomorphism.

Proof. First we show that the map $f\colon \mathcal{G} \to X$ defined as $f(s) = sx$ is open. Indeed, let O be an open set in $\mathcal{G}$ and let $x \in f(O)$. Then there is $s \in O$ such that $x = f(s)$. Also, the map defined by

$$t \mapsto st^{-1}t$$

is continuous and $se^{-1}e = s \in O$. So that there is a neighborhood U of e such that

$$s\overline{U}^{-1}U \subset O.$$

The collection $\{tU\}_{t\in\mathcal{G}}$ forms an open cover of $\mathcal{G}$ and as $\mathcal{G}$ has a countable base, then there exists a sequence $\{t_n\}_{n\geq 1}$ such that $\{t_nU\}_{n\geq 1}$ covers $\mathcal{G}$ as well. Therefore,

$$\mathcal{G} = \bigcup_{n\geq 1} t_nU = \bigcup_{n\geq 1} t_n\overline{U}.$$

If we set

$$X_n = f(t_n\overline{U})$$

then we get

$$X = f(\mathcal{G}) = \bigcup_{n\geq 1} X_n. \tag{3.3}$$

Now for each n, $t_n\overline{U}$ is compact and so its image under the continuous map f is compact, i.e., X_n is compact. However, X is a Hausdorff space since it is regular, so that X_n is closed for all n. Hence, X is a countable union of closed sets and according to Lemma 1.5.2, there is n_0 such that X_{n_0} contains an open set W, i.e.,

$$W \subset X_{n_0} = f(t_{n_0}\overline{U}) = t_{n_0}\overline{U}x.$$

So,

$$f(\overline{U}) = \overline{U}x = t_{n_0}^{-1}f(t_{n_0}\overline{U}).$$

Since the map $x \mapsto t_{n_0}^{-1}x$ is clearly a homeomorphism and that $f(t_{n_0}\overline{U})$ contains an open set W, then $f(\overline{U})$ contains an open set $S = t_{n_0}^{-1}W$. Let $y \in S$, then $y \in f(\overline{U})$ so that there is $z \in \overline{U}$ such that $y = f(z) = zx$. Thus, $f(s) = sz^{-1}zx \in sz^{-1}S$. On the other hand, since $sz^{-1}S$ is contained in $sz^{-1}f(\overline{U}) = sz^{-1}\overline{U}x$ and

$$sz^{-1}\overline{U} \subset s\overline{U}^{-1}U \subset O,$$

then $sz^{-1}S \subseteq Ox = f(O)$. We find an open set $sz^{-1}S$ containing $f(s)$ which is in $f(O)$. So f is an open map. Finally, we show that g is an open map. Indeed, let U be an open set in $\mathcal{G}/H$, then

$$g(U) = f(\kappa^{-1}(U))$$

which is clearly open. Consequently, g is a homeomorphism. □

Proposition 3.2.10. *Let $\mathcal{G}$ be a locally compact group and totally disconnected. Then for all neighborhoods U of e, there is an open compact subgroup H such that $H \subset U$.*

Proof. As $\mathcal{G}$ is totally disconnected, $\{e\}$ is the connected component of e which is compact as long as $\mathcal{G}$ is locally compact. Then by Prop.1.5.11, there is an open and compact neighborhood O of e such that $O \subset U$. Let $W = \{w \in \mathcal{G} : Ow \subset O\}$, then W is non-empty since $e \in O$. Let $H = W \cap W^{-1}$. We show that H is an open, compact subgroup. Indeed,

- let $w \in W$ and $x \in O$, then $xw \in O$. Since O is open, there are neighborhoods U_x and V_x of x and w, respectively, such that $U_xV_x \subset O$. We also have, $O \subset \bigcup_{x\in O} U_x$. O being compact, then there is a finite subcovering $\{U_x\}_{x\in O}$ of O, i.e., $O \subset \bigcup_{i=1}^{n} U_{x_i}$. Let $V = \cap_{i=1}^{n} V_{x_i}$, then $OV \subset O$ and so $V \subset W$. Therefore, W is open. Consequently, H is open by Prop.2.3.2.

- By definition of W, we can see that $W \subset O$. Indeed, let $w \in W$, then $ew = w \in O$. Since O is compact and W is closed in O, then W is compact. Similarly, it is easy to prove that W^{-1} is compact since it is closed in O. Therefore, H is compact.

- Finally, H is subgroup of $\mathcal{G}$. Indeed, let $h_1, h_2 \in H$, $h_1 \in W$ and $h_2^{-1} \in W$. Hence, $O(h_1h_2^{-1}) = (Oh_1)h_2^{-1} \subset O$. We get that $h_1h_2^{-1} \in H$.

