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Characterization and Analysis of Properties in Optical Step-Index Fiber

A graduation project submitted to the Department of Physics in partial fulfillment of the requirements for the degree of Bachelor of Science in Applied Physics

By

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المستخلص باللغة العربية

الألياف الضوئية هي الأساس للعديد من التطبيقات سريعة التوسع، مثل الاتصالات السلكية واللاسلكية، وأجهزة الاستشعار، وأجهزة قياس الجرعات. في هذا المشروع، نستعرض اهم المبادئ النظرية لأبسط الألياف الضوئية. نقدم في البداية المفاهيم الرئيسية من النظرية الكهرومغناطيسية، والتي تشكل أساس دراستنا، ونفحص انتشار الموجات الكهرومغناطيسية عند الحدود، وهو جانب مهم من الديناميكا الكهربائية. أخيرًا، نقدم التحليل النظري لمكونات المجال وتوزيع الطاقة الضوئية في قلب وغلاف الألياف الضوئية ذات مؤشر الخطوة، مع إجراء عمليات محاكاة باستخدام MAPLE .

Abstract

Optical fibers are the foundation for many rapidly expanding applications, such as telecommunications, sensors, and dosimeters. In this work, we delve into the theoretical principles behind optical fibers. We revisit key concepts from electromagnetic theory, which forms the basis of our study, and examine the propagation of electromagnetic waves at boundaries, an important aspect of electrodynamics. Lastly, we present the theoretical analysis of the field components and optical power distribution in the core and cladding of step-index optical fibers, with simulations performed using MAPLE.

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Introduction

Optical Fiber refers to a specialized dielectric waveguide designed to guide light waves [1]. It serves as the core component in optical fiber communication systems [2,3], widely recognized as the most advanced and promising transmission medium for various forms of digital and data communication. Unlike traditional conductors that rely on electrical currents, optical fibers allow electromagnetic light waves to propagate through transparent materials [4], such as glass or plastic, via the principle of total internal reflection [5]. This enables efficient light transmission like the way electromagnetic waves propagate through metallic conductors [6].

The performance of an optical fiber is crucial in determining the efficiency of an optical communication system [7].

Several key questions arise when exploring optical fibers, such as:

1. What is the structure of an optical fiber?
2. How does light propagate within the fiber?
3. What materials are used in fiber fabrication?
4. How are optical fibers manufactured?
5. How are fibers incorporated into cable structures?
6. What causes signal loss or attenuation in the fiber?
7. How and why does signal distortion occur during transmission?

This project aims to address these fundamental questions, providing a comprehensive understanding of the physical structure and waveguiding properties of optical fibers.

This project will introduce the key characteristics of optical fibers, discussing relevant theoretical concepts and practical applications. Additionally, it will provide essential background information and define important terms frequently used in the field of optical fiber technology.

Furthermore, the project will explore the motivation behind the development of optical fiber systems, particularly in the context of waveguide optics, which have become increasingly attractive for sensing applications due to their:

1. High sensitivity.
2. Immunity to electromagnetic interference (EMI).
3. Small size, lightweight structure, and flexibility.
4. Ability to withstand harsh chemical environments.
5. Easy integration with fiber optic networks.
6. Fast detection time.

By examining these aspects, this project aims to offer valuable insights into the design, operation, and potential applications of optical fibers [8-14].

This project presents a comprehensive review of recent advancements in optical fibers, focusing on guided light propagation, which has seen substantial and rapid progress.

Chapter 1 introduces the fundamental concepts and equations of electromagnetic (EM) wave theory that are essential for understanding light-wave propagation in optical waveguides.

In Chapter 2, Maxwell's equations and boundary conditions are detailed, as they form the foundation for the analysis presented throughout the project.

Chapter 3 delves into the theoretical framework and supporting calculations necessary to comprehend the basic properties of step-index optical fibers. The derivation of dispersion equations is also explained in detail, offering insight into the dispersion characteristics of optical fibers.

Chapter 4 explores the interesting results of the required numerical methods. Numerical calculations, performed using MATLAB software, apply the beam propagation method to facilitate mode analysis of optical fibers. We conclude with some important remarks.

Chapter 1

Electromagnetic Theory

1.1 Introduction

The Scottish physicist James Clerk Maxwell (1831-1879) is viewed as the founder of electromagnetic theory in its present form. Maxwell's celebrated work managed the discovery of electromagnetic waves. Through his theoretical efforts when he was between 35 and 40 years old, Maxwell published the first unified theory of electricity and magnetism. The theory comprised all previously known results, both experimental and theoretical, on electricity and magnetism. It further introduced displacement current and predicted the existence of electromagnetic waves. Maxwell's equations were not fully accepted by many scientists until 1888 when they were confirmed by Heinrich Rudolf Hertz (1857-1894). The German physicist was successful in generating and detecting radio waves. The laws of electromagnetism that Maxwell put together in the form of four equations. The integral form of Maxwell's equations depicts the underlying physical laws, whereas the differential form is used more frequently in solving problems. For a field to "qualify" as an electromagnetic field, it must satisfy all four of Maxwell's equations. The importance of Maxwell's equations cannot be overemphasized because they summarize all known laws of electromagnetism. We shall often refer to them in the remainder of this text. Since this section is meant to be a compendium of our discussion in this text, it is worthwhile to mention other equations that go hand in hand with Maxwell's equations. The Lorentz force equation

$$\mathbf{F} = \mathrm{Q}(\mathbf{E} + \boldsymbol{u} \times \mathbf{B}), \tag{1.1}$$

where **F** is the force exerted on the charge Q, **E** is the electric field intensity, **u** is the velocity of the charge Q, and **B** is the magnetic flux density.

1.2 Maxwell's Equations

Eq. (1.1) is associated with the electromagnetic laws that are governed by the called Maxwell's equations, which can be written in the following forms.

1.2.1 Maxwell's Equations in Integral Form

$$\oint_{s} \mathbf{E}\,.\,d\mathbf{S} = \int_{v} \frac{\rho_v}{\varepsilon_o}\,dv, \tag{1.2}$$

$$\oint_{s} \mathbf{H}\,.\,d\mathbf{S} = 0, \tag{1.3}$$

$$\oint_{L} \mathbf{E}\,.\,d\boldsymbol{l} = -\int_{s} \mu_o \frac{\partial \mathbf{H}}{\partial t}\,.\,d\mathbf{S}, \tag{1.4}$$

$$\oint_{L} \mathbf{H}\,.\,d\boldsymbol{l} = \int_{s} \left(\mathbf{J} + \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}\right).\,d\mathbf{S}, \tag{1.5}$$

where ρ_v is the volume charge density, the constant ε_o is known as the permittivity of free space, $\boldsymbol{\nabla}$ is the vector differential operator, **B** is related to the magnetic field intensity (**H**) in free space as $\mathbf{B} = \mu_o\mathbf{H}$, and μ_o is a constant known as the permeability of free space, $\mathbf{J}$ is the conduction current density ($\mathbf{J} = \sigma \boldsymbol{E}$), σ is the conductivity, and $\varepsilon_o \frac{\partial \mathbf{E}}{\partial t}$ is the displacement current density ($\boldsymbol{J}_d = \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}$).

1.2.2 Maxwell's Equations in Differential Form

$$\boldsymbol{\nabla} \,.\, \mathbf{E} = \frac{\rho_v}{\varepsilon_o}, \tag{1.6}$$

$$\boldsymbol{\nabla} \,.\, \mathbf{H} = 0, \tag{1.7}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t}, \tag{1.8}$$

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathrm{J} + \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}, \tag{1.9}$$

1.2.3 Continuity Equation

Due to the principle of charge conservation, thus current I_{out} coming out of the closed surface is

$$I_{out} = \oint_s \mathbf{J} . d\mathbf{S} = -\frac{dQ_{in}}{dt}, \quad (1.10)$$

where Q_{in} is the total charge enclosed by the closed surface. Invoking the divergence theorem

$$\oint_s \mathbf{J} . d\mathbf{S} = \int_v \nabla . \mathbf{J} \, dv, \quad (1.11)$$

$$-\frac{dQ_{in}}{dt} = -\int_v \frac{d\rho_v}{dt} . d\mathbf{S}, \quad (1.12)$$

Substituting Eqs. (1.11) and (1.12) into Eq.(1.10)gives

$$\nabla . \mathbf{J} = -\frac{\partial \rho_v}{\partial t}, \quad (1.13)$$

which is called the continuity of current equation or just continuity equation.

1.3 Constitutive Relations

The concepts of linearity, isotropy, and homogeneity of a material medium still apply for time-varying fields; in a linear, homogeneous, and isotropic medium characterized by σ, ε, and μ, the constitutive relations

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_o \mathbf{E} + \mathbf{P}, \quad (1.14)$$

where **D** is the electric flux density, which is related to the electric field intensity (**E**), in free space as $\mathbf{D} = \varepsilon_o \mathbf{E}$, ε is the permittivity of the medium ($\varepsilon = \varepsilon_o \varepsilon_r$), ε_r is the dielectric constant or relative permittivity, ε_o is the permittivity of free space, **P** is the polarization (**P**=Q**d**), **d** is the distance vector from -Q to +Q of the dipole,

$$\mathbf{B} = \mu \mathbf{H} = \mu_o (\mathbf{H} + \mathbf{M}), \quad (1.15)$$

where μ is the permeability of the material ($\mu = \mu_o \mu_r$), μ_r is the relative permeability of the material, μ_o is the permeability of free space, $\mathbf{M}$ is the magnetization $\mathbf{M} = \chi_m \mathbf{H}$, χ_m is the magnetic susceptibility of the medium,

$$\mathbf{J} = \sigma \mathbf{E} + \rho_v \boldsymbol{u}, \tag{1.16}$$

hold for time-varying fields.

1.4 Boundary Conditions

These boundary conditions can be proved and here we will give a simple form that can be used in practice. The boundary conditions remain valid for time-varying fields, where $\mathbf{a}_n$ is the unit normal vector to the boundary.

$$E_{1t} - E_{2t} = 0 \text{ or } (\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{a}_n = 0, \tag{1.17a}$$

$$H_{1t} - H_{2t} = K \text{ or } (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_n = \mathbf{K}, \tag{1.18}$$

$$D_{1n} - D_{2n} = \rho_s \text{ or } (\mathbf{D}_1 - \mathbf{D}_2).\,\mathbf{a}_n = \rho_s, \tag{1.19}$$

$$B_{1n} - B_{2n} = 0 \text{ or } (\mathbf{B}_2 - \mathbf{B}_1).\,\mathbf{a}_n = 0, \tag{1.20}$$

where ρ_s is the free charge density placed deliberately at the boundary, and K is the surface current on the boundary.

