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Title

Quantum Entanglement and its applications

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Abstract

Quantum entanglement, a cornerstone of quantum mechanics, manifests as nonlocal correlations between quantum systems and is pivotal across multiple physics disciplines, including quantum optics, statistical mechanics, nuclear physics, and condensed matter physics. This project report offers an introductory exploration of quantum entanglement and its practical applications, with a focus on quantum information processing through the concept of spin of particles. The report commences with a foundational review of quantum mechanics principles—state vectors, operators, and measurements—essential for comprehending entanglement. It subsequently examines entanglement in bipartite systems for spin-1/2 particles, employing concurrence as a measure to quantify entanglement strength, and highlights quantum teleportation as a key application, where entangled states facilitate the transmission of quantum information described in the state of particle. The fidelity of quantum teleportation is analyzed to assess the quality of information transfer, revealing optimal conditions for entanglement.

الملخص باللغة العربية

التشابك الكمي، أحد الأعمدة الأساسية لميكانيكا الكم، يتجلى في صورة ارتباطات غير محلية بين الأنظمة الكمية، وهو يحتل مكانة محورية في عدة تخصصات فيزيائية، بما في ذلك البصريات الكمية، الميكانيكا الإحصائية، الفيزياء النووية، وفيزياء المادة المكثفة. يقدم هذا التقرير بحثًا تمهيديًا حول التشابك الكمي وتطبيقاته العملية، مع التركيز على معالجة المعلومات الكمية من خلال مفهوم السبين للجسيمات. يبدأ التقرير بمراجعة أساسية لمبادئ ميكانيكا الكم - متجهات الحالة، المؤثرات، والقياسات - وهي عناصر ضرورية لفهم التشابك. يتناول التقرير بعد ذلك التشابك في الأنظمة الثنائية لجسيمات ذات سبين 1/2، مستخدمًا مقياس التزامن لقياس قوة التشابك، ويسلط الضوء على النقل الكمي كتطبيق رئيسي، حيث تسهل الحالات المتشابكة نقل المعلومات الكمية الموصوفة في حالة الجسيم. في الأخير، يتم تحليل دقة النقل الكمي لتقييم جودة نقل المعلومات، الذي يكشف عن الظروف المثلى للتشابك.

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Contents

Chapter 1

Introduction

Quantum entanglement, a cornerstone of quantum mechanics, emerges in composite quantum systems comprising multiple subsystems. These subsystems are deemed entangled when they lack independent quantum states, existing instead in a shared state. Physically, entanglement manifests as nonlocal correlations between measurements on the subsystems—correlations that cannot be reproduced by local operations on individual subsystems. These inherently quantum-mechanical correlations defy explanation through local realistic properties, distinguishing them from classical correlations. Until recently, the foundational elements of quantum theory were primarily the domain of epistemological inquiry. Although profound debates sought to unravel the ultimate nature of reality, experimental resolution of these questions appeared elusive. The EPR paradox, introduced by Einstein, Podolsky, and Rosen in 1935 [1], shifted the physics community's focus to potential deficiencies in the nascent quantum mechanics framework. In their seminal paper, they proposed a "local realism" worldview, attributing independent and objective reality to the physical properties of spatially separated subsystems within a composite quantum system. Then EPR applied the criterion of local realism to predictions associated with an entangled state, a state that cannot be described solely in terms of the properties of its subsystems, to conclude that quantum mechanics is incomplete. EPR criticism was the source of many discussions concerning fundamental differences between quantum and classical description of nature. Schrödinger [1], regarding the EPR paradox, did not see a conflict with quantum mechanics. Schrödinger proposed that nonlocality, or "Verschränkung" (the German term for entanglement), is a defining characteristic of quantum mechanics. He was the first to explore the connection between entanglement and information theory, noting that, prior to measurement-induced resolution of entanglement, only a collective description of the entangled subsystems exists within a higher-dimensional state space. Consequently, knowledge about individual subsystems may diminish significantly, potentially to zero, while the information about the composite system remains fully preserved. Best possible knowledge of a whole does not include best possible knowledge of its parts - and that is what keeps coming back to haunt us. Schrödinger thus identified a profound non-classical relation between the information that an entangled state gives about the whole system and the corresponding information that is given to us about the subsystems. The most significant progress toward the resolution of this "academic" EPR problem was made by Bell [1] in the 60s who proved that the local realism implies constraints on the predictions of spin correlations in the form of inequalities (called Bell's inequalities) which can be violated by quantum mechanical predictions for the system. Subsequent experiments validated Schrödinger's perspective on entanglement, as detailed in the forthcoming section on Bell inequalities. Over time, physicists have come to appreciate entanglement—a fundamental feature of nature—as a transformative resource for technological advancements. This phenomenon underpins novel paradigms, enabling breakthroughs such as accelerated quantum computing, unconditionally secure quantum cryptography, and quantum teleportation, which are unattainable within the constraints of classical physics.

This report is structured as follows. Chapter 2 provides a foundational overview of essential quantum mechanics concepts critical for subsequent discussions. Chapter 3 explores the phenomenon of quantum entanglement in bipartite systems, delving into its application in quantum teleportation and introducing fidelity as a metric to evaluate the quality of information transfer. The report concludes in Chapter 4 with a comprehensive summary of the project's findings and implications.

Chapter 2

Basic Concepts in Quantum Mechanics

2.1 Introduction

Quantum mechanics emerged through the pioneering contributions of scientists including Max Born, Paul A. M. Dirac, Erwin Schrödinger, and Werner Heisenberg. Its establishment, spanning 1923 to 1927, resolved longstanding theoretical ambiguities. During this period, matrix mechanics, developed by Heisenberg, and wave mechanics, formulated by Schrödinger's, were introduced concurrently and later demonstrated to be mathematically equivalent representations of quantum mechanics, providing a unified framework for the theory.