□

Proposition 3.2.11. *Let $\mathcal{G}$ be a compact and disconnected group, then for all neighborhoods U of e, there exists an open normal subgroup N of $\mathcal{G}$ such that $N \subset U$*

Proof. By the proposition above, there is a compact open subgroup H of $\mathcal{G}$ such that $e \in H \subset U$. Let $N = \cap_{x\in\mathcal{G}} x^{-1}Hx$, we show that N is a normal subgroup and open in $\mathcal{G}$. Indeed,

- let $y \in \mathcal{G}$, then $y^{-1}Ny = \cap_{x\in\mathcal{G}} y^{-1}x^{-1}Hxy$. Since the map $x \mapsto xy^{-1}$ is a homeomorphism, we have $y^{-1}Ny = \cap_{x\in\mathcal{G}} x^{-1}Hx = N$. It follows that N is a normal subgroup.

- We have $x^{-1}ex = e \in N$ for all $x \in \mathcal{G}$. Therefore, there are neighborhoods U_x and O_x of e and x, respectively, such that $O_x^{-1}U_xO_x \subset H$. $\mathcal{G}$ being compact and $\mathcal{G} = \cup_{x\in\mathcal{G}} O_x$, then there is $n \in \mathbb{N}$ such that $\mathcal{G} = \cup_{i=1}^n O_{x_i}$. Now let $U = \cap_{i=1}^n U_{x_i}$ so U contains e and we get

$$x^{-1}Ux \subset O_{x_i}^{-1}UO_{x_i} \subset O_{x_i}^{-1}U_{x_i}O_{x_i} \subset H \quad \text{for all } x \in \mathcal{G}.$$

  So that $U \subset N$ and thus $U_{x_i} \subset N$ for all $i = 1, 2, \cdots, n$. Hence, N is open.

□

Chapter 4

Introduction to Haar Measure on Locally Compact Groups

In this chapter, we introduce a measure which is analogous to the Lebesgue measure on locally compact groups. Namely, on any locally compact group there is an essential unique regular Borel measure which is invariant under translations called a **Haar measure**. We focus on two problems, i.e., the existence and the uniqueness of Haar measures. First, we study a general result concerning the existence of a Haar measure on a locally compact group. In addition, the Riesz representation theorem, see [3], will be required to achieve the uniqueness.

4.1 Basics of Measure and Integration Theories

In this section, we give a quick review to the concepts of measure and integration theories with some prelimaries useful results.

Definition 4.1.1 (σ-algebra). *Let X be a set. A collection $\mathcal{A}$ of subsets of X is called a **σ-algebra** on X if the following axioms hold.*

(i) $X \in \mathcal{A}$,

(ii) $X \setminus A \in \mathcal{A}$ *provided that* $A \in \mathcal{A}$,

(iii) if $\{A_n\}$ *is a finite or infinite countable collection of sets in* $\mathcal{A}$, *then* $\bigcup_{n\geq 1} A_n \in \mathcal{A}$.

A set X equipped with $\mathcal{A}$ is called a **measurable space**, denoted by $(X, \mathcal{A})$, and a subset A of $\mathcal{A}$ is said to be a **measurable set**.

Remark 4.1.1. One can deduce from the preceding axioms more properties for measurable spaces. For instance, (i) and (ii) imply that the empty set is measurable, i.e., $\emptyset \in \mathcal{A}$. Also, from all conditions above, we see that the countable intersection of finite or infinite family of measurable sets is measurable, i.e., if $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ either finite or infinite, then $\bigcap_{n\geq 1} A_n \in \mathcal{A}$.

Definition 4.1.2 (Borel σ-algebra). *Let X be a topological space. The smallest σ-algebra containing all open subsets of X is called a **Borel σ-algebra**, denoted as $\mathcal{B}(X)$.*

In this case, we say that $\mathcal{B}(X)$ is **generated** by a topology on X. The members of $\mathcal{B}(X)$ is called **Borel sets**.

Remark 4.1.2. Since all open sets are Borel sets, then in view of (ii) in Def.4.1.1, we deduce that all closed sets are also Borel sets.

Now we go through into the notion of measures. For convenience, we will deal with non-negative measures. This types of functions can reach ∞ in the extended real numbers which includes $-\infty$ and $+\infty$, i.e., $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Further, the arithmetic and order properties inherit as the real line system with some exceptions, i.e., we define the product of 0 and $\pm\infty$ to be 0 while the difference of two ∞ is undefined.

Definition 4.1.3 (Measure). *Let $(X, \mathcal{A})$ be a measurable space. A function $\mu \colon \mathcal{A} \to [0, \infty]$ is said to be a* ***measure*** *on $\mathcal{A}$ if*

(i) $\mu(\emptyset) = 0$, *and*

(ii) if $\{A_n\}_{n=1}^{\infty}$ is a collection of disjoint sets in $\mathcal{A}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

A measurable space together with a measure μ is called a **measure space** and denoted by $(X, \mathcal{A}, \mu)$.