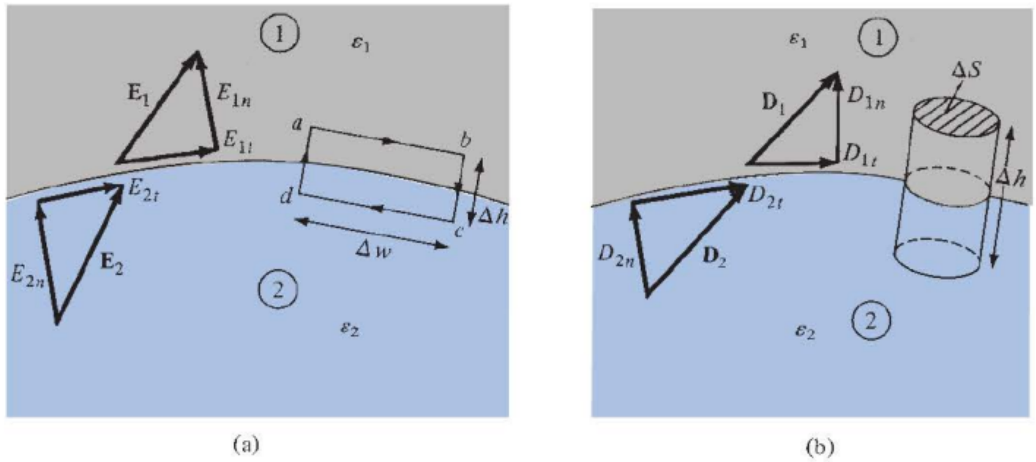

Fig 1.1 Boundary conditions between two electric media: (a) for $\mathbf{E} = \mathbf{E_t} + \mathbf{E_n}$ the tangential and normal components of **E** to the interface, (b) for **D**.

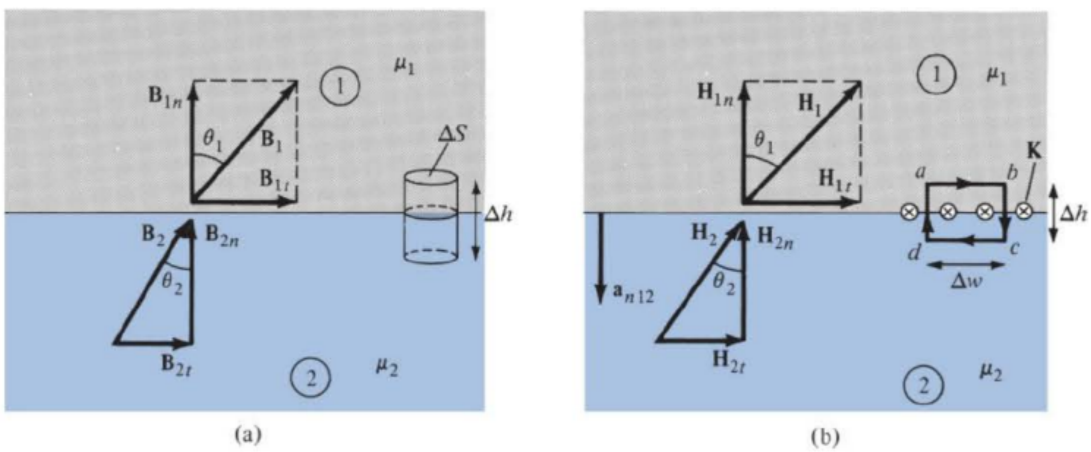

Fig 1.2 Boundary conditions between two magnetic media: (a) for **B**, (b) for **H**.

1.5 Maxwell's Equations in Free Space

For free space, we know ($\rho_v = 0$, $J = 0$), Maxwell's equations in differential form can be written for **E** and **H** in terms of free charges and currents as

$$\nabla \cdot \mathbf{E} = 0 \qquad (1.18)$$

$$\nabla \cdot \mathbf{H} = 0 \qquad (1.19)$$

$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t} , \quad (1.20)$$

$$\nabla \times \mathbf{H} = \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} . \quad (1.21)$$

These equations can be used to derive some of the basic properties of a light wave.

1.6 Wave Equation in Free Space

We take the curl of Eq. (1.20) and use Eq. (1.21),

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\mu_o \frac{\partial \mathbf{H}}{\partial t}\right), \quad (1.22)$$

$$\nabla(\nabla . \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_o \frac{\partial}{\partial t}(\nabla \times \mathbf{H}), \quad (1.23)$$

because free space is free of charge, $\nabla . \mathbf{E} = 0$, giving us

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad (1.21)$$

we can use the same procedure to obtain

$$\nabla^2 \mathbf{H} = \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}, \quad (1.22)$$

these equations are wave equations, the velocity of the wave in a vacuum,

$$c = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = 3 \times 10^8 \, [\mathrm{m}s^{-1}]. \quad (1.23)$$

1.7 Electromagnetic Wave Propagation

our major goal is to solve Maxwell's equations and describe EM wave motion in the following media:

1. Free space ($\sigma = 0, \varepsilon = \varepsilon_o, \mu = \mu_o$)
2. Lossless dielectrics ($\sigma \simeq 0, \varepsilon = \varepsilon_o\varepsilon_r, \mu = \mu_o\mu_r$, or $\sigma \ll \omega\varepsilon$)
3. Lossy dielectrics ($\sigma \neq 0, \varepsilon = \varepsilon_o\varepsilon_r, \mu = \mu_o\mu_r$)
4. Good conductors ($\sigma \simeq \infty, \varepsilon = \varepsilon_o\varepsilon_r, \mu = \mu_o\mu_r$, or $\sigma \gg \omega\varepsilon$)

where ω is the angular frequency of the wave. Case 3, for lossy dielectrics, is the most general case and will be considered first. Once this general case has been solved, we simply derive the other cases (1, 2, and 4) from it as special cases by changing the values of σ, μ, and ε.

1.7.1 Wave propagation in lossy dielectrics

A lossy dielectric is a medium in which an EM wave, as it propagates, loses power owing to an imperfect dielectric. In other words, a lossy dielectric is a partially conducting medium (imperfect dielectric or imperfect conductor) with $\sigma \neq 0$, as distinct from a lossless dielectric (perfect dielectric) in which $\sigma = 0$.

Consider a linear, isotropic, homogeneous, lossy dielectric medium that is charge-free (macroscopic $\rho_v = 0$). Assuming and suppressing the time factor $e^{i\omega t}$, Maxwell's equations become

$$\nabla \cdot \mathbf{E}_s = 0, \tag{1.27}$$

$$\nabla \cdot \mathbf{H}_s = 0, \tag{1.28}$$

$$\nabla \times \mathbf{E}_s = -i\omega\mu\mathbf{H}_s, \tag{1.29}$$

$$\nabla \times \mathbf{H}_s = (\sigma + i\omega\varepsilon)\mathbf{E}_s, \tag{1.30}$$

where $\mathbf{E}_s$, $\mathbf{H}_s$ is the phasor form of **E**, **H**. Taking the curl of both sides of eq. (1.29) gives

$$\nabla \times (\nabla \times \mathbf{E}_s) = -i\omega\mu(\nabla \times \mathbf{H}_s), \tag{1.31}$$

$$\nabla(\nabla \cdot \mathbf{E}_s) - \nabla^2\mathbf{E}_s = -i\omega\mu(\sigma + i\omega\varepsilon)\mathbf{E}_s, \tag{1.32}$$

$$\nabla^2\mathbf{E}_s - \gamma^2\mathbf{E}_s = 0, \tag{1.33}$$

or

$$\nabla^2 \mathbf{E}_s - \gamma^2 \mathbf{E}_s = 0, \tag{1.33}$$

where

$$\gamma^2 = i\omega\mu(\sigma + i\omega\varepsilon), \tag{1.34}$$

and γ, in reciprocal meters, is called the propagation constant of the medium. By a similar procedure, it can be shown that for the **H** field

$$\nabla^2 \mathbf{H}_s - \gamma^2 \mathbf{H}_s = 0, \tag{1.35}$$

Since γ in Eqs. (1.33) to (1.35) is a complex quantity, we may let

$$\gamma = \alpha + i\beta, \tag{1.36}$$

We obtain α and β from Eqs. (1.34) and (1.36)

$$\alpha = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}, \tag{1.37}$$

$$\beta = \omega \sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]}, \tag{1.38}$$

where α is known as the attenuation constant of the medium. It is a measure of the spatial rate of decay of the wave in the medium, measured in nepers per meter (Np/m) and can be expressed in decibels per meter (dB/m). An attenuation of 1 neper denotes a reduction to e^{-1} of the original value, whereas an increase of 1 neper indicates an increase by a factor of e. Hence, for voltages

$$1\text{Np} = 20\log_{10}e = 8.686\ \text{dB}. \tag{1.39}$$

We notice that if $\sigma = 0$, as is the case for a lossless medium and free space, $\alpha = 0$ and the wave is not attenuated as it propagates. The quantity β is a measure of the phase shift per unit length in radians per meter and is called the phase constant or wave number. In terms of β, the wave velocity u and wavelength λ are, respectively, given by

$$u = \frac{\omega}{\beta}, \qquad \lambda = \frac{2\pi}{\beta}. \tag{1.40}$$

Without loss of generality, if we assume that a wave propagates along $+\mathbf{a}_z$ and that $\mathbf{E}_s$ has only an x-component, then

$$\mathbf{E}_s = E_{xs}(z)\mathbf{a}_x, \tag{1.41}$$

We then substitute into eq. (1.33), which yields

$$(\nabla^2 - \gamma^2)E_{xs}(z) = 0, \tag{1.42}$$

if we assume that a wave propagates in an unbounded medium along $\mathbf{a}_z$ and that **E** has only an x-component that does not vary with x and y, then

$$\frac{d^2E_{xs}(z)}{dz^2} - \gamma^2E_{xs}(z) = 0, \tag{1.43}$$

This is a scalar wave equation, a linear homogeneous differential equation, with a solution

$$E_{xs}(z) = E_o e^{-\gamma z} + E'_o e^{\gamma z}, \tag{1.44}$$

where E_o and E'_o are constants. The fact that the field must be finite at infinity requires that $E'_o = 0$. Inserting the time factor $e^{i\omega t}$ into eq. (1.44) and using eq. (1.36), we obtain

$$\mathbf{E}(z,t) = \mathrm{Re}\left[E_o e^{-\alpha z} e^{i(\omega t-\beta z)} \mathbf{a}_x\right] = E_o e^{-\alpha z} \cos(\omega t - \beta z)\, \mathbf{a}_x. \tag{1.45}$$

A sketch of $|\mathbf{E}|$ at times $t = 0$ and $t = \Delta t$ is portrayed in Figure1.3, where it is evident that **E** has only an x-component and it is traveling in the +z-direction.