In this chapter, we present the foundational concepts of quantum mechanics essential for the analyses throughout this report. We provide a concise introduction to the mathematical framework underpinning quantum state descriptions, focusing on key tools such as the representation of quantum states as vectors in a Hilbert space, Hermitian operators, and the principles of quantum measurements.

2.2 Bra and ket vectors

In the theory of quantum mechanics, a physical state is represented by a state vector in a complex vector space. This state vector contains all the information about the physical state. Following Dirac [2], we call such a state vector a ket vector, or simply ket, denoted by a simple symbol $|\,\rangle$. If we want to specify a ket by a label, α say, we insert it in the middle, thus $|\alpha\rangle$. For a set of state vectors describes a particular physical system that forms a complex inner product space is known as a state space or ket space for the system. This ket space is closed under two operations: vector addition and scalar multiplication.

Kets may be added together and may be multiplied by a complex number to give other ket

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle\,, \tag{2.1}$$

$$c\,|\alpha\rangle = |\xi\rangle\,, \tag{2.2}$$

where c is any complex number.

The principle of linear superposition of quantum mechanics applies to the state kets. This requires us to consider that between these states there exist peculiar relationships such that whenever the system is definitely in one state. The original state should be considered as the result of a kind of superposition of the two or more new states, in a way that cannot be conceived on classical ideas. Any state ket may be regarded as the result of a superposition of two or more other states kets. Reciprocally, any two or more state ket may be superposed to give a new state. If the state vectors $|\alpha_1\rangle\,,\ |\alpha_2\rangle\,,\ \cdots$ represent physically possible states of the system, then a linear combination

$$|\alpha\rangle = c_1 |\alpha_1\rangle + c_2 |\alpha_2\rangle + \cdots \tag{2.3}$$

also represents a physically possible state of the quantum system.

In any mathematical theory, when we have a set of vectors we can always set up a set of vectors, which call the dual vectors. In the framework of the linear algebra, a dual space can be associated with any vector space, where we can postulate for every ket there exists a unique bra in the bra space, denoted by $\langle\alpha|$. There is a one-to-one correspondence between bras and kets, such that:

$$|\alpha\rangle \longleftrightarrow \langle\alpha| , \tag{2.4}$$

$$|\alpha\rangle + |\beta\rangle \longleftrightarrow \langle\alpha| + \langle\beta| , \tag{2.5}$$

$$c_\alpha |\alpha\rangle + c_\beta |\beta\rangle \longleftrightarrow c_\alpha^* \langle\alpha| + c_\beta^* \langle\beta| . \tag{2.6}$$

We now introduce the inner product of a bra and ket, where the bra being on the left and the ket being on the right, and the two vertical lines being contracted to one for brevity

$$(\langle\alpha|) . (|\beta\rangle) = \langle\alpha|\beta\rangle. \tag{2.7}$$

In general, the inner product is a complex number.

The inner products of kets and bras satisfy two fundamental properties. First

$$\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*, \tag{2.8}$$

where $\langle\alpha|\beta\rangle$ and $\langle\beta|\alpha\rangle$ are the complex conjugates of each other.

The second property is

$$\langle\alpha|\alpha\rangle \geq 0, \tag{2.9}$$

where the equality holds for a null ket.

We say that two kets are orthogonal if their inner product is zero. The norm of a ket is defined as the square root of the number $\langle\alpha|\alpha\rangle$. When a quantum state is represented by a ket $|\alpha\rangle$, the ket $c|\alpha\rangle$ represents the same physical state and the factor c should be chosen so that the ket is of norm unity. This procedure is called normalization and the ket so chosen is said to be normalized.

2.3 Operators

In the preceding section, we introduced the concept of of ket and bra vectors. We shall now consider a ket vector in the ket space that will lead to the concept of an operator.

An operator acts on a ket from the left side

$$X.\left(|\alpha\rangle\right) = X|\alpha\rangle, \tag{2.10}$$

which results another ket. The operator X is said to be completely defined when the result of its application to every ket in the ket space is given.

An operator X acts on a bra from the right side

$$\left(\langle\alpha|\right).X = \langle\alpha|X, \tag{2.11}$$

which results another bra. In general, $X|\alpha\rangle$ and $\langle\alpha|X$ are not dual to each other and we have

$$X\,|\alpha\rangle \longleftrightarrow \langle\alpha|\,X^{+}, \tag{2.12}$$

where X^{+} is called the Hermitian adjoint of X. The operator X is called Hermitian if

$$X^{+} = X. \tag{2.13}$$

An observable, such as coordinates, momentum and spin components of particles, can be represented by an operator A in the vector space. There exist an important set of ket vectors, $\{|a^{'}\rangle\}$, known as eigenkets of the operator A with the property

$$A\,|a^{'}\rangle = a^{'}\,|a^{'}\rangle\,, \tag{2.14}$$

where, $a^{'}$ is any number, called an eigenvalue of A. Here, the application of the observable A to an eigenket leads to the same eigenket apart from a multiplicative number. The physical state that corresponds to an eigenket is called an eigenstate.

The eigenvalues of a Hermitian operator A are real and its corresponding eigenkets to different eigenvalues are orthogonal.