Remark 4.1.3. Note that all measures satisfy the subadditive property

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n),$$

where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of sets while the equality holds whenever $\{A_n\}_{n\in\mathbb{N}}$ is a pairwise disjoint family.

Example 4.1.1. Recall that the **Lebesgue measure** defined on $\mathbb{R}$ is the unique measure such that the measure of any interval (open or closed) is the distance of its endpoints.

Definition 4.1.4 (Borel Measure). *A* ***Borel measure*** *on a topological space X is a measure whose domain is $\mathcal{B}(X)$ and each compact subset has a finite measure.*

Definition 4.1.5 (Regular Measure). *Let X be a topological space together with a σ-algebra $\mathcal{A}$ such that $\mathcal{B}(X) \subset \mathcal{A}$. Then a Borel measure is* ***regular*** *if the following hold.*

(i) $\mu(A) = \inf\{\mu(U) : U$ *is open and* $A \subset U\}$ *for all* $A \in \mathcal{A}$, *and*

(ii) $\mu(U) = \sup\{\mu(K) : K$ *is compact and* $K \subset U\}$ *for all open subsets U of X.*

A measure satisfying (i) is called **outer regular** and a measure is said to be **inner regular** if (ii) holds.

Definition 4.1.6 (Outer Measure). *An* ***outer measure*** *on a set X is a function $\lambda \colon \mathcal{P}(X) \to [0, \infty]$ such that the following hold.*

(i) $\lambda(\emptyset) = 0$,

(ii) $\lambda(A) \leq \lambda(B)$ *whenever* $A \subseteq B$ *(monotoncity),*

(iii) countably subadditive property.

Definition 4.1.7 (λ-measurable Set). *Let λ be an outer measure on a set X. A subset B of X is said to be* ***λ-measurable*** *if*

$$\lambda(A) = \lambda(A \cap B) + \lambda(A \cap (X \setminus B))$$

holds for any subset A in X.

Proposition 4.1.1 ([11]). *Let λ be an outer measure on a set X and $\mathcal{M}$ be the collection of all λ-measurable subsets of X. Then*

(i) $\mathcal{M}$ is a σ-algebra on X.

(ii) The restriction of λ on $\mathcal{M}_\lambda$ is a measure on $\mathcal{M}_\lambda$.

Definition 4.1.8 (Measurable and Borel Measurable Functions). *Let $(X, \mathcal{A})$ and $(Y, \mathcal{C})$ be two measurable spaces. A function $f\colon X \to Y$ is said to be* ***measurable*** *if $f^{-1}(C)$ is measurable in X for all $C \in \mathcal{C}$. If f assigns to each Borel set in X a Borel subset in Y, then it is called a* ***Borel measurable*** *function.*

Proposition 4.1.2 ([11]). *Let X and Y topological spaces and let $f\colon X \to Y$ be a continuous function. Then f is Borel measurable.*

Definition 4.1.9. *Let (X, μ) be a measure space and let f be a measurable function. Then the* ***integral*** *of f over X, $\int_X f d\mu$, exists if it is either finite or equal to ∞ (or $-\infty$) while it is said to be* ***μ-integrable*** *if the integral of f is finite.*

Proposition 4.1.3. *Let (X, μ) be a measure space, $\alpha \in \mathbb{R}$ and f, g be two real-valued functions which both are μ-integrable on X. Then*

(i) $f + g$, αf are μ-integrable.

(ii) $\int_X \alpha f d\mu = \alpha \int_X f d\mu$ (homogoneity).

(iii) $\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$ (linearity).

(iv) $\int_X f d\mu \leq \int_X g d\mu$ whenever $f \leq g$ on X (monotoncity).

(v) $|f|$ is μ-integrable if and only if f is μ-integrable. Moreover,

$$\left| \int_X f d\mu \right| \leq \int_X |f| \, d\mu \text{ (triangal inequality).}$$

Definition 4.1.10 (Sections of a Function). *Let X and Y be two sets. The* ***sections*** *f_x and f^y of a function f defined on $X \times Y$ are given by*

$$f_x(y) = f(x, y) \quad \text{and} \quad f^y(x) = f(x, y).$$

Given two sets X and Y. Recall that the **support** of a function $f\colon X \to Y$ is the closure of the set where f does not vanish and denoted by $supp(f)$. In other words,

$$supp(f) = \overline{\{x \in X : f(x) \neq 0\}}.$$

We denote the set of continuous functions from X into Y with compact support by $C_c(X, Y)$.

Remark 4.1.4. Note that $C_c(X)$, the set of continuous real-valued functions, is a vector space over $\mathbb{R}$ and each function is bounded on X. Futher, since f is continuous, then by Prop.4.1.2, f is Borel measurable. Also, if $f \in C_c(X)$ and μ is a regular Borel measure on X, then f is μ-integrable. In addtion, as μ is regular and $supp(f)$ is compact, it follows that $supp(f)$ has a finite measure.