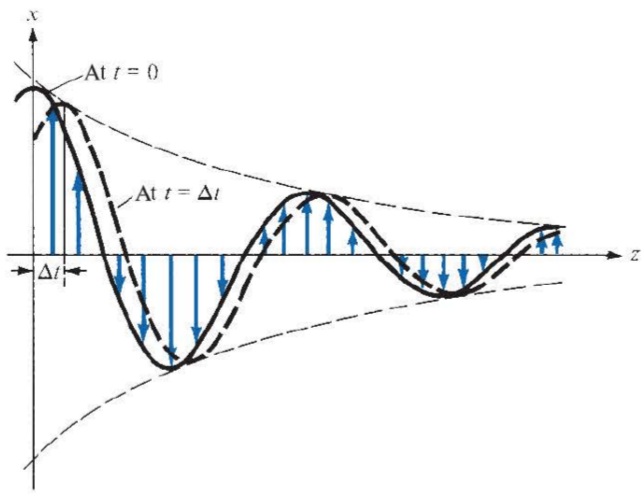

Fig 1.3 An E-field with an x-component traveling in the +z-direction at times $t = 0$ and $t = \Delta t$; arrows indicate instantaneous values of E.

We obtain $\mathbf{H}(z,t)$ by using eq. (1.45) in conjunction with Maxwell's equations

$$\mathbf{H}(z,t) = \mathrm{Re}\left[H_o e^{-\alpha z} e^{i(\omega t-\beta z)} \mathbf{a}_y\right], \tag{1.46}$$

where

$$H_o = \frac{E_o}{\eta}, \tag{1.47}$$

and η is a complex quantity known as the intrinsic impedance, in ohms, of the medium.

$$\eta = \sqrt{\frac{i\omega\mu}{\sigma + i\omega\varepsilon}} = |\eta| e^{i\theta_\eta}, \tag{1.48}$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}, \qquad \tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}, \tag{1.49}$$

where $0 \leq \theta_\eta \leq 45^\circ$. Substituting eqs. (1.47) and (1.48) into eq. (1.46) gives

$$\mathbf{H} = \mathrm{Re}\left[\frac{E_o}{|\eta| e^{i\theta_\eta}} e^{-\alpha z} e^{i(\omega t - \beta z)} \mathbf{a}_y\right], \tag{1.50}$$

$$\mathbf{H} = \frac{\mathrm{E_o}}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)\, \mathbf{a}_y, \tag{1.51}$$

Notice from eqs. (1.45) and (1.51) that as the wave propagates along $\mathbf{a}_z$, it decreases or attenuates in amplitude by a factor $e^{-\alpha z}$.

1.7.2 Lossless dielectric medium

In a lossless dielectric case, $\sigma \ll \omega\varepsilon$. It is a special case of lossy dielectric except that

$$\sigma \simeq 0, \qquad \varepsilon = \varepsilon_o \varepsilon_r, \qquad \mu = \mu_o \mu_r, \tag{1.52}$$

Substituting these into Eqs. (1.37) and (1.38) gives

$$\sigma = 0, \qquad \beta = \omega\sqrt{\mu\varepsilon}, \tag{1.53}$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \qquad \lambda = \frac{2\pi}{\beta}, \tag{1.54}$$

also

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}, \qquad \theta_\eta = 0^\circ, \tag{1.55}$$

and thus, **E** and **H** are in the time phase with each other.

1.7.3 Free Space medium

In free space

$$\sigma = 0, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o, \tag{1.56}$$

This may also be regarded as a special case of the lossless dielectric medium

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu_o \varepsilon_o} = \frac{\omega}{c}, \tag{1.57}$$

$$u = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = c, \qquad \lambda = \frac{2\pi}{\beta}, \tag{1.58}$$

By substituting the constitutive parameters in eq. (1.56) into eq. (1.49), $\theta_\eta = 0^o$ and $\eta = \eta_o$, where η_o is called the intrinsic impedance of free space and is given by

$$\eta = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi \simeq 377\Omega. \tag{1.59}$$

$$\mathbf{E} = \mathrm{E}_o \cos(\omega t - \beta z)\mathbf{a}_x, \tag{1.60}$$

then

$$\mathbf{H} = \frac{\mathrm{E}_o}{\eta_o} \cos(\omega t - \beta z)\, \mathbf{a}_y, \tag{1.61}$$

The plots of **E** and **H** are shown in Figure 1.4. In general, if $\mathbf{a}_E$, $\mathbf{a}_H$, and $\mathbf{a}_k$ are unit vectors along the **E** field, the **H** field, and the direction of wave propagation

$$\mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_H, \qquad \mathbf{a}_k \times \mathbf{a}_H = -\mathbf{a}_E, \qquad \mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_k, \tag{1.62}$$

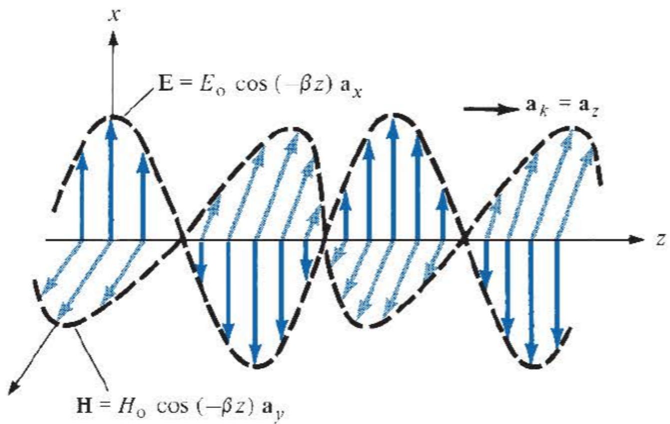

Fig 1.4 Plots of **E** and **H** as functions of z at t = 0.

1.7.4 Perfect and good conductor medium

A perfect, or good conductor, is one in which $\sigma \gg \omega\varepsilon$, so that $\frac{\sigma}{\omega\varepsilon} \gg 1$; that is

$$\sigma \simeq \infty, \qquad \varepsilon = \varepsilon_o\varepsilon_r, \qquad \mu = \mu_o\mu_r, \tag{1.63}$$

Hence, eqs. (1.37) and (1.38) become

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}, \tag{1.64}$$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \qquad \lambda = \frac{2\pi}{\beta}, \tag{1.65}$$

Also, from eq. (1.48),

$$\eta = \sqrt{\frac{i\omega\mu}{\sigma}}, \qquad \theta_\eta = 45^o, \tag{1.66}$$

and thus **E** leads **H** by 45°. If

$$\mathbf{E} = E_o e^{-\alpha z}\cos(\omega t - \beta z)\mathbf{a}_x, \tag{1.67}$$

then

$$\mathbf{H} = \frac{\mathrm{E_o}}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\alpha z} \cos(\omega t - \beta z - 45°)\, \mathbf{a}_y, \qquad (1.68)$$

Therefore, as the **E** (or **H**) wave travels in a conducting medium, its amplitude is attenuated by the factor $e^{-\alpha z}$.

Chapter 2

Conservation Laws

2.1 Introduction

In this chapter, we study the conservation of energy, momentum, and angular momentum, in electrodynamics. But we want to start with the conservation of charge because it is the paradigm for all conservation laws. Formally, the continuity equation is

$$\frac{\partial \rho_v}{\partial t} = -\nabla \cdot \mathbf{J}, \tag{2.1}$$

the precise mathematical statement of local conservation of charge.

The purpose of this chapter is to develop the corresponding equations for the local conservation of energy and momentum. In the process, we will show how to express the energy density and the momentum density (the analogs to ρ_v), as well as the energy "current" and the momentum "current" (analogous to **J**).

2.2 Energy

2.2.1 Poynting's Theorem

The work necessary to assemble a static charge distribution is

$$W_E = \frac{\varepsilon_o}{2} \int E^2 dv. \tag{2.2}$$

Likewise, the work required to get currents going is

$$W_M = \frac{\mu_o}{2} \int H^2 dv. \tag{2.3}$$

This suggests that the total energy stored in electromagnetic fields, per unit volume, is

$$W_{EM} = \frac{1}{2}(\varepsilon_o E^2 + \mu_o H^2). \tag{2.4}$$

we will confirm eq. (2.4) and develop the energy conservation law for electrodynamics.

Suppose we have some charge and current configuration which, at time t, produces fields **E** and **B**. In the next instant, dt, the charges move around a bit. According to the Lorentz force law, the work done on a charge Q is

$$\mathbf{F} \cdot d\boldsymbol{l} = Q(\mathbf{E} + \boldsymbol{u} \times \mathbf{B}) \cdot \boldsymbol{u}\, dt = Q\mathbf{E} \cdot \boldsymbol{u}\, dt. \tag{2.5}$$

In terms of the charge and current densities, $Q \to \rho_v dv$ *and* $\rho_v \boldsymbol{u} \to \mathbf{J},$ so the rate at which work is done on all the charges in volume v is

$$\frac{dW}{dt} = \int_v (\mathbf{E} \cdot \mathbf{J})\, dv. \tag{2.6}$$

Evidently $\mathbf{E} \cdot \mathbf{J}$ is the work done per unit time, per unit volume, which is to say, the power delivered per unit volume. We can express this quantity in terms of the fields alone, by using the Ampère-Maxwell law to eliminate $\mathbf{J}$:

$$\mathbf{E} \cdot \mathbf{J} = \mathbf{E} \cdot (\nabla \times \mathbf{H}) - \varepsilon_o \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}. \tag{2.7}$$

But for any vector fields **A** and **B**

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}), \tag{2.8}$$

letting $\mathbf{A} = \mathbf{E}$ and $\mathbf{B} = \mathbf{H}$ gives

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}). \tag{2.9}$$

Invoking Faraday's law ($\nabla \times \mathbf{E} = -\mu_o\, \partial \mathbf{H}/\partial t$), it follows that

$$\mathbf{E}\cdot(\nabla\times\mathbf{H}) = -\mu_o\mathbf{H}\cdot\frac{\partial\mathbf{H}}{\partial t} - \nabla\cdot(\mathbf{E}\times\mathbf{H}). \tag{2.10}$$

Meanwhile

$$\mathbf{H}\cdot\frac{\partial\mathbf{H}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(H^2), \qquad \text{and} \quad \mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2), \tag{2.11}$$

$$\mathbf{E}\ \cdot\ \mathbf{J} = -\frac{1}{2}\frac{\partial}{\partial t}(\varepsilon_o E^2 + \mu_o H^2) - \nabla\cdot(\mathbf{E}\times\mathbf{H}). \tag{2.12}$$

Putting this into eq. (2.7), and applying the divergence theorem to the second term, we have

$$\frac{dW}{dt} = -\frac{d}{dt}\int_v \frac{1}{2}\,(\varepsilon_o E^2 + \mu_o H^2)\,dv\ - \oint_s (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S}, \tag{2.13}$$

where s is the surface bounding v. This is Poynting's theorem; it is the "work energy theorem" of electrodynamics. The first integral on the right is the total energy stored in the fields. The second term evidently represents the rate at which energy is transported out of v, across its boundary surface, by the electromagnetic fields. Poynting's theorem says, then, that the work done on the charges by the electromagnetic force is equal to the decrease in energy remaining in the fields, less the energy that flowed out through the surface.