Proof. We will start with assuming that the operator A has two eigenkets with two eigenvalues

$$A\,|a^{'}\rangle = a^{'}\,|a^{'}\rangle\,, \tag{2.15}$$

$$A\,|a^{''}\rangle = a^{''}\,|a^{''}\rangle \quad \longleftrightarrow \quad \langle a^{''}|\,A^{+} = a^{''*}\,\langle a^{''}|\,. \tag{2.16}$$

Using the fact that the operator A is Hermitian, we have

$$\langle a^{''}|\,A = a^{''*}\,\langle a^{'}|\,. \tag{2.17}$$

If we multiply both sides of Eq. (2.15) by $\langle a^{''}|$ on the left and multiply both sides of Eq. (2.16) by $|a^{'}\rangle$ on the right, we obtain

$$\langle a^{''}|\,A\,|a^{'}\rangle = a^{'}\langle a''|a'\rangle \tag{2.18}$$

$$\langle a^{''}|\,A\,|a^{'}\rangle = a^{''*}\langle a''|a'\rangle. \tag{2.19}$$

Subtract Eq. (2.18) from Eq. (2.19), we get

$$\left(a^{'} - a^{''*}\right) \langle a''|a'\rangle = 0. \tag{2.20}$$

Now, we consider two cases separately:
Case 1: For $a^{'} = a^{''}$, we have

$$\left(a^{'} - a^{'*}\right) \langle a^{'}|a^{'}\rangle = 0. \tag{2.21}$$

Since $\langle a'|a'\rangle > 0$, by assuming that $|a^{'}\rangle$ is not a null ket, we must have

$$a^{'} = a^{'*}. \tag{2.22}$$

Hence, the eigenvalues are always real.
Case 2: In the case of $a^{'} \neq a^{''}$ and by considering the reality condition, we have

$$\left(a^{'} - a^{''}\right) \langle a^{''}|a^{'}\rangle = 0. \tag{2.23}$$

This implies that

$$\langle a^{''}|a^{'}\rangle = 0, \tag{2.24}$$

which demonstrates the orthogonality condition of the eigenkets of A.

2.4 Base kets of ket space

It is convenient to consider the set of the eigenkets of A, $\{|a^{'}\rangle\}$, to be an orthonoromal set that satisfies

$$\langle a''|a'\rangle = \delta_{a''a'}, \tag{2.25}$$

where $\delta_{a''a'}$ is the Kronecker delta symbol defined by

$$\delta_{a''a'} = \begin{cases} 1 & \text{if } a^{''} = a^{'} \\ 0 & \text{if } a^{''} \neq a^{'} . \end{cases} \tag{2.26}$$

Thus, we can consider that the ket space is spanned by the eigenkets, as base kets, of A that forms a complete orthonormal set.

Any arbitrary ket $|\alpha\rangle$ in the ket space spanned by the eigenkets of A can be expanded as a linear combination

$$|\alpha\rangle = \sum_{a^{'}} c_{a'} \, |a^{'}\rangle \, , \tag{2.27}$$

with $a^{(1)}$, $a^{(2)}$,$\cdots$, $a^{(N)}$, where $c_{a'}$ are complex coefficients. Here, we have considered an N-dimensional vector space spanned by N eigenkets of the operator A.

Let us attempt to find an expression of the expansion coefficient $c_{a'}$. If we multiply $\langle a^{''}|$ on the left of Eq. (2.27), we obtain

$$\langle a''|\alpha\rangle = \sum_{a^{'}} c_{a'} \langle a''|a'\rangle. \tag{2.28}$$

By using the orthonormality condition given by Eq. (2.26), we get

$$\langle a''|\alpha\rangle = c_{a''}. \tag{2.29}$$

Hence, $c_{a'}$ also can be expressed as $c_{a'} = \langle a'|\alpha\rangle$. If the ket $|\alpha\rangle$ is normalized, $\langle \alpha|\alpha\rangle = 1$, the coefficients in the expansion of $|\alpha\rangle$ must verify the condition

$$\langle \alpha|\alpha\rangle = \sum_{a^{'}} \left|\langle a^{'}|\alpha\rangle\right|^2 = \sum_{a^{'}} |c_{a'}|^2 = 1. \tag{2.30}$$

2.5 Matrix representations

In the proceeding section, we introduced the following products: $\langle\alpha|\beta\rangle$, $X|\alpha\rangle$ and $\langle\beta|X$. In order to represent the operators in the framework of matrices, we need to consider the product of kets and bras. The resulting product

$$(|\alpha\rangle)\,.\,(\langle\beta|) = |\alpha\rangle\langle\beta|, \tag{2.31}$$

is called the outer product of $|\alpha\rangle$ and $\langle\beta|$. This product is seen as an operator which is different from the inner product.

We consider a ket space spanned by the base kets $\{|a'\rangle\}$. Using the completeness relation

$$\sum_{a'} |a'\rangle\,\langle a'| = \mathbb{1}, \tag{2.32}$$

an operator X can be written as

$$\begin{aligned} X &= \mathbb{1}.X.\mathbb{1} \\ &= \sum_{a''}\sum_{a'} |a''\rangle\,\langle a''|X|a'\rangle\,\langle a'|\,, \end{aligned} \tag{2.33}$$

where $\mathbb{1}$ represents the identity operator.