Proposition 4.1.4 ([11]). *Let X and Y be two locally compact spaces, let μ and ν be two regular Borel measures on X and Y, respectively. Let $f \in C_c(X \times Y)$.*

(i) For each $x \in X$ and each $y \in Y$, the sections f_x and f^y are continuous and whose supports are compact in Y and X, respectively, i.e., $f_x \in C_c(Y)$ and $f^y \in C_c(X)$.

(ii) The functions

$$x \mapsto \int_Y f_x(y)d\nu(y) \quad \text{and} \quad y \mapsto \int_X f^y(x)d\mu(x)$$

are in $C_c(X)$ and $C_c(Y)$, respectively.

(iii) The equality

$$\int_X \int_Y f(x,y)d\nu(y)d\mu(x) = \int_Y \int_X f(x,y)d\mu(x)d\nu(y)$$

holds.

Proposition 4.1.5 ([11]). *Let μ be a regular Borel measure on a locally compact group $\mathcal{G}$ and let $f \in C_c(\mathcal{G})$. Then the functions*

$$x \mapsto \int_{\mathcal{G}} f(xy)d\mu(y) \quad \text{and} \quad x \mapsto \int_{\mathcal{G}} f(yx)d\mu(y)$$

are continuous.

We say that a scalar-valued function f is a **linear functional** on a scalar vector space X if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for any scalars α and β and any $x, y \in X$.

Now we state an important result which guarantees the existence of a unique regular Borel measure on any locally compact space.

Theorem 4.1.1 (Riesz Representation Theorem). *Let X be a locally compact space and let P be a positive linear continuous functional on $C_c(X)$. Then there is a unique regular Borel measure μ on X such that*

$$P(f) = \int_X f d\mu$$

for any $f \in C_c(X)$.

4.2 Existence and Uniqueness of a Haar Measure

In this section, we give the definition of the Haar measure illustrated by some examples and then we state and prove the existence and uniqueness.

Definition 4.2.1 (Haar Measure). *Let $\mathcal{G}$ be a locally compact group and μ be a non-zero regular Borel measure on $\mathcal{G}$. Then μ is called a **left** Haar measure if it is invariant under left translation, i.e.,*

$$\mu(xA) = \mu(A) \quad \text{for all } x \in \mathcal{G} \text{ and } A \in \mathcal{B}(\mathcal{G}).$$

*Similarly, we can define a **right** Haar measure. A left or right Haar measure is simply said to be a **Haar measure**.*

Remark 4.2.1. This definition is well defined since xA and Ax are open for any open (Borel) set A. Hence, both are Borel sets.

Example 4.2.1. The following examples are Haar measures.

1. The Lebesgue measure on the additive group $\mathbb{R}$ (or on $\mathbb{R}^n$).

2. The counting measure c defined as

$$c(A) = \begin{cases} \infty, & \text{if } card(A) = \infty \\ card(A), & \text{if } card(A) < \infty \\ 0, & \text{if } A \text{ is empty} \end{cases}$$

   on all groups in the discrete topology is a Haar measure.

3. The measure ν defined as

$$\nu(A) = \int_A \frac{1}{x} \mu(dx)$$

   where μ is the Lebesgue measure on $\mathbb{R}$, is a Haar measure on $\mathbb{R} \backslash \{0\}$ under multiplication.

When we talk about a Haar measure, we have to mention whether it is a left or right Haar measure. In our consideration, we focus on the left Haar measure since the study on the right one is similar.

Now we state the proof of the existence of Haar measures on locally compact groups. The strategy of the construction will be given through it.

Theorem 4.2.1 (Existence)**.** *There exists a left Haar measure on any locally compact group.*

Proof.

Step 1: (Defining $(K\colon V)$). Let K be a compact subset of a locally compact group $\mathcal{G}$, and let V be a subset of $\mathcal{G}$ whose interior is non-empty, i.e., $\mathring{V} \neq \emptyset$. Then $\{x\mathring{V}\}_{x \in \mathcal{G}}$ is an open cover of K. K being compact, there is a finite subfamily $\{x_i\}_{i=1}^n$ of elements of $\mathcal{G}$ such that

$$K \subseteq \bigcup_{i=1}^n x_i \mathring{V} \subseteq \bigcup_{i=1}^n x_i V.$$

Let $(K\colon V)$ be the smallest non-negative integer n for which such a collection $\{x_i\}_{i=1}^n$ exists. Define $(K\colon V) = 0$ if and only if $K = \emptyset$.