The energy per unit time, per unit area, transported by the fields is called the Poynting vector:

$$\boldsymbol{\mathcal{P}} \equiv (\mathbf{E}\times\mathbf{H}). \tag{2.14}$$

Specifically, $\boldsymbol{\mathcal{P}}\cdot d\mathbf{S}$ is the energy per unit time crossing the infinitesimal surface [Equation] the energy flux (so $\boldsymbol{\mathcal{P}}$ is the energy flux density). Using the Poynting vector to express Poynting's theorem compactly:

$$\frac{dW}{dt} = -\frac{d}{dt}\int_v W_{EM}\,dv\ - \oint_s \boldsymbol{\mathcal{P}}\cdot d\mathbf{S}. \tag{2.15}$$

If no work is done on the charges in v, for example, we are in a region of empty space, where there is no charge. In that case $dW/dt\ =\ 0$, so

$$\int_v \frac{\partial W_{EM}}{\partial t}\,dv\ = -\oint_s \boldsymbol{\mathcal{P}}\cdot d\mathbf{S} = -\int_v (\nabla\ \cdot\boldsymbol{\mathcal{P}})\,dv, \tag{2.16}$$

and hence

$$\frac{\partial W_{EM}}{\partial t} = -\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{P}}. \tag{2.17}$$

This is the "continuity equation" for energy [Equation] (energy density) plays the role of [Equation] (volume charge density), and [Equation] takes the part of [Equation] (current density). It expresses local conservation of electromagnetic energy.

The fields do work on the charges, and the charges create fields energy is tossed back and forth between them.

2.3 Momentum

Let's calculate the total electromagnetic force on the charges in volume v

$$\mathbf{F} = \int_v (\mathbf{E} + \boldsymbol{u} \times \mathbf{B})\, \rho_v \, dv = \int_v [\, \rho_v \mathbf{E} + \mu_o (\mathbf{J} \times \mathbf{H})]\, dv. \tag{2.18}$$

The force per unit volume is

$$\boldsymbol{\mathcal{F}} = \rho_v \mathbf{E} + \mathbf{J} \times \mu_o (\mathbf{J} \times \mathbf{H}). \tag{2.19}$$

We modify to express eq. (2.19) in terms of fields alone, eliminating ρ_v and $\mathbf{J}$ by using Maxwell's equations

$$\boldsymbol{\mathcal{F}} = \varepsilon_o (\boldsymbol{\nabla} \cdot \mathbf{E})\mathbf{E} + \mu_o \left(\boldsymbol{\nabla} \times \mathbf{H} - \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{H}. \tag{2.20}$$

Therefore,

$$\boldsymbol{\mathcal{F}} = \varepsilon_o [(\boldsymbol{\nabla} \cdot \mathbf{E})\mathbf{E} + (\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{E}] + \mu_o [(\boldsymbol{\nabla} \cdot \mathbf{H})\mathbf{H} + (\mathbf{H} \cdot \boldsymbol{\nabla})\mathbf{H}] - \frac{1}{2}\boldsymbol{\nabla}(\varepsilon_o E^2 + \mu_o H^2) - \varepsilon_o \mu_o \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{H}). \tag{2.21}$$

But it can be simplified by introducing the Maxwell stress tensor,

$$T_{ij} \equiv \varepsilon_o \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \mu_o \left(H_i H_j - \frac{1}{2} \delta_{ij} H^2 \right). \tag{2.22}$$

The indices i and j refer to the coordinates x, y, and z, so the stress tensor has a total of nine components (T_{xx}, T_{yy}, T_{xz}, T_{yx}, and so on). The Kronecker delta, δ_{ij}, is 1 if the indices are the same ($\delta_{xx} = \delta_{yy} = \delta_{zz} = 1$) and zero otherwise.

Because it carries two indices, where a vector has only one, T_{ij} is sometimes written with a double arrow $\overleftrightarrow{\mathbf{T}}$. In particular, the divergence of $\overleftrightarrow{\mathbf{T}}$ has as its jth component

$$\begin{aligned}\left(\nabla \cdot \overleftrightarrow{\mathbf{T}}\right)_j = \varepsilon_o &\left[(\nabla \cdot \mathbf{E})E_j + (\mathbf{E} \cdot \nabla)E_j - \frac{1}{2}\nabla_j E^2\right] \\ &+ \mu_o \left[(\nabla \cdot \mathbf{H})H_j + (\mathbf{H} \cdot \nabla)H_j - \frac{1}{2}\nabla_j H^2\right].\end{aligned} \tag{2.23}$$

Thus the force per unit volume eq. (2.28) can be written in the much tidier form

$$\boldsymbol{\mathcal{F}} = \nabla \cdot \overleftrightarrow{\mathbf{T}} - \varepsilon_o \mu_o \frac{\partial \boldsymbol{\mathcal{P}}}{\partial t}, \tag{2.31}$$

where $\boldsymbol{\mathcal{P}}$ is the Poynting vector eq. (2.14).

The total electromagnetic force on the charges in v eq. (2.18), used the divergence theorem to convert the first term to a surface integral is

$$\mathbf{F} = \oint_s \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{S} - \varepsilon_o \mu_o \frac{d}{dt} \int_v \boldsymbol{\mathcal{P}}\, dv. \tag{2.32}$$

In the static case the second term drops out, and the electromagnetic force on the charge configuration can be expressed entirely in terms of the stress tensor at the boundary

$$\mathbf{F} = \oint_s \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{S}. \tag{2.33}$$

Physically, $\overleftrightarrow{\mathbf{T}}$ is the force per unit area (or stress) acting on the surface. More precisely, T_{ij} is the force (per unit area) in the ith direction acting on an element of surface oriented in the jth direction "diagonal" elements (T_{xx}, T_{yy}, T_{zz}) represent pressures, and "off-diagonal" elements $(T_{xy}, T_{xz}, \text{etc.})$ are shears.

According to Newton's second law, the force on an object is equal to the rate of change of its momentum

$$\mathbf{F} = \frac{d\mathbf{p}_{mech}}{dt}. \tag{2.34}$$

Eq. (2.31) can therefore be written in the form

$$\frac{d\mathbf{p}_{mech}}{dt} = -\varepsilon_o\mu_o \frac{d}{dt}\int_v \boldsymbol{\mathcal{P}}\, dv + \oint_s \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{S}, \tag{2.35}$$

where $\mathbf{p}_{mech}$ is the (mechanical) momentum of the particles in volume v. This expression is similar in structure to Poynting's theorem eq. (2.15), and it invites an analogous interpretation: The first integral represents momentum stored in the fields

$$\mathbf{p} = \varepsilon_o\mu_o \int_v \boldsymbol{\mathcal{P}}\, dv, \tag{2.36}$$

while the second integral is the momentum per unit time flowing in through the surface.

Eq. (2.35) is the statement of conservation of momentum in electrodynamics: If the mechanical momentum increases, either the field momentum decreases, or else the fields are carrying momentum into the volume through the surface. The momentum density in the fields is evidently

$$\mathbf{g} = \varepsilon_o\mu_o\boldsymbol{\mathcal{P}} = \varepsilon_o\mu_o(\mathbf{E} \times \mathbf{H}), \tag{2.37}$$

and the momentum flux transported by the fields is $-\overleftrightarrow{\mathbf{T}}$ (specifically, $-\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{S}$ is the electromagnetic momentum per unit time passing through the area $d\mathbf{S}$). If the mechanical momentum in v is not changing (for example, region of empty space), then

$$\int_v \frac{\partial \mathbf{g}}{\partial t}\, dv = \oint_s \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{S} = \int_v \nabla \cdot \overleftrightarrow{\mathbf{T}}\, dv, \tag{2.38}$$

and hence

$$\frac{\partial \mathbf{g}}{\partial t} = \nabla \cdot \overleftrightarrow{\mathbf{T}}. \tag{2.39}$$

This is the "continuity equation" for electromagnetic momentum, with **g** (momentum density) in the role of ρ_v (volume charge density) and $-\overleftrightarrow{\mathbf{T}}$ playing the part of $\mathbf{J}$; it expresses the local conservation of field momentum. But in general (when there are charges around) the field momentum by itself, and the mechanical momentum by itself, are not conserved charges and fields exchange momentum, and only the total is conserved.

Notice that the Poynting vector has appeared in two quite different roles: $\boldsymbol{\mathcal{P}}$ itself is the energy per unit area, per unit time, transported by the electromagnetic fields, while $\varepsilon_o \mu_o \boldsymbol{\mathcal{P}}$ is the momentum per unit volume stored in those fields. Similarly, $\overleftrightarrow{\mathbf{T}}$ plays a dual role: $\overleftrightarrow{\mathbf{T}}$ itself is the electromagnetic stress (force per unit area) acting on a surface, and $-\overleftrightarrow{\mathbf{T}}$ describes the flow of momentum (it is the momentum current density) carried by the fields.

2.4 Angular Momentum

By now, the electromagnetic fields (which started out as mediators of forces between charges) have taken on a life of their own. They carry energy eq. (2.4)

$$W_{EM} = \frac{1}{2}(\varepsilon_o E^2 + \mu_o H^2), \tag{2.40}$$

and momentum eq. (2.37)

$$\mathbf{g} = \varepsilon_o \mu_o (\mathbf{E} \times \mathbf{H}), \tag{2.41}$$

and, for that matter, angular momentum

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{g} = \varepsilon_o \mu_o [\mathbf{r} \times (\mathbf{E} \times \mathbf{H})]. \quad (2.42)$$

Even perfectly static fields can harbor momentum and angular momentum, as long as $\mathbf{E} \times \mathbf{H}$ is nonzero, and it is only when these field contributions are included that the conservation laws are sustained.