There are N^2 element of $\langle a''|X|a'\rangle$, where N is the dimensionality of the ket space. The operator X can be represented as

$$X \doteq \begin{pmatrix} \langle a^{(1)}|X|a^{(1)}\rangle & \langle a^{(1)}|X|a^{(2)}\rangle & \langle a^{(1)}|X|a^{(3)}\rangle & \cdots \\ \langle a^{(2)}|X|a^{(1)}\rangle & \langle a^{(2)}|X|a^{(2)}\rangle & \langle a^{(2)}|X|a^{(3)}\rangle & \cdots \\ \langle a^{(3)}|X|a^{(1)}\rangle & \langle a^{(3)}|X|a^{(2)}\rangle & \langle a^{(3)}|X|a^{(3)}\rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.34}$$

A ket $|\alpha\rangle$ is represented by a column matrix as follows:

$$|\alpha\rangle \doteq \begin{pmatrix} c_{a^{(1)}} \\ c_{a^{(2)}} \\ c_{a^{(3)}} \\ \vdots \end{pmatrix} = \begin{pmatrix} \langle a^{(1)}|\alpha\rangle \\ \langle a^{(2)}|\alpha\rangle \\ \langle a^{(3)}|\alpha\rangle \\ \vdots \end{pmatrix}. \tag{2.35}$$

A bra $\langle\alpha|$ is represented by a row matrix as follows:

$$\begin{aligned} \langle\alpha| &\doteq \begin{pmatrix} c^*_{a^{(1)}} & c^*_{a^{(2)}} & c^*_{a^{(3)}} & \cdots \end{pmatrix} \\ &= \begin{pmatrix} \langle\alpha|a^{(1)}\rangle & \langle\alpha|a^{(2)}\rangle & \langle\alpha|a^{(3)}\rangle & \cdots \end{pmatrix}. \end{aligned} \tag{2.36}$$

The matrix representing the inner product $\langle\beta|\alpha\rangle$ is given by

$$\begin{aligned} \langle\beta|\alpha\rangle &= \sum_{a'} \langle\beta|a'\rangle\langle a'|\alpha\rangle \\ &\doteq \begin{pmatrix} \langle\beta|a^{(1)}\rangle & \langle\beta|a^{(2)}\rangle & \langle\beta|a^{(3)}\rangle & \cdots \end{pmatrix} \begin{pmatrix} \langle a^{(1)}|\alpha\rangle \\ \langle a^{(2)}|\alpha\rangle \\ \langle a^{(3)}|\alpha\rangle \\ \vdots \end{pmatrix}. \end{aligned} \tag{2.37}$$

The matrix representing the outer product $|\beta\rangle\langle\alpha|$ is given by

$$
\begin{aligned}
|\beta\rangle\langle\alpha| &\doteq \begin{pmatrix} \langle a^{(1)}|\beta\rangle \\ \langle a^{(2)}|\beta\rangle \\ \langle a^{(3)}|\beta\rangle \\ \vdots \end{pmatrix} \begin{pmatrix} \langle\alpha|a^{(1)}\rangle & \langle\alpha|a^{(2)}\rangle & \langle\alpha|a^{(3)}\rangle & \cdots \end{pmatrix} \\
&= \begin{pmatrix} \langle a^{(1)}|\beta\rangle\langle\alpha|a^{(1)}\rangle & \langle a^{(1)}|\beta\rangle\langle\alpha|a^{(2)}\rangle & \langle a^{(1)}|\beta\rangle\langle\alpha|a^{(3)}\rangle & \cdots \\ \langle a^{(2)}|\beta\rangle\langle\alpha|a^{(1)}\rangle & \langle a^{(2)}|\beta\rangle\langle\alpha|a^{(2)}\rangle & \langle a^{(2)}|\beta\rangle\langle\alpha|a^{(3)}\rangle & \cdots \\ \langle a^{(3)}|\beta\rangle\langle\alpha|a^{(1)}\rangle & \langle a^{(3)}|\beta\rangle\langle\alpha|a^{(2)}\rangle & \langle a^{(3)}|\beta\rangle\langle\alpha|a^{(3)}\rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
\end{aligned}
\tag{2.38}
$$

The matrix representation of an observable becomes simple if its eigenkets are used as the base kets:

$$
\begin{aligned}
A &= \sum_{a''}\sum_{a'} |a''\rangle\langle a''|A|a'\rangle\langle a'|, \\
&= \sum_{a'} a'|a'\rangle\langle a'| \\
&\doteq \begin{pmatrix} a^{(1)} & 0 & 0 & \cdots \\ 0 & a^{(2)} & 0 & \cdots \\ 0 & 0 & a^{(3)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},
\end{aligned}
\tag{2.39}
$$

where $\{a^{(1)}, a^{(2)}, \cdots\}$ represent the eigenvalues of A.

2.6 Measurements

A measurement always causes the quantum system to jump into an eigenstate of the dynamical variable that is being measured. Suppose that the system is represented, before performing a measurement of the observable A, by the ket

$$|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle. \tag{2.40}$$

When the measurement is performed, the system is thrown into one of the eigenkets, $|a'\rangle$, of A and the result of measurement yields one of the eigenvalues of the observable. The probability for finding the system in the eigentate $|a'\rangle$ is defined by

$$P(a') = \left|\langle a'|\alpha\rangle\right|^2. \tag{2.41}$$

A selective measurement can be performed by applying the operator $M(a') = |a'\rangle\langle a'|$ on the ket $|\alpha\rangle$

$$M(a')|\alpha\rangle = |a'\rangle\langle a'|\alpha\rangle, \tag{2.42}$$

where $M(a')$ is called the projection operator. This process selects only one eigenkets of the observable A.

We define the expectation value of A with respect to the state ket $|\alpha\rangle$ as

$$\begin{aligned} \langle A\rangle &= \langle\alpha|A|\alpha\rangle, \\ &= \sum_{a'} a' \left|\langle a'|\alpha\rangle\right|^2, \end{aligned} \tag{2.43}$$

where a' is the measured value and $\left|\langle a'|\alpha\rangle\right|^2$ is the probability for obtaining a'.