Step 2: (Defining h_U). Let C be a compact subset of $\mathcal{G}$ such that its interior is non-empty. Our goal is to measure an arbitrary compact subset K of $\mathcal{G}$ by finding the ratio $(K\colon U)/(C\colon U)$ for each neighborhood U of e and then computing a limit of this proportion as the neighborhood U becomes smaller. After that, we use this limit to construct an outer measure λ on $\mathcal{G}$. Finally, we show that the restriction of λ on $\mathcal{B}(\mathcal{G})$ is identically the left Haar measure. Let now $\mathscr{C}$ and $\mathscr{U}$ be collections of all compact subsets of $\mathcal{G}$ and all neighborhoods of e, respectively. Thus, for each $U \in \mathscr{U}$, define $h_U\colon \mathscr{C} \to \mathbb{R}$ by

$$h_U(K) = \frac{(K\colon U)}{(C\colon U)}.$$

Step 3: (Properties of h_U). h_U has many features which are introduced below.

Lemma 4.2.1 ([11])**.** *Let $U \in \mathscr{U}$ and $K, K_0, K_1 \in \mathscr{C}$ with $x \in \mathcal{G}$. Then*

(i) $0 \leq h_U(K) \leq (K\colon C)$.

(ii) $h_U(C) = 1$.

(iii) $h_U(xK) = h_U(K)$.

(iv) $h_U(K_0) \leq h_U(K_1)$ *provided that* $K_0 \subseteq K_1$.

(v) $h_U(K_0 \cup K_1) \leq h_U(K_0) + h_U(K_1)$.

(vi) *If* $K_0U^{-1} \cap K_1U^{-1} = \emptyset$, *then* $h_U(K_0 \cap K_1) = h_U(K_0) + h_U(K_1)$.

Step 4: (Identifying the limit of h_U). We discuss the limit of the ratios h_U. Indeed, the way used here is to build a product space containing all the functions h_U, and then by compactness argument, we can obtain the so-called a limit function. For each $K \in \mathscr{C}$, let I_K be the closed interval $[0, (K\colon C)]$ in $\mathbb{R}$ and let

$$X = \prod_{K \in \mathscr{C}} I_K$$

be their product. By Lemma 4.2.1 (i), $h_U(K) \in I_K$ for all $K \in \mathscr{C}$. It follows that $h_U = (h_U(K))_{K\in\mathscr{C}} \in X$ for all $U \in \mathscr{U}$. Moreover, since each interval I_K is closed and bounded in $\mathbb{R}$, it is compact. Hence, according to Theorem 1.5.2, X is compact. Now, for each neighborhood V of e, let $S(V)$ be the closure of the set $\{h_U \in X : U \in \mathscr{U}, U \subseteq V\}$ in X. If $V_1, V_2, \cdots, V_n \in \mathscr{U}$ and $V = \bigcap_{i=1}^n V_i$, then $h_V \in \bigcap_{i=1}^n S(V_i)$. Since the sets $V_1, V_2, \cdots, V_n \in \mathscr{U}$ are arbitrary, we deduce that the collection $\{S(V)\}_{V\in\mathscr{U}}$ has the finite intersection property. By Prop.1.5.9, $\bigcap_{V\in\mathscr{U}} S(V)$ is non-empty as long as X is compact. Therefore, there is $h_0 \in \bigcap_{V\in\mathscr{U}} S(V)$, and h_0 will be the limit of h_U.

Step 5: (Properties of h_0). We give some properties of h_0 before moving to the construction of an outer measure derived from h_0.

Lemma 4.2.2 ([11])**.** *Let* $K, K_0, K_1 \in \mathscr{C}$ *and* $x \in \mathcal{G}$. *Then*

(i) $0 \leq h_0(K)$.

(ii) $h_0(C) = 1$.

(iii) $h_0(\emptyset) = 0$.

(iv) $h_0(xK) = h_0(K)$.

(v) $h_0(K_0) \leq h_0(K_1)$ *provided that* $K_0 \subseteq K_1$.

(vi) $h_0(K_0 \cup K_1) \leq h_0(K_0) + h_0(K_1)$.

(vii) *If* $K_0 \cap K_1 = \emptyset$, *then* $h_0(K_0 \cap K_1) = h_0(K_0) + h_0(K_1)$.

Step 6: (Constructing the left Haar measure on $\mathcal{G}$). Firstly, we show that the function $\lambda\colon \mathscr{U} \to [0, \infty]$ defined as

$$\lambda(U) = \sup\{h_0(K) : K \subseteq U, K \in \mathscr{C}\} \tag{4.1}$$

is an outer measure on $\mathcal{G}$ and then extend it to all subsets A of $\mathcal{G}$ by

$$\lambda(A) = \inf\{\lambda(U) : A \subseteq U, U \in \mathscr{U}\}. \tag{4.2}$$