Chapter 3

Optical Fibers

Introduction

Silica-based optical fibers are the most important transmission medium for long-distance and large-capacity optical communication systems. The most distinguished feature of optical fiber is its low-loss characteristics. Together with such low loss properties, low dispersion is also required for signal transmission. Signal distortion due to dispersion of the fiber is closely related to the guiding structure of optical fibers. In this chapter, the analysis of step-index fiber is presented, to understand the basic properties of optical fibers.

3.1. Basic Equations

The electromagnetic fields in optical fibers are expressed in cylindrical coordinates as

$$\mathbf{E} = \mathbf{E}(\boldsymbol{r}, \boldsymbol{\phi})\, e^{-\gamma z} e^{i\omega \mathrm{t}}, \tag{3.1a}$$

$$\mathbf{H} = \mathbf{H}(\boldsymbol{r}, \boldsymbol{\phi})\, e^{-\gamma z} e^{i\omega \mathrm{t}}. \tag{3.1b}$$

From a given Helmholtz equation

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0, \tag{3.2a}$$

$$\nabla^2 \mathbf{H} + k^2 \mathbf{H} = 0, \tag{3.2b}$$

$$k^2 = \omega^2 \mu\varepsilon = k_{\mathrm{o}}^2 \mathrm{n}^2, \tag{3.3}$$

where ε and μ are related to their respective values in a vacuum by

$$\varepsilon = \varepsilon_{o} n^2, \tag{3.4}$$

$$\mu = \mu_{o}, \tag{3.5}$$

n is the refractive index, $k_{o} = \omega\sqrt{\varepsilon_{o}\mu_{o}} = \omega/c$ is the wavenumber in a vacuum, which is related to the angular frequency ω, and c is the light velocity in a vacuum.

The scalar Helmholtz equation in cylindrical coordinates for the z-component, we obtain two sets of wave equations

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} + k_{o}^2 \mathrm{n}(r,\phi)^2 E_z = 0, \tag{3.6a}$$

$$\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r}\frac{\partial H_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_z}{\partial \phi^2} + \frac{\partial^2 H_z}{\partial z^2} + k_{o}^2 \mathrm{n}(r,\phi)^2 H_z = 0, \tag{3.6b}$$

By separating variables:

$$E_z(r,\phi,z) = \mathrm{R}(r)\,\Phi(\phi)\,\mathrm{Z}(z), \tag{3.10}$$

so we let

$$\frac{\mathrm{d}^2 Z(z)}{\mathrm{d}z^2} - \gamma^2 Z(z) = 0, \tag{3.11a}$$

$$\frac{\mathrm{d}^2 \Phi(\phi)}{\mathrm{d}\phi^2} + n^2 \Phi(\phi) = 0, \tag{3.11b}$$

$$\frac{\mathrm{d}^2 \mathrm{R}(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{dR}(r)}{\mathrm{d}r} + \left[k_{o}^2 \mathrm{n}(r)^2 + \gamma^2 - \frac{n^2}{r^2}\right]\mathrm{R}(r) = 0. \tag{3.11c}$$

In axially symmetric optical fibers, the refractive-index distribution is not dependent on ϕ and is expressed by $\mathrm{n}(r)$. Then the transverse electromagnetic fields are related to E_z and H_z as follows:

$$E_r = -\frac{i}{[k_o^2 n(r)^2 + \gamma^2]}\left(-i\gamma\frac{\partial E_z}{\partial r} + \frac{\omega\mu_o}{r}\frac{\partial H_z}{\partial \phi}\right), \tag{3.12a}$$

$$E_\phi = -\frac{i}{[k_o^2 n(r)^2 + \gamma^2]}\left(\frac{-i\gamma}{r}\frac{\partial E_z}{\partial \phi} - \omega\mu_o\frac{\partial H_z}{\partial r}\right), \tag{3.12b}$$

$$H_r = \frac{i}{[k_o^2 n(r)^2 + \gamma^2]}\left(\frac{\omega\varepsilon_o n^2}{r}\frac{\partial E_z}{\partial \phi} + i\gamma\frac{\partial H_z}{\partial r}\right), \tag{3.12c}$$

$$H_\phi = -\frac{i}{[k_o^2 n(r)^2 + \gamma^2]}\left(\omega\varepsilon_o n^2\frac{\partial E_z}{\partial r} - \frac{i\gamma}{r}\frac{\partial H_z}{\partial \phi}\right). \tag{3.12d}$$

The azimuthal dependency of the electromagnetic fields in axially symmetric fibers is expressed by $\cos(n\phi + \psi)$ or $\sin(n\phi + \psi)$, where n is an integer and $\psi = \omega t - \beta z$ denotes the phase. The mode in an optical fiber consists of TE modes ($E_z = 0$), TM modes ($H_z = 0$) and hybrid modes ($E_z \neq 0,\ H_z \neq 0$), respectively. In the following, electromagnetic fields, dispersion equations, and propagation characteristics of optical fibers are described in detail for step-index fibers as shown in Fig.3.1, which have a uniform refractive index in the core.

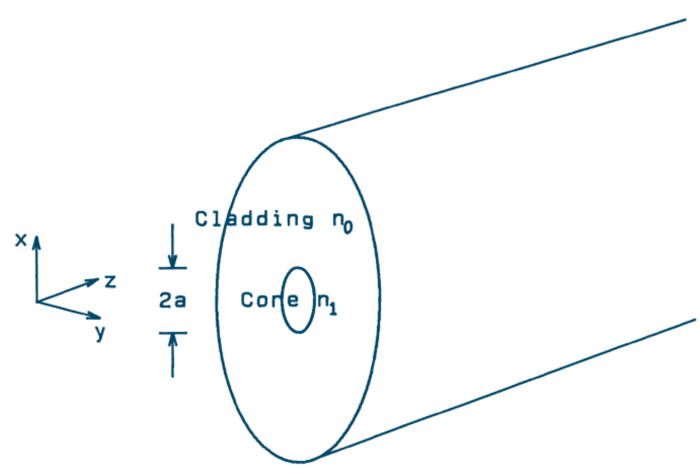

Fig.3.1 Waveguide structure of optical fiber.

3.2 Wave Theory of Step-Index Fibers

3.2.1 TE Modes

When we have $E_z = 0$, we have the following set of equations for the TE mode:

$$E_r = -\frac{i\omega\mu_o}{[k_o^2 n(r)^2 + \gamma^2]}\frac{1}{r}\frac{\partial H_z}{\partial \phi}, \tag{3.13a}$$

$$E_\phi = \frac{i\omega\mu_o}{[k_o^2 n(r)^2 + \gamma^2]}\frac{\partial H_z}{\partial r}, \tag{3.13b}$$

$$H_r = -\frac{\gamma}{[k_o^2 n(r)^2 + \gamma^2]}\frac{\partial H_z}{\partial r}, \tag{3.13c}$$

$$H_\phi = -\frac{\gamma}{[k_o^2 n(r)^2 + \gamma^2]}\frac{1}{r}\frac{\partial H_z}{\partial \phi}. \tag{3.13d}$$

The magnetic field in the core and cladding are expressed as

$$H_z = \begin{Bmatrix} g(r) \\ h(r) \end{Bmatrix} \cos(n\phi + \psi) \quad \begin{matrix} (0 \le r \le a) \\ (r > a) \end{matrix}. \tag{3.14}$$

From the boundary condition that the tangential field components H_z and H_ϕ should be continuous at the core-cladding interface $r = a$, we obtain the following two conditions:

$$g(a) = h(a), \tag{3.15a}$$

$$\frac{-\gamma}{[k_o^2 n(a)^2 + \gamma^2]}\frac{n}{a} g(a)\sin(n\phi + \psi) = \frac{-\gamma}{[k_o^2 n_0^2 + \gamma^2]}\frac{n}{a} h(a)\sin(n\phi + \psi). \tag{3.15b}$$

Since in the step-index fiber, the refractive index at the boundary of the core $n(a) = n_1$ (n_1 is the refractive index of the core), $n(a)$ is not equal to the refractive index of the cladding n_0. Then, it is known that integer n should be zero for Eq. (3.15b) to be satisfied with arbitrary values of ϕ. Consequently, the azimuthal dependency in the TE modes $\partial/\partial\phi = 0$, and the wave equation and related electromagnetic fields are obtained as

$$\frac{d^2H_z}{dr^2}+\frac{1}{r}\frac{dH_z}{dr}+[k_o^2 n(r)^2+\gamma^2]H_z=0, \tag{3.16}$$

$$E_\phi=\frac{i\omega\mu_o}{[k_o^2 n(r)^2+\gamma^2]}\frac{\partial H_z}{\partial r}, \tag{3.17a}$$

$$H_r=-\frac{\gamma}{[k_o^2 n(r)^2+\gamma^2]}\frac{\partial H_z}{\partial r}, \tag{3.17b}$$

$$H_\phi=E_r=E_z=0. \tag{3.17c}$$

Here, we define the wave number in the core and cladding along the transversal direction as

$$\kappa=\sqrt{k_o^2 n(r)^2+\gamma^2}, \tag{3.18a}$$

$$\zeta=\sqrt{-[k_o^2 n(r)^2+\gamma^2]}. \tag{3.18b}$$

The wave Eq. (3.16) for the field in the core $H_z=g(r)$ is obtained as

$$\frac{d^2g(r)}{dr^2}+\frac{1}{r}\frac{dg(r)}{dr}+\kappa^2 g(r)=0 \qquad (0\le r\le a), \tag{3.19}$$

and the wave equation for the field in the cladding $H_z=h(r)$ is given by

$$\frac{d^2h(r)}{dr^2}+\frac{1}{r}\frac{dh(r)}{dr}-\upsilon^2 h(r)=0 \qquad (r>a). \tag{3.20}$$

The solutions for Eq. (3.19) are the 0th-order Bessel function $J_0(\kappa\, r)$ and the 0th-order Neumann function $N_0(\kappa\, r)$, respectively. However, $N_0(\kappa\, r)$ diverges infinitely at $r=0$. Therefore $J_0(\kappa\, r)$ is the proper solution in the core. The solutions for Eq. (3.20) are modified Bessel function of the first kind $I_0(\zeta\, r)$ and modified Bessel functions of the second kind $K_0(\zeta\, r)$, respectively.