2.7 Conclusion

This chapter aimed to elucidate the core mathematical framework for describing quantum mechanical systems. We explored the representation of quantum states using ket and bra vectors within a Hilbert space, clarifying their interrelationships. Additionally, we defined operators, detailing the properties of their eigenkets and eigenvalues. Operators were further examined through their matrix representations. Moreover, we analyzed the quantum measurement process, highlighting its role in selectively projecting a system into a desired state.

Chapter 3

Entanglement and Teleportation

3.1 Introduction

Since the early 1980s, a significant advancement has been the ability of physicists to manipulate and observe individual quantum entities—such as photons, atoms, and ions—beyond merely studying the collective behavior of large ensembles. This capability to control and measure single quantum objects forms the cornerstone of quantum information and computation, enabling their use as physical carriers of quantum bits. Entanglement, characterized by nonlocal correlations, arises in quantum systems involving two or more parties, necessitating the formalism of composite systems. This discussion centers on bipartite systems.

In this chapter, we describe the phenomenon of entanglement for bipartite system. We introduce the entanglement measure of the bipartite entanglement and determine whether the state is separable and entangled. Finally, we consider a practical application of quantum entanglement and fidelity to examine the quality of teleportation in the framework of spin-1/2 particles.

3.2 Composite systems and entanglement

We consider a bipartite composite system comprising two subsystems, labeled a and b. The Hilbert space of the composite system is defined as $H = H_a \otimes H_b$, where H_a and H_b represent the Hilbert spaces of subsystems a and b, respectively. This composite Hilbert space H is spanned by the tensor product basis kets $\{|i\rangle_a \otimes |j\rangle_b\}$, with $\{|i\rangle_a\}$ and $\{|j\rangle_b\}$ denoting the basis kets for H_a and H_b, respectively.

The state ket of the bipartite system, $a \oplus b$, is given in terms of the base kets as

$$|\Psi\rangle_{ab} = \sum_{i,j} c_{ij} |i\rangle_a \otimes |j\rangle_b \ . \tag{3.1}$$

The focus is on the bipartite case here, but the generalization to more than two subsystems proceeds the same manner.

For example if the subsystem a and subsystem b both have a spin-1/2 state. The space associated to the whole system is spanned by $\{|\uparrow\rangle_a \otimes |\uparrow\rangle_b,\ |\uparrow\rangle_a \otimes |\downarrow\rangle_b, |\downarrow\rangle_a \otimes |\uparrow_b\rangle, |\downarrow\rangle_a \otimes |\downarrow\rangle_b\}$ and the state of the system is given by

$$|\Psi\rangle_{ab} = c_{00}|\uparrow\uparrow\rangle_{ab} + c_{01}|\uparrow\downarrow\rangle_{ab} + c_{10}|\downarrow\uparrow\rangle_{ab} + c_{11}|\downarrow\downarrow\rangle_{ab}, \tag{3.2}$$

where the tensor product symbol is often omitted from the base kets.

The state given by Eq. (3.1) is said to be separable if it can be written as

$$|\Psi\rangle_{ab} = |\Phi\rangle_a \otimes |\varphi\rangle_b, \tag{3.3}$$

where $|\Phi\rangle_a \in H_a$ and $|\varphi\rangle_b \in H_b$. Nonseparable states are called entangled states that verifies

$$|\Psi\rangle_{ab} \neq |\Phi\rangle_a \otimes |\varphi\rangle_b. \tag{3.4}$$

The orthonormal basis of the space leads for example to the following entangled states

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \tag{3.5}$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \tag{3.6}$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \tag{3.7}$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right), \tag{3.8}$$

where $|0\rangle \equiv |\uparrow\rangle$ and $|1\rangle \equiv |\downarrow\rangle$. These states are called the Bell states and they play an important role in many protocols of the theory of quantum information. When discussing the entanglement, it is useful to imagine that the subsystem a and subsystem b are separated by a distance.

3.3 Measure of entanglement

The state ket of the bipartite system, $a \oplus b$, is given in terms of the base kets as

$$|\Psi\rangle_{ab} = c_{00}|\uparrow\uparrow\rangle_{ab} + c_{01}|\uparrow\downarrow\rangle_{ab} + c_{10}|\downarrow\uparrow\rangle_{ab} + c_{11}|\downarrow\downarrow\rangle_{ab}, \tag{3.9}$$

For a pure bipartite state of two particles, such as the state defined in equation (3.9), the concurrence is a standard measure of entanglement, as proposed by Wootters [3]. The concurrence C quantifies the degree of entanglement, with $C = 0$ for separable states and $C = 1$ for maximally entangled states. For a pure state $|\Psi\rangle_{ab}$ written in the computational basis as

$$|\Psi\rangle_{ab} = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle,$$

where $|0\rangle \equiv |\uparrow\rangle$, $|1\rangle \equiv |\downarrow\rangle$, and the state is normalized ($|c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1$), the concurrence is given by [3]

$$C = 2|c_{00}c_{11} - c_{01}c_{10}|. \tag{3.10}$$

This expression is derived by considering the spin-flipped state $|\tilde{\Psi}\rangle = (\sigma_y \otimes \sigma_y)|\Psi^*\rangle$, where σ_y is the Pauli Y matrix, and $|\Psi^*\rangle$ is the complex conjugate of $|\Psi\rangle$ in the computational basis. The concurrence is then $C = |\langle\Psi|\tilde{\Psi}\rangle|$, which simplifies to equation (3.10) for the state in equation (3.9).