According to Lemma 4.2.2, it clear that λ is non-negative, monotonic and $\lambda(\emptyset) = 0$. We check that λ is countably subadditive. Indeed, in view of (4.2), it is sufficient to show this for any $U \in \mathscr{U}$. Let $\{U_i\}_{i=1}^\infty$ be a countable collection of open subsets of $\mathcal{G}$ and let K be a compact subset of $\mathcal{G}$ such that $K \subseteq \bigcup_{i=1}^\infty U_i$. K being compact and covering by $\{U_i\}_{i=1}^\infty$, then there is positive number n such that $K \subseteq \bigcup_{i=1}^n U_i$. In addition, by Remark 1.5.2, there are compact subsets K_i such that $K_i \subseteq U_i$ for all $i = 1, 2, \cdots, n$. By Lemma 4.2.2 (vi) and (4.1), we obtain

$$h_0(K) \leq \sum_{i=1}^n h_0(K_i) \leq \sum_{i=1}^n \lambda(U_i) \leq \sum_{i=1}^\infty \lambda(U_i).$$

Since K is arbitrary compact subset of $\bigcup_{i=1}^{\infty} U_i$ and in view of (4.1), it follows that λ is countably subadditive and thus λ is an outer measure on $\mathcal{G}$. Secondly, we show that every Borel set in $\mathcal{G}$ is λ-measurable. Indeed, by Prop.4.1.1 (i), the family of λ-measurable sets is σ-algebra on $\mathcal{G}$ and by the definition of the Borel σ-algebra, it suffices to show that if U and V are open subsets of $\mathcal{G}$ such that $\lambda(V) < \infty$, then

$$\lambda(V) \geq \lambda(V \cap U) + \lambda(V \cap (\mathcal{G} \setminus U)).$$

Let $\epsilon > 0$ and choose a compact subset K of $V \cap U$ and a compact subset C of $V \cap (\mathcal{G} \setminus K)$ such that

$$h_0(K) > \lambda(V \cap U) - \epsilon \quad \text{and} \quad h_0(C) > \lambda(V \cap (\mathcal{G} \setminus K)) - \epsilon.$$

Then K and C are disjoint and since $V \cap (\mathcal{G} \setminus U) \subseteq V \cap (\mathcal{G} \setminus K)$, by Lemma 4.2.2 (v), we have

$$h_0(C) > \lambda(V \cap (\mathcal{G} \setminus U)) - \epsilon.$$

By applying Lemma 4.2.2 (vii), we get

$$h_0(K \cup C) = h_0(K) + h_0(C) \geq \lambda(V \cap U) + \lambda(V \cap (\mathcal{G} \setminus U)) - 2\epsilon.$$

Since ϵ is arbitrary and $h_0(K \cup C) \leq \lambda(V)$, we deduce that every Borel set is λ-measurable. Therefore, $\mathcal{B}(\mathcal{G})$ is contained in $\mathcal{M}_\lambda$ and consequently, the restriction of λ on $\mathcal{M}_\lambda$ is a measure (see Prop.4.1.1 (ii)). Thirdly, let $\mu = \lambda|_{\mathcal{B}(\mathcal{G})}$ and we show that μ is regular. Indeed, if U is open and K is compact in $\mathcal{G}$ such that $K \subseteq U$, then $h_0(K) \leq \mu(U)$. This implies by using (4.2)

$$h_0(K) \leq \mu(K). \tag{4.3}$$

Further, if U has a compact closure $\overline{U}$, then by Prop.1.5.16, we obtain

$$h_0(C) \leq h_0(\overline{U})$$

for each compact subset C of U. Thus,

$$\mu(K) \leq \mu(U) \leq h_0(\overline{U}).$$

That is, μ is finite on the compact subsets of $\mathcal{G}$. Moreover, μ is outer and inner regular since it satisfies (4.1), (4.2) and (4.3). Finally, Lemma 4.2.2 (i) and (iv) together with (4.1) and (4.2) ensure that μ is a non-zero and translation invariant. Hence, μ is the required left Haar measure.

The proof of theorem is now complete. □

Proposition 4.2.1 ([11])**.** *Let $\mathcal{G}$ be a locally compact group and $x \in \mathcal{G}$. If μ is a left Haar measure on $\mathcal{G}$ and f is a non-negative or a μ-integrable function defined on a Borel subset of $\mathcal{G}$, then*

$$\int_{\mathcal{G}} f(x^{-1}y)d\mu(y) = \int_{\mathcal{G}} f d\mu.$$

The following lemma ensures that the integral with respect to Haar measure of a non-zero continuous function with compact support cannot be zero. This allows us to deal with the proof of the uniqueness result.

Lemma 4.2.3 ([11])**.** *Let $\mathcal{G}$ be a locally compact group and let μ be a left Haar measure on $\mathcal{G}$. Then each non-empty open subset U of $\mathcal{G}$ satisfies $\mu(U) > 0$ and each non-negative continuous function f in $C_c(\mathcal{G})$ which is not identically zero satisfiess*

$$\int_{\mathcal{G}} f d\mu > 0.$$

Now we state the uniqueness of a Haar measure.