However, $I_0(\zeta\, r)$ diverges infinitely at $r = \infty$. Therefore $K_0(\zeta\, r)$, is the proper solution in the cladding. Then the magnetic fields for the TE mode are given by

$$H_z = \begin{cases} \mathrm{A}\, J_0(\kappa\, r) & (0 \leq r \leq a) \\ \mathrm{B}\, K_0(\zeta\, r) & (r > a) \end{cases}, \tag{3.21}$$

where A and B are constants. The boundary conditions are given by the conditions that H_z and E_ϕ should be continuous at $r = a$:

$$H_z = \mathrm{A}\, J_0(\kappa\, a) = \mathrm{B}\, K_0(\zeta\, a), \tag{3.22a}$$

$$E_\phi = \frac{\mathrm{A}}{\kappa}\, J_0'(\kappa\, a) = -\frac{\mathrm{B}}{\zeta}\, K_{n0}'(\zeta\, a), \tag{3.22b}$$

By using the normalized transverse wave numbers

$$\nu = \kappa a = a\sqrt{k_0^2 \mathrm{n}_1^2 + \gamma^2} \tag{3.23a}$$

$$\mathrm{w} = \zeta a = a\sqrt{-[k_0^2 n_0^2 + \gamma^2]} \tag{3.23b}$$

Where. $J_0'(\kappa\, a) = -J_1(\kappa\, a)$, and $K_0'(\zeta\, a) = -K_1(\zeta\, a)$. Eqs. (4.22a) and (3.22b) can be reduced to the following dispersion equation:

$$\frac{J_1(\nu)}{\nu\, J_0(\nu)} = -\frac{K_1(\mathrm{w})}{\mathrm{w}\, K_0(\mathrm{w})}, \tag{3.24}$$

Transverse wave numbers ν and w are related, from eq. (4.23), as

$$\nu^2 + \mathrm{w}^2 = k_0^2 a^2 (\mathrm{n}_1^2 - \mathrm{n}_0^2) = \upsilon^2, \tag{3.25}$$

Therefore, when the normalized frequency υ is given, the transverse wave numbers ν and ω are determined from Eqs. (3.24) and (3.25). Substituting Eqs. (3.21) and (3.22a) into (3.17), the electromagnetic fields for TE mode are obtained

$$E_r = E_z = H_\phi = 0, \quad (3.26)$$

a. Fields in the core ($0 \leq r \leq a$):

$$E_\phi = -\frac{i\omega\mu_o}{\kappa} \mathrm{A}\, J_1(\kappa\, r), \quad (3.27a)$$

$$H_r = -\frac{\gamma}{\kappa} \mathrm{A}\, J_1(\kappa\, r), \quad (3.27b)$$

$$H_z = \mathrm{A}\, J_0(\kappa\, r), \quad (3.27c)$$

b. Fields in the cladding ($r > a$):

$$E_\phi = \frac{i\omega\mu_o}{\zeta} \frac{J_0(\kappa\, a)}{K_0(\zeta\, a)} \mathrm{A}\, K_1(\zeta\, r), \quad (3.28a)$$

$$H_r = -\frac{\gamma}{\zeta} \frac{J_0(\kappa\, a)}{K_0(\zeta\, a)} \mathrm{A}\, K_1(\zeta\, r), \quad (3.28b)$$

$$H_z = \frac{J_0(\kappa\, a)}{K_0(\zeta\, a)} \mathrm{A}\, K_0(\zeta\, r). \quad (3.28c)$$

The constant A is determined in the relationship from the power P_{tr} carried by the mode.

3.2.2 TM Modes

When we set ($H_z = 0$) in Eqs. (3.9) and (3.12) for TM modes and follow the same procedure as in the TE mode, we know that $n = 0$ holds also for the TM mode. Then the wave equation and other electromagnetic field components are expressed as

$$\frac{d^2 E_z}{dr^2} + \frac{1}{r}\frac{dE_z}{dr} + [k_o^2 n(r)^2 + \gamma^2]E_z = 0, \tag{3.29}$$

$$E_r = -\frac{\gamma}{[k_o^2 n(r)^2 + \gamma^2]}\frac{\partial E_z}{\partial r}, \tag{3.30a}$$

$$H_\phi = -\frac{i\omega\varepsilon_o n^2}{[k_o^2 n(r)^2 + \gamma^2]}\frac{\partial E_z}{\partial r}, \tag{3.30b}$$

$$E_\phi = H_r = H_z = 0. \tag{3.30c}$$

Solutions for Eq. (4.29) are given by the 0th-order Bessel functions as

$$E_z = \begin{cases} A\, J_0(\kappa\, r) & (0 \le r \le a) \\ B\, K_0(\zeta\, r) & (r > a) \end{cases}, \tag{3.31}$$

where A and B areconstants. Applying the boundary conditions that E_z and H_ϕ should be continuous at the core-cladding interface $r = a$, we obtain

$$E_z = A\, J_0(\kappa\, a) = B\, K_0(\zeta\, a), \tag{3.32a}$$

$$H_\phi = n_1^2 \frac{A}{\kappa}\, J_1(\kappa\, a) = -n_0^2 \frac{B}{\upsilon}\, K_1(\zeta\, a), \tag{3.32b}$$

$$\frac{J_1(\nu)}{\nu\, J_0(\nu)} = -\left(\frac{n_0}{n_1}\right)^2 \frac{K_1(w)}{w\, K_0(w)}. \tag{3.32b}$$

The electromagnetic fields for TM mode are summarized as

$$E_\phi = H_r = H_z = 0, \tag{3.33}$$

a. Fields in the core $(0 \le r \le a)$:

$$E_r = \frac{\gamma}{\kappa} A\, J_1(\kappa\, r), \tag{3.34a}$$

$$E_z = A\, J_0(\kappa\, r), \tag{3.34b}$$

$$H_\phi = -\frac{i\omega \varepsilon_o n_1^2}{\kappa} A\, J_1(\kappa\, r), \tag{3.34c}$$

b. Fields in the cladding ($r > a$):

$$E_r = -\frac{\gamma}{\zeta} \frac{J_0(\kappa\, a)}{K_0(\zeta\, a)} A\, K_1(\zeta\, r), \tag{3.35a}$$

$$E_z = \frac{J_0(\kappa\, a)}{K_0(\zeta\, a)} A\, K_0(\zeta\, r), \tag{3.35b}$$

$$H_\phi = -\frac{i\omega \varepsilon_o n_0^2}{\zeta} \frac{J_0(\kappa\, a)}{K_0(\zeta\, a)} A\, K_1(\zeta\, r). \tag{3.35c}$$

3.2.3. Hybrid Modes

In hybrid modes the axial electromagnetic field components E_z and H_z are not zero. Therefore, solutions for eq. (3.6) are given by the product of nth-order Bessel functions and $\cos(n\phi + \psi)$ or $\sin(n\phi + \psi)$. E_z and H_z should be continuous at $r = a$ Also it is known from eq. (3.7) that $\partial E_z/\partial r$ and $\partial H_z/\partial \phi$ (or $\partial E_z/\partial \phi$ and $\partial H_z/\partial r$) have the same ϕ dependencies. Taking these into consideration, the z-components of the electromagnetic field are expressed by

$$E_z = \begin{cases} \mathrm{A}\, J_n(\kappa\, r)\cos(n\phi + \psi) & (0 \le r \le a) \\ \mathrm{A}\dfrac{J_n(\kappa\, a)}{K_n(\zeta a\,)}\, K_n(\zeta\, r)\cos(n\phi + \psi) & (r > a) \end{cases}, \qquad (3.36a)$$

$$H_z = \begin{cases} \mathrm{C}\, J_n(\kappa\, r)\sin(n\phi + \psi) & (0 \le r \le a) \\ \mathrm{C}\dfrac{J_n(\kappa\, a)}{K_n(\zeta a\,)}\, K_n(\zeta\, r)\sin(n\phi + \psi) & (r > a) \end{cases}, \qquad (3.36b)$$

a. Fields in the core $(0 \le r \le a)$:

$$E_r = -\frac{i}{\kappa^2}\left[-\mathrm{A}i\gamma\kappa\, \mathrm{J}_n'(\kappa\, r) + C\omega\mu_o \frac{n}{r}\mathrm{J}_n(\kappa\, r)\right]\cos(n\phi + \psi), \qquad (3.37\ a)$$

$$E_\phi = -\frac{i}{\kappa^2}\left[\mathrm{A}i\gamma \frac{n}{r}\mathrm{J}_n(\kappa\, r) - C\omega\mu_o\kappa\, \mathrm{J}_n'(\kappa\, r)\right]\sin(n\phi + \psi)\,, \qquad (3.37b)$$

$$H_r = -\frac{i}{\kappa^2}\left[\mathrm{A}\omega\varepsilon_o \mathrm{n}_1^2 \frac{n}{r}\mathrm{J}_n(\kappa\, r) - Ci\gamma\kappa\, \mathrm{J}_n'(\kappa\, r)\right]\sin(n\phi + \psi)\,, \qquad (3.37c)$$

$$H_\phi = -\frac{i}{\kappa^2}\left[\mathrm{A}\omega\varepsilon_o \mathrm{n}_1^2\kappa\, \mathrm{J}_n'(\kappa\, r) - Ci\gamma \frac{n}{r}\mathrm{J}_n(\kappa\, r)\right]\cos(n\phi + \psi)\,, \qquad (3.37d)$$

b. Fields in the cladding $(r > a)$:

$$E_r = \frac{i}{\zeta^2}\left[-\mathrm{A}i\gamma\zeta\, \mathrm{K}_n'(\zeta\, r) + C\omega\mu_o \frac{n}{r}\mathrm{K}_n(\zeta\, r)\right]\frac{\mathrm{J}_n(\kappa\, a)}{\mathrm{K}_n(\zeta\, a)}\cos(n\phi + \psi), \qquad (3.38a)$$

$$E_\phi = \frac{i}{\zeta^2}\left[\mathrm{A}i\gamma \frac{n}{r}\mathrm{K}_n(\zeta\, r) - C\omega\mu_o\zeta\, \mathrm{K}_n'(\zeta\, r)\right]\frac{\mathrm{J}_n(\kappa\, a)}{\mathrm{K}_n(\zeta\, a)}\sin(n\phi + \psi)\,, \qquad (3.38b)$$

$$H_r = \frac{i}{\upsilon^2}\left[\mathrm{A}\omega\varepsilon_o \mathrm{n}_0^2 \frac{n}{r}\mathrm{K}_n(\upsilon\, r) - Ci\gamma\upsilon\, \mathrm{K}_n'(\upsilon\, r)\right]\frac{\mathrm{J}_n(\kappa\, a)}{\mathrm{K}_n(\upsilon\, a)}\sin(n\phi + \psi)\,, \qquad (3.38c)$$

$$H_\phi = \frac{i}{\zeta^2}\left[\mathrm{A}\omega\varepsilon_o \mathrm{n}_0^2\zeta\, \mathrm{K}_n'(\zeta\, r) - Ci\gamma \frac{n}{r}\mathrm{K}_n(\zeta\, r)\right]\frac{\mathrm{J}_n(\kappa\, a)}{\mathrm{K}_n(\zeta\, a)}\cos(n\phi + \psi). \qquad (3.38d)$$