For example, for a Bell state such as $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, the coefficients are $c_{00} = c_{11} = \frac{1}{\sqrt{2}}$, $c_{01} = c_{10} = 0$. Substituting into equation (3.10), we obtain

$$C = 2\left|\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - 0 \cdot 0\right| = 2 \cdot \frac{1}{2} = 1,$$

indicating maximal entanglement. For a separable state, such as $|00\rangle$, where $c_{00} = 1$, $c_{01} = c_{10} = c_{11} = 0$, the concurrence is

$$C = 2|1 \cdot 0 - 0 \cdot 0| = 0,$$

indicating no entanglement.

3.4 Quantum Teleportation

The discussion in the preceding section establishes the fundamental nature of entanglement. In this section, we show that the entanglement can be exploited for accomplishing certain quantum information processes. We consider an example of such processes: quantum teleportation.

The original protocol of teleportation was proposed by Bennett et al. [4] that is implemented through bell states. The teleportation is an interesting application of entangled states that leads to quantum information transfer.

Suppose that Alice has a particle in an unknown state that she would like to transmit to Bob. Here Alice wishes to send to Bob the information on the state of a spin-1/2 particle

$$|\psi\rangle_{A'} = \alpha|0\rangle_{A'} + \beta|1\rangle_{A'}, \tag{3.11}$$

without directly transmitting this particle to Bob. They share the entangled state $|\phi^+\rangle$ and they have to communicate classical information, which is illustrated schematically in Fig. 3.1. The principle of information transfer consists of using an auxiliary pair of entangled states A and B of spin-1/2 by Alice and Bob.

The joint state of the three particle, Alice's and Bob's particles, is written as

$$\begin{aligned} |\Phi\rangle_{A'AB} &= (\alpha|0\rangle_{A'} + \beta|1\rangle_{A'}) \otimes \frac{1}{\sqrt{2}} (|0\rangle_A \otimes (|0\rangle_B + |1\rangle_A \otimes |1\rangle_B) \\ &= \frac{\alpha}{\sqrt{2}}|0\rangle_{A'} \otimes (|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) \\ &+ \frac{\beta}{\sqrt{2}}|1\rangle_{A'} \otimes (|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)\,, \end{aligned} \tag{3.12}$$

which by using Eqs. (3.5)–(3.8), can be re-expressed as

$$\begin{aligned}
|\Phi\rangle_{A'AB} &= \left(\frac{\alpha}{\sqrt{2}}|0\rangle_{A'}\otimes|0\rangle_A + \frac{\beta}{\sqrt{2}}|1\rangle_{A'}\otimes|0\rangle_A\right)\otimes|0\rangle_B \\
&+ \left(\frac{\alpha}{\sqrt{2}}|0\rangle_{A'}\otimes|1\rangle_A + \frac{\beta}{\sqrt{2}}|1\rangle_{A'}\otimes|1\rangle_A\right)\otimes|1\rangle_B \\
&= \frac{1}{2}\Big(|\phi^+\rangle_{A'A}\otimes(\alpha|0\rangle_B+\beta|1\rangle_B) + |\phi^-\rangle_{A'A}\otimes(\alpha|0\rangle_B-\beta|1\rangle_B) \\
&+ |\psi^+\rangle_{A'A}\otimes(\beta|0\rangle_B+\alpha|1\rangle_B) + |\psi^-\rangle_{A'A}\otimes(-\beta|0\rangle_B+\alpha|1\rangle_B)\Big).
\end{aligned} \tag{3.13}$$

This implies that if Alice performs a Bell measurement on her particles A and A' in the Bell basis, her measurement outcomes are one of the four Bell states $|\phi^+\rangle_{A'A}, |\phi^-\rangle_{A'A}, |\psi^+\rangle_{A'A}, |\psi^-\rangle_{A'A}$, each of which occurs with probability 1/4. Furthermore, it can be seen from Eq. (3.13) that the subsequent state of Bob's particle corresponding to Alice's measurement outcomes is

$$\begin{aligned}
\alpha|0\rangle_B+\beta|1\rangle_B &= U_1|\phi\rangle_B \\
\alpha|0\rangle_B-\beta|1\rangle_B &= U_2|\phi\rangle_B \\
\beta|0\rangle_B+\alpha|1\rangle_B &= U_3|\phi\rangle_B \\
-\beta|0\rangle_B+\alpha|1\rangle_B &= U_4|\phi\rangle_B,
\end{aligned} \tag{3.14}$$

where the U_i operators are given by

$$U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ U_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, U_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, U_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.15}$$

Now suppose that Alice sends to Bob her measurements via a classical channel, then the Eq. (3.13) implies that Bob can recover the desired state $|\psi\rangle_{A'}$ by applying the corresponding unitary transformation to his particle B.