Theorem 4.2.2 (Uniqueness). *Let μ and ν be two left Haar measures on a locally compact group. Then there is a positive real number c such that $\nu = c\mu$.*

Proof.

Step 1: (Showing the ratio independence on the Haar measure μ). Let g be a non-zero and non-negative continuous function with compact support on a locally compact group $\mathcal{G}$. We shall fix g and take f be an arbitrary continuous function with compact support on $\mathcal{G}$. Then by Lemma 4.2.3, we get

$$\int_{\mathcal{G}} g d\mu \neq 0,$$

and consider the ratio

$$\frac{\int_{\mathcal{G}} f d\mu}{\int_{\mathcal{G}} g d\mu}. \tag{4.4}$$

One can show that if ν is another Haar measure, then

$$\frac{\int_{\mathcal{G}} f d\nu}{\int_{\mathcal{G}} g d\nu} = \frac{\int_{\mathcal{G}} f d\mu}{\int_{\mathcal{G}} g d\mu}$$

or equivalently

$$\int_{\mathcal{G}} f d\nu = c \int_{\mathcal{G}} f d\mu = \int_{\mathcal{G}} f d(c\mu) \tag{4.5}$$

where $c = \int_{\mathcal{G}} g d\nu / \int_{\mathcal{G}} g d\mu$. Since the integral is positive linear functional on the vector space $C_c(\mathcal{G})$ and (4.5) is true for any $f \in C_c(\mathcal{G})$, by Theorem 4.1.1, we obtain $\nu = c\mu$.

Step 2: If $h \in C_c(\mathcal{G} \times \mathcal{G})$, then by Prop.4.2.1 (iii),

$$\int_{\mathcal{G}} \int_{\mathcal{G}} h(x, y) d\mu(x) d\nu(y) = \int_{\mathcal{G}} \int_{\mathcal{G}} h(x, y) d\nu(y) d\mu(x).$$

As both μ and ν are Haar measures, using their translation invariance and applying Prop.4.2.1, we get by exchanging x with $y^{-1}x$, then reversing the order of integration, and finally replacing y with xy, that

$$\begin{aligned}
\int_{\mathcal{G}\times\mathcal{G}} h(x, y) d\nu(y) d\mu(x) &= \int_{\mathcal{G}} \int_{\mathcal{G}} h(x, y) d\nu(y) d\mu(x) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} h(x, y) d\mu(x) d\nu(y) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} h(y^{-1}x, y) d\mu(x) d\nu(y) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} h(y^{-1}x, y) d\nu(y) d\mu(x) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} h(y^{-1}, xy) d\nu(y) d\mu(x).
\end{aligned} \tag{4.6}$$

Now consider the function $h \colon \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ defined by

$$h(x, y) = \frac{f(x) g(yx)}{\int_{\mathcal{G}} g(tx) d\nu(t)}, \tag{4.7}$$

and its corresponding function $h(y^{-1}, xy) \colon \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ given by

$$h(y^{-1}, xy) = \frac{f(y^{-1}) g(x)}{\int_{\mathcal{G}} g(ty^{-1}) d\nu(t)}. \tag{4.8}$$

Step 3: (Showing indeed that $h \in C_c(\mathcal{G} \times \mathcal{G})$). By Prop.4.1.5, the function $x \mapsto \int_{\mathcal{G}} g(tx)d\nu(t)$ is continuous and Lemma 4.2.3 asserts that it never equals to zero. Also, $f(x)$ and $g(yx)$ are continuous so, $h(x, y)$ is well-defined and continuous. Note that if $K = supp(f)$ and $C = supp(g)$, then $supp(h) \subseteq K \times CK^{-1}$. The set $K \times CK^{-1}$ is compact according to Prop.3.2.1 and Theorem 1.5.2, thus, $supp(h)$ is compact. This proves that $h \in C_c(\mathcal{G} \times \mathcal{G})$.

Step 4: (Showing the ratio dependence only on the functions f and g). Take the integrals of (4.7) and (4.8) over $\mathcal{G}$ with respect to ν and y, we get

$$\int_{\mathcal{G}} h(x, y)d\nu(y) = \int_{\mathcal{G}} \frac{f(x)g(yx)}{\int_{\mathcal{G}} g(tx)d\nu(t)} d\nu(y) = f(x)\frac{\int_{\mathcal{G}} g(yx)d\nu(y)}{\int_{\mathcal{G}} g(tx)d\nu(t)} = f(x) \tag{4.9}$$

and

$$\int_{\mathcal{G}} h(y^{-1}, xy)d\nu(y) = \int_{\mathcal{G}} \frac{f(y^{-1})g(x)}{\int_{\mathcal{G}} g(ty^{-1})d\nu(t)} d\nu(y) = g(x) \int_{\mathcal{G}} \frac{f(y^{-1})}{\int_{\mathcal{G}} g(ty^{-1})d\nu(t)} d\nu(y). \tag{4.10}$$