The boundary condition at $r = a$ that E_ϕ and H_ϕ should be continuous brings two relations, one is from eqs. (3.37b) and (3.38b) as

$$-Ai\gamma\left(\frac{1}{\kappa^2}+\frac{1}{\zeta^2}\right)\frac{n}{a} = -C\omega\mu_o\left[\frac{J_n'(\kappa\, a)}{\kappa\, J_n(\kappa\, a)}+\frac{K_n'(\zeta\, a)}{\upsilon K_n(\zeta\, a)}\right], \tag{3.39}$$

and the others from Eqs. (4.37d) and (4.38d):

$$A\omega\varepsilon_o\left[n_1^2\frac{J_n'(\kappa\, a)}{\kappa\, J_n(\kappa\, a)}+n_0^2\frac{K_n'(\zeta\, a)}{\upsilon K_n(\zeta\, a)}\right] = Ci\gamma\left(\frac{1}{\kappa^2}+\frac{1}{\zeta^2}\right)\frac{n}{a}. \tag{3.40}$$

We obtain the dispersion equation from eqs. (3.39) and (3.40) in the form

$$\left[\frac{J_n'(\kappa\, a)}{\kappa\, J_n(\kappa\, a)}+\frac{K_n'(\zeta\, a)}{\zeta K_n(\zeta\, a)}\right]\left[n_1^2\frac{J_n'(\kappa\, a)}{\kappa\, J_n(\kappa\, a)}+n_0^2\frac{K_n'(\zeta\, a)}{\zeta K_n(\zeta\, a)}\right] = \frac{-\gamma^2}{k_o}\left(\frac{1}{\kappa^2}+\frac{1}{\zeta^2}\right)^2\left(\frac{n}{a}\right)^2. \tag{4.41}$$

Substituting the following relation, which is derived from eq. (3.18):

$$-\frac{\gamma^2}{k_o}\left(\frac{1}{\kappa^2}+\frac{1}{\zeta^2}\right) = \frac{n_1^2}{\kappa^2}+\frac{n_0^2}{\zeta^2}, \tag{3.42}$$

Eq.(3.41) may be rewritten

$$\left[\frac{J_n'(\kappa\, a)}{\kappa\, J_n(\kappa\, a)}+\frac{K_n'(\zeta\, a)}{\zeta K_n(\zeta\, a)}\right]\left[\frac{J_n'(\kappa\, a)}{\kappa\, J_n(\kappa\, a)}+\left(\frac{n_0}{n_1}\right)^2\frac{K_n'(\zeta\, a)}{\zeta K_n(\zeta\, a)}\right] = \left(\frac{1}{\kappa^2}+\frac{1}{\zeta^2}\right)\left[\frac{1}{\kappa^2}+\left(\frac{n_0}{n_1}\right)^2\frac{1}{\zeta^2}\right]\left(\frac{n}{a}\right)^2. \tag{4.43}$$

The propagation constant of the hybrid modes is calculated by solving eq. (3.43) using the $\nu-\omega$ relation of eq. (3.25). Although eq. (3.43) is the strict solution of the hybrid modes in step-index fibers, it is rather difficult to investigate the propagation properties of optical fibers with eq. (3.43). In practical fibers, the refractive-index difference $\Delta = \frac{n_1^2 - n_0^2}{2n_1^2}$ is of the order of 1%, so we can approximate $n_1 \cong n_0$ in some cases. The constant C in the electromagnetic field expressions (3.37) and (3.38) can be written from eq. (3.39) as

$$C = A\frac{i\gamma}{\omega\mu_o}s, \tag{3.44a}$$

$$s = \frac{\left(\frac{1}{\kappa^2} + \frac{1}{\zeta^2}\right)\frac{n}{a}}{\left[\frac{J_n'(\kappa\, a)}{\kappa\, J_n(\kappa\, a)} + \frac{K_n'(\zeta\, a)}{\zeta K_n(\zeta\, a)}\right]}. \tag{3.44b}$$

Where when we apply the following recurrence relations for the Bessel functions

$$J_n'(z) = \frac{1}{2}[J_{n-1}(z) - J_{n+1}(z)], \tag{3.45a}$$

$$\frac{n}{z}J_n(z) = \frac{1}{2}[J_{n-1}(z) + J_{n+1}(z)], \tag{3.45b}$$

$$K_n'(\zeta\, a) = -\frac{1}{2}[K_{n-1}(z) + K_{n+1}(z)], \tag{3.45c}$$

$$\frac{n}{z}K_n(\zeta\, a) - \frac{1}{2}[K_{n-1}(z) - K_{n+1}(z)], \tag{3.45d}$$

Eqs. (3.37) and (3.38) become the following.

a. Core region ($0 \le r \le a$):

$$E_r = \frac{-\gamma A}{\kappa}\left[\frac{(1-s)}{2}J_{n-1}(\kappa r) - \frac{(1+s)}{2}J_{n+1}(\kappa r)\right]\cos(n\phi+\psi), \tag{3.46 a}$$

$$E_\phi = \frac{\gamma A}{\kappa}\left[\frac{(1-s)}{2}J_{n-1}(\kappa r) + \frac{(1+s)}{2}J_{n+1}(\kappa r)\right]\sin(n\phi+\psi), \tag{3.46b}$$

$$E_z = AJ_n(\kappa r)\cos(n\phi+\psi), \tag{3.46c}$$

$$H_r = \frac{-iA\omega\varepsilon_o n_1^2}{\kappa}\left[\frac{(1-s_1)}{2}J_{n-1}(\kappa r) + \frac{(1+s_1)}{2}J_{n+1}(\kappa r)\right]\sin(n\phi+\psi), \tag{3.46d}$$

$$H_\phi = \frac{-iA\omega\varepsilon_o n_1^2}{\kappa}\left[\frac{(1-s_1)}{2}J_{n-1}(\kappa r) - \frac{(1+s_1)}{2}J_{n+1}(\kappa r)\right]\cos(n\phi+\psi), \tag{3.46e}$$

$$H_z = A\frac{i\gamma}{\omega\mu_o}sJ_n(\kappa r)\sin(n\phi+\psi), \tag{3.46f}$$

b. Cladding region ($r > a$):

$$E_r = \frac{-\gamma A}{\zeta}\frac{J_n(\kappa a)}{K_n(\zeta a)}\left[\frac{(1-s)}{2}K_{n-1}(\zeta r) + \frac{(1+s)}{2}K_{n+1}(\zeta r)\right]\cos(n\phi+\psi), \tag{3.47a}$$

$$E_\phi = \frac{\gamma A}{\zeta}\frac{J_n(\kappa a)}{K_n(\zeta a)}\left[\frac{(1-s)}{2}K_{n-1}(\zeta r) - \frac{(1+s)}{2}K_{n+1}(\zeta r)\right]\sin(n\phi+\psi), \tag{4.47b}$$

$$E_z = A\frac{J_n(\kappa a)}{K_n(\zeta a)}K_n(\zeta r)\cos(n\phi+\psi), \tag{3.47c}$$

$$H_r = \frac{-iA\omega\varepsilon_o n_0^2}{\zeta}\frac{J_n(\kappa a)}{K_n(\zeta a)}\left[\frac{(1-s_0)}{2}K_{n-1}(\zeta r) - \frac{(1+s_0)}{2}K_{n+1}(\zeta r)\right]\sin(n\phi+\psi), \tag{3.47d}$$

$$H_\phi = \frac{-iA\omega\varepsilon_o n_0^2}{\zeta}\frac{J_n(\kappa a)}{K_n(\zeta a)}\left[\frac{(1-s_0)}{2}K_{n-1}(\zeta r) + \frac{(1+s_0)}{2}K_{n+1}(\zeta r)\right]\cos(n\phi+\psi), \tag{3.47e}$$

$$H_z = \mathrm{A}\frac{i\gamma}{\omega\mu_o} s \frac{\mathrm{J}_n(\kappa\, a)}{\mathrm{K}_n(\zeta\, a)} \mathrm{K}_n(\zeta\, r)\sin(n\phi + \psi), \tag{3.47f}$$

$$s_1 = -\frac{\gamma^2}{k_0^2 \mathrm{n}_1^2} s, \tag{3.48a}$$

$$s_0 = -\frac{\gamma^2}{k_0^2 \mathrm{n}_0^2} s. \tag{3.48b}$$

3.3. Optical Power Carried by Each Mode

The time averaged Poynting vector component along the z-axis per unit area is expressed, as

$$\mathcal{P}_{\text{ave}}(z) = \frac{1}{2}\left(E_r H_\phi^* - E_\phi H_r^*\right). \tag{3.49}$$

The power carried by the optical fiber is then given by

$$P_{tr} = \int_{\phi=0}^{2\pi}\int_{r=0}^{\infty} \mathcal{P}_{\text{ave}}(z)\, r\, \mathrm{d}r\mathrm{d}\phi = \frac{1}{2}\int_0^{2\pi}\int_0^{\infty}\left(E_r H_\phi^* - E_\phi H_r^*\right) r\, \mathrm{d}r\mathrm{d}\phi. \tag{3.50}$$

The analytical expressions for the transmission power in each mode are described in the following.