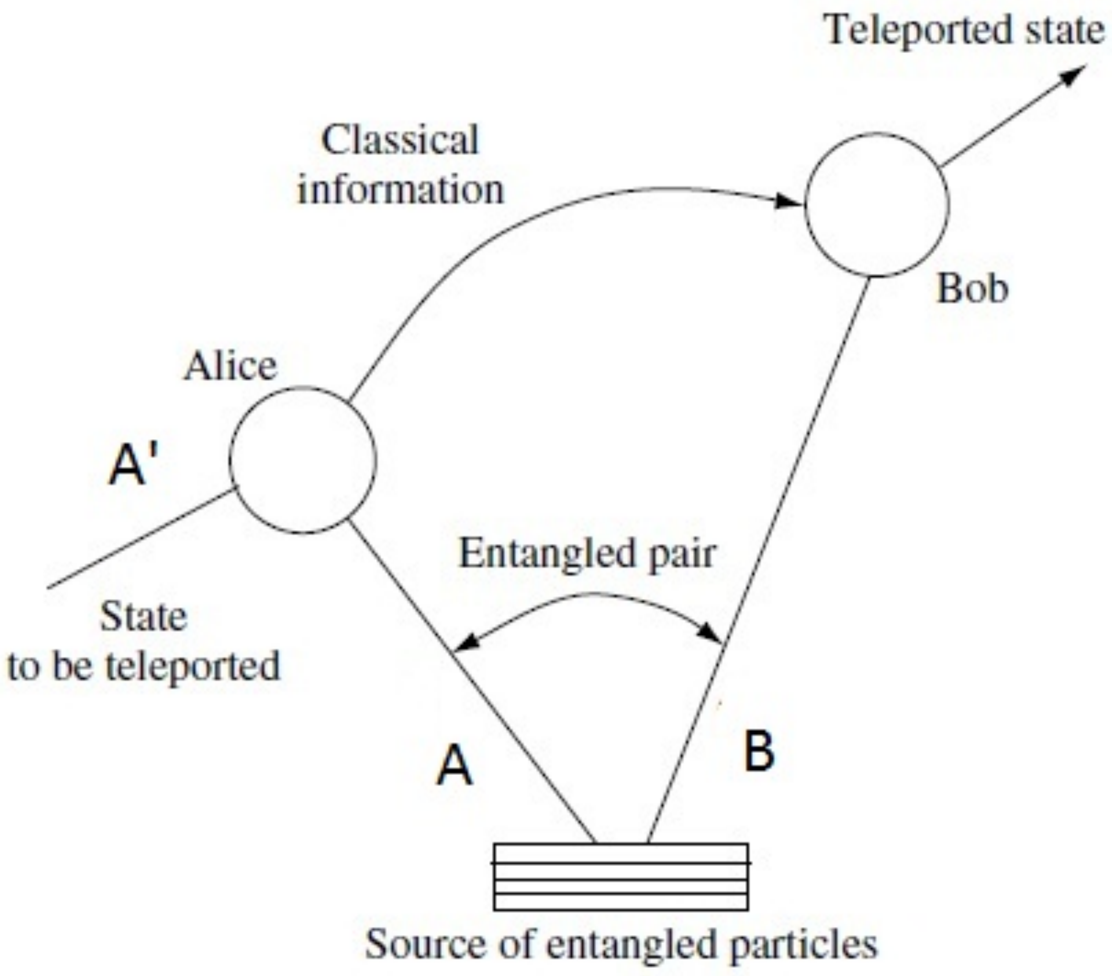

FIGURE 3.1: Schematic diagram of the teleportation protocol.

3.5 Fidelity of quantum teleportation

The quality of quantum teleportation can be evaluated by calculating the fidelity. The fidelity is defined as

$$F = {}_{A'}\langle\phi|\rho_B|\phi\rangle_{A'} \tag{3.16}$$

This quantity gives the information of how close the teleported state ρ_B out is to the state $\rho_{A'} = |\phi\rangle_{A'}\langle\phi|$ in to be teleported, i.e., they are equal when $F = 1$ and orthogonal when $F = 0$. In the present work, we would like to investigate the variation of fidelity when the teleportation is executed in the presence of $|\psi\rangle_{AB} = \delta|00\rangle_{AB} + \gamma|11\rangle_{AB}$.

If the state projects the whole system into the state $|\Phi_+\rangle\langle\Phi_+|$

$$|\Phi^+\rangle_{A'A}\langle\Phi^+| \otimes \left[{}_{A'A}\langle\Phi_+| \left(|\phi\rangle_{A'}\langle\phi| \otimes \rho_{AB}\right)|\Phi_+\rangle_{A'A}\right], \tag{3.17}$$

where

$$\begin{aligned} |\phi\rangle_{A'}\langle\phi| &= \left(\alpha|0\rangle_{A'} + \beta|1\rangle_{A'}\right)\left(\alpha^\star_{A'}\langle 0| + \beta^\star_{A'}\langle 1|\right) \\ &= |\alpha|^2|0\rangle_{A'}\langle 0| + \alpha\beta^\star|0\rangle_{A'}\langle 1| + \beta\alpha^\star|1\rangle_{A'}\langle 0| + |\beta|^2|1\rangle_{A'}\langle 1|, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \rho_{AB} &= \left(\gamma|00\rangle_{AB} + \delta|11\rangle_{AB}\right)\left(\gamma^\star\ {}_{AB}\langle 00| + \delta^\star_{AB}\langle 11|\right) \qquad (3.19) \\ &= |\gamma|^2|0\rangle_A\langle 0| \otimes |0\rangle_B\langle 0| + \gamma\delta^\star|0\rangle_A\langle 1| \otimes |0\rangle_B\langle 1| + \delta\gamma^\star|1\rangle_A\langle 0| \otimes |1\rangle_B\langle 0| + |\delta|^2|1\rangle_A\langle 1| \otimes |1\rangle_B\langle 1|/. \end{aligned}$$

By the tedious calculations, we can obtain

$$\begin{aligned} &|\Phi^+\rangle_{A'A}\langle\Phi^+| \otimes \left[{}_{A'A}\langle\Phi_+| \left(|\phi\rangle_{A'}\langle\phi| \otimes \rho_{AB}\right)|\Phi_+\rangle_{A'A}\right] = \\ &|\Phi^+\rangle_{A'A}\langle\Phi^+| \otimes \Big[\frac{|\alpha|^2|\gamma|^2}{2}|0\rangle_B\langle 0| + \frac{\alpha\beta^\star\gamma\delta^\star}{2}|0\rangle_B\langle 1| + \frac{\beta\alpha^\star\delta\gamma^\star}{2}|1\rangle_B\langle 0| + \frac{|\beta|^2|\delta|^2}{2}|1\rangle_B\langle 1|\Big]. \end{aligned} \tag{3.20}$$