Thus, integrate (4.9) and (4.10) over $\mathcal{G}$ with respect to μ and x, we obtain

$$\int_{\mathcal{G}} \int_{\mathcal{G}} h(x, y)d\nu(y)d\mu(x) = \int_{\mathcal{G}} f d\mu \tag{4.11}$$

and

$$\int_{\mathcal{G}} \int_{\mathcal{G}} h(y^{-1}, xy)d\nu(y)d\mu(x) = \int_{\mathcal{G}} g d\mu \int_{\mathcal{G}} \frac{f(y^{-1})}{\int_{\mathcal{G}} g(ty^{-1})d\nu(t)} d\nu(y). \tag{4.12}$$

Substituting (4.11) and (4.12) into (4.6) gives

$$\int_{\mathcal{G}} f d\mu = \int_{\mathcal{G}} g d\mu \int_{\mathcal{G}} \frac{f(y^{-1})}{\int_{\mathcal{G}} g(ty^{-1})d\nu(t)} d\nu(y).$$

This concludes the proof because it shows that the ratio in (4.4) depends on f and g, but not on μ.

The proof is now complete. □

Remark 4.2.2. If $\mathcal{G}$ is abelian, then the uniqueness proof becomes simple. Indeed, let $f, g \in C_c(\mathcal{G})$ be defined as above. Then

$$\begin{aligned}
\int_{\mathcal{G}} f d\mu \int_{\mathcal{G}} g d\nu &= \int_{\mathcal{G}} \int_{\mathcal{G}} f(x)g(y)d\mu(x)d\nu(y) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} f(xy)g(y)d\mu(x)d\nu(y) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} f(y)g(yx^{-1})d\nu(x)d\mu(y) \\
&= \int_{\mathcal{G}} \int_{\mathcal{G}} f(y)g(x^{-1})d\mu(x)d\nu(y) \\
&= \int_{\mathcal{G}} f d\nu \int_{\mathcal{G}} g(x^{-1})d\mu(x).
\end{aligned}$$

Now define $c = \int_{\mathcal{G}} g d\nu / \int_{\mathcal{G}} g(x^{-1})d\mu(x)$, we get (4.5). Hence, by Theorem 4.1.1, $\nu = c\mu$ as desired.

References

[1] E. Alfsen, *A simplified constructive proof of existence and uniqueness of Haar measure*, Math. Scand., Vol 12, 1963, 106-116.

[2] H. Cartan, *Sur la mésure de Haar*, C. R. Acad. Ser. A, Vol 211, 1940, 759-762.

[3] D.L. Cohn, *Measure Theory*, 2^{nd} ed., Springer, 2013.

[4] T. tom Dieck, *Transformation Groups*, de Gruyter Studies in Mathematics, Vol 8, Walter de Gruyter, Berlin, 1987.

[5] D.S. Dummit and R.M. Foote, *Abstract Algebra*, 3^{rd} ed., John Wiley & Sons, Inc., 2004.

[6] J.A. Gallian, *Contemporary Abstract Algebra*, 7^{th} ed., Brooks/Cole, Cengage Learning, 2010.

[7] A. Haar, *Der massbegriff in der theorie der kontinuierlichen gruppen*, Ann. Math., Vol 34, 1933, 147-169.

[8] T. Husain, *Introduction to Topological Groups*, Saunders Mathematics Books, 1966.

[9] P. Jossen, *Groupes Topologiques I*, École Polytechnique Fédérale, Institut de Mathématiques, Lausanne, June 2004.

[10] D. Kohler, *Groupes Topologiques II*, Projet de semestre, École Polytechnique Fédérale, Institut de Mathématiques, Lausanne, hiver 2003-2004.

[11] S. Lundgren, *The Existence and Uniqueness of the Haar Measure*, Umeå University, Spring 2021.

[12] S.A. Morris, *Topological Groups, Advances, Surveys, and Open Questions*, Special Issue Published in Axioms, 2019.

[13] J. Munkres, *Topology*, 2^{nd} ed., Pearson, 2000.

[14] J. von Neumman, *Die einführung analytischer parameter in topologischen gruppen*, Ann. Math., Vol 34, 1933, 170-179.

[15] Jean-Paul Pier, *L'apparition de la théorie des groupes topologiques*, Cahiers du séminaire d'histoire des mathématiques, tome 9, 1988, 1-21.

[16] L.S. Pontryagin, *Topological Groups*, Princeton University Press, 1946.

[17] W. Rudin, *Fourier Analysis on Groups*, Interscience publishers, New York 1967.

[18] A. Weil, *L'intégration dans les Groupes Topologiques et ses Applications*, 2ème édition, Actualités scientifiques et industrielles, Hermann, Paris 1965.