3.3.1 TE Modes

The transmission power in the core and cladding are calculated from eqs. (3.26)–(3.28) and (3.50) as follows:

$$P_{core} = \frac{\pi}{2}\omega\mu_o|\mathrm{A}|^2 \frac{a^2}{\kappa^2}\left[J_1^2(\kappa\, a) - J_0(\kappa\, a) J_2(\kappa\, a)\right], \tag{3.51}$$

$$P_{clad} = -\frac{\pi}{2} i\gamma\omega\mu_o|\mathrm{A}|^2 \frac{a^2}{\zeta^2}\frac{\mathrm{J}_0^2(\kappa\, a)}{\mathrm{K}_0^2(\zeta\, a)}\left[K_0(\zeta\, a)K_2(\zeta\, a) - K_1^2(\zeta\, a)\right], \tag{3.52}$$

Which can be written as

$$P_{core} = \frac{\pi}{2}\omega\mu_o|A|^2\frac{a^2}{\kappa^2}J_1^2(\kappa\, a)\left[1+\frac{\zeta^2}{\kappa^2}\frac{K_0(\zeta\, a)K_2(\zeta\, a)}{K_1^2(\zeta\, a)}\right], \tag{3.53}$$

$$P_{clad} = -\frac{\pi}{2}i\gamma\omega\mu_o|A|^2\frac{a^2}{\zeta^2}J_1^2(\kappa\, a)\left[\frac{K_0(\zeta\, a)K_2(\zeta\, a)}{K_1^2(\zeta\, a)}-1\right]. \tag{3.54}$$

The total power carried by the TE mode is given by

$$P_{tr} = P_{core}+P_{clad} = -\frac{\pi}{2}i\gamma\omega\mu_o|A|^2\frac{(\kappa^2+\zeta^2)a^2}{\kappa^2}J_1^2(\kappa\, a)\left[\frac{K_0(\zeta\, a)K_2(\zeta\, a)}{K_1^2(\zeta\, a)}\right]. \tag{3.55}$$

The unknown constant A can be determined from eq. (3.55) when we specify the total power flow P_{tr} in optical fiber.

3.3.2 TM Modes

Substituting Eqs. (3.33)–(3.35) into (3.50) and applying similar procedures to those for the TE mode, the power in the core and cladding for the TM mode is given by

$$P_{core} = -\frac{\pi}{2}i\gamma\omega\varepsilon_o n_1^2|A|^2\frac{a^2}{\kappa^2}J_1^2(\kappa\, a)\left[1+\frac{n_1^2}{n_0^2}\frac{\zeta^2}{\kappa^2}\frac{K_0(\zeta\, a)K_2(\zeta\, a)}{K_1^2(\zeta\, a)} + \left(1-\frac{n_0^2}{n_1^2}\right)\frac{J_0^2(\kappa\, a)}{J_1^2(\kappa\, a)}\right], \tag{3.56}$$

$$P_{clad} = \frac{\pi}{2}\omega\varepsilon_o n_1^2|A|^2\frac{a^2}{\kappa^2}J_1^2(\kappa\, a)\frac{n_1^2}{n_0^2}\left[\frac{K_0(\zeta\, a)K_2(\zeta\, a)}{K_1^2(\zeta\, a)}-1\right], \tag{3.57}$$

$$P_{tr} = P_{core} + P_{clad} \tag{3.58}$$

$$= -\frac{\pi}{2} i\gamma\omega\varepsilon_o n_1^2 |A|^2 \frac{a^2}{\kappa^2} J_1^2(\kappa\, a) \left[\frac{n_1^2}{n_0^2}\frac{(\kappa^2+\zeta^2)}{\kappa^2}\frac{K_0(\zeta\, a)K_2(\zeta\, a)}{K_1^2(\zeta\, a)} + \left(1-\frac{n_1^2}{n_0^2}\right)\left(1-\frac{n_0^2}{n_1^2}\frac{J_0^2(\kappa\, a)}{J_1^2(\kappa\, a)}\right)\right].$$

When the weakly guiding approximation $n_1/n_0 \cong 1$ is satisfied, Eqs. (3.56) – (3.58) are simplified into equations like Eqs. (3.53) – (3.55) for the TE mode.

Chapter 4

Results

This chapter presents the results of step-index fiber to understand the basic properties of optical fibers. The refractive indices of the fiber and the vacuum cladding are $n_1 = 1.5$ and $n_0 = 1$. The relations between the transverse wavenumbers ν and ɯ, which are calculated via the dispersion equation itself and $\nu^2 + ɯ^2 = \upsilon^2$ for υ = 5, a=1 (arbitrary unit).

4.1 TE Modes

We consider first the TE modes, and we present in Fig 4.1, the radial dependency of the components of the electric and magnetic fields for the TE modes in the step-index fiber. The expressions of the components of the electric and magnetic fields are given by

a- Fields in the core ($0 \leq r \leq a$):

$$E_\phi = -\frac{i\omega\mu_o}{\kappa} \mathrm{A}\, J_1(\kappa\, r), \tag{4.1}$$

$$H_r = -\frac{\gamma}{\kappa} \mathrm{A}\, J_1(\kappa\, r), \tag{4.2}$$

$$H_z = \mathrm{A}\, J_0(\kappa\, r), \tag{4.3}$$

b- Fields in the cladding ($r > a$):

$$E_\phi = \frac{i\omega\mu_o}{\zeta} \frac{J_0(\kappa\, a)}{K_0(\zeta\, a)} \mathrm{A}\, K_1(\zeta\, r), \tag{4.4}$$

$$H_r = -\frac{\gamma}{\zeta} \frac{J_0(\kappa\, a)}{K_0(\zeta\, a)} \mathrm{A}\, K_1(\zeta\, r), \tag{4.5}$$

$$H_z = \frac{J_0(\kappa\, a)}{K_0(\zeta\, a)} \mathrm{A}\, K_0(\zeta\, r). \tag{4.6}$$

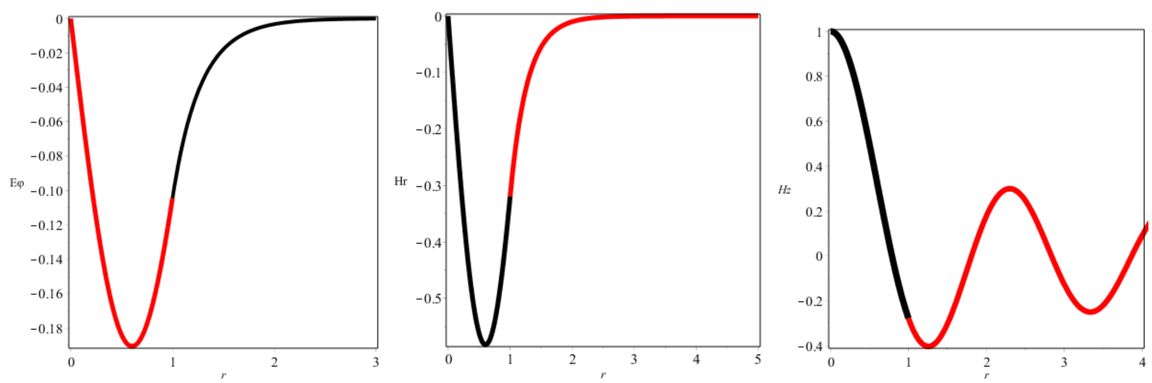

Fig. 4-1 The radial variation of the components of the electric and magnetic fields for the TE mode.

4.2 TM Modes

We consider first the TE modes, and we present in Fig 4.2, the radial dependency of the components of the electric and magnetic fields for the TM modes in the step-index fiber. The expressions of the components of the electric and magnetic fields are given by

a- Fields in the core ($0 \leq r \leq a$):

$$E_r = \frac{\gamma}{\kappa} \mathrm{A}\, J_1(\kappa\, r), \tag{4.7}$$

$$E_z = \mathrm{A}\, J_0(\kappa\, r), \tag{4.8}$$

$$H_\phi = -\frac{i\omega \varepsilon_o n_1^2}{\kappa} \mathrm{A}\, J_1(\kappa\, r), \tag{4.9}$$

b- Fields in the cladding ($r > a$):

$$E_r = -\frac{\gamma}{\zeta}\frac{J_0(\kappa\, a)}{K_0(\zeta\, a)} \mathrm{A}\, K_1(\zeta\, r), \tag{4.10}$$

$$E_z = \frac{J_0(\kappa\, a)}{K_0(\zeta\, a)} \mathrm{A}\, K_0(\zeta\, r), \tag{4.11}$$

$$H_\phi = -\frac{i\omega\varepsilon_o n_0^2}{\zeta}\frac{J_0(\kappa\, a)}{K_0(\zeta\, a)} \mathrm{A}\, K_1(\zeta\, r). \tag{4.12}$$

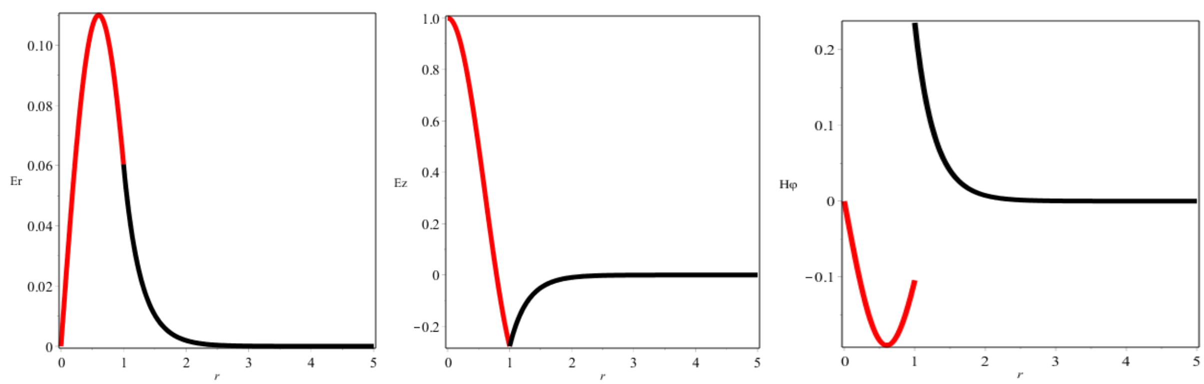

Fig 4.2 The radial dependency of the components of the electric and magnetic fields for TM modes.

Conclusion and Outlook

As we progress further into the twenty-first century, often referred to as the "Information Technology," there is no question that the rapid expansion of information technology has been driven by advancements in modern telecommunications systems. Among these, optical fiber communication has played a pivotal role in enabling high-speed and high-quality data transmission. Today, optical fibers are not only integral to telecommunication networks but are also widely employed in the Internet backbone and local area networks (LANs), providing unparalleled data rates and bandwidth.

Given the significance of optical fiber technology in modern communication, this project has delved into the underlying theory and principles that highlight its crucial role in the development of advanced telecommunication systems. By exploring key aspects such as wave propagation, dispersion, and mode analysis, we have gained a deeper understanding of the fundamental characteristics that make optical fibers essential in this field.

Looking ahead, several promising areas of research in optical fiber technology remain to be explored. One such area is the design and development of planar waveguide sensor structures, which represent an exciting and innovative trend. These sensors offer new possibilities for precise detection in a variety of applications. Another emerging research direction involves the detection of adulteration in petroleum-based products—such as petrol, kerosene, and diesel—using embedded planar waveguide technology.

As optical fiber technology continues to evolve, it will undoubtedly remain a cornerstone of modern communication systems and expand into new, impactful applications across various industries.

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