By the tracing over the first two particles, we can obtain

$$\rho_B = \frac{1}{2Z}\left[|\alpha|^2|\gamma|^2|0\rangle_B\langle 0| + \alpha\beta^\star\gamma\delta^\star|0\rangle_B\langle 1| + \beta\alpha^\star\delta\gamma^\star|1\rangle_B\langle 0| + |\beta|^2|\delta|^2|1\rangle_B\langle 1|\right], \tag{3.21}$$

where

$$Z = \frac{1}{2}\left(|\alpha|^2|\gamma|^2 + |\beta|^2|\delta|^2\right). \tag{3.22}$$

such that

$$|\alpha|^2 + |\beta|^2 = 1 \qquad\qquad |\delta|^2 + |\gamma|^2 = 1 \tag{3.23}$$

From Eq. (3.21), it is very easy to obtain the fidelity of the teleported state

$$\begin{aligned} F &= {}_{A'}\langle\phi|\rho_B|\phi\rangle_{A'} \\ &= \begin{pmatrix} \alpha^\star & \beta^\star \end{pmatrix} \begin{pmatrix} \frac{|\alpha|^2|\gamma|^2}{2Z} & \frac{\alpha\beta^\star\gamma\delta^\star}{2Z} \\ \frac{\alpha^\star\beta\gamma^\star\delta}{2Z} & \frac{|\beta|^2|\delta|^2}{2Z} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \frac{1}{2Z}\left(|\alpha|^4|\gamma|^2 + |\beta|^4|\delta|^2\right) + \frac{|\alpha|^2|\beta|^2}{2Z}\left(\gamma^\star\delta + \gamma\delta^\star\right). \end{aligned}$$

The fidelity is plotted as a function of δ for $\alpha = \beta = 1/\sqrt{2}$ in Fig. 3.2. From the figure, the maximum of the fidelity is obtained at $\gamma = \delta = 1/\sqrt{2}$, which corresponds to maximal entangled state.

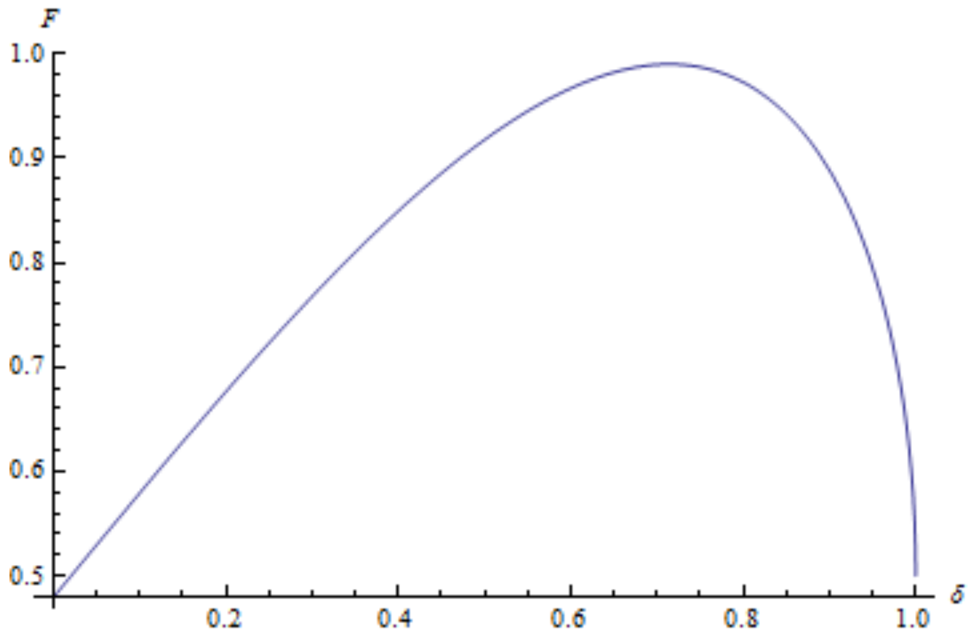

FIGURE 3.2: Fidelity for teleported state in the case of $|\Phi_+\rangle$ with $\alpha = \beta = 1/\sqrt{2}$.

3.6 Conclusion

In this chapter, we explored the phenomenon of quantum entanglement, focusing on nonlocal correlations in bipartite systems. We first established the mathematical framework necessary to describe composite quantum systems and quantify entanglement. We then highlighted a key application of entanglement, quantum teleportation, and detailed its operational procedure. Additionally, we analyzed the fidelity metric to assess the quality of information transfer in teleportation. In conclusion, we showcased quantum teleportation as a pivotal application of two-level systems and entanglement, evaluating the efficiency of information transmission through fidelity across system parameters.

Chapter 4

Conclusion

We have provided an introductory exploration of quantum entanglement and its applications, with a particular focus on quantum teleportation through the concept of spin of particles. We began with a foundational review of quantum mechanics principles—state vectors, operators, and measurements—which are essential for understanding the phenomenon of entanglement. We established the mathematical framework necessary for describing quantum systems, including the use of state vectors to represent quantum states, operators to describe physical observables, and the measurement postulate to predict experimental outcomes. We have examined entanglement in bipartite systems for spin-1/2 particles. Finally, quantum teleportation was presented as a key application, illustrating how entangled states enable the transmission of quantum information described in the quantum states and the analysis of teleportation fidelity by demonstrating the optimal conditions.

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