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المملكة العربية السعودية
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قسم الرياضيات والإحصاء

Math Help Center Workbook

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Student's Name	
Student's Number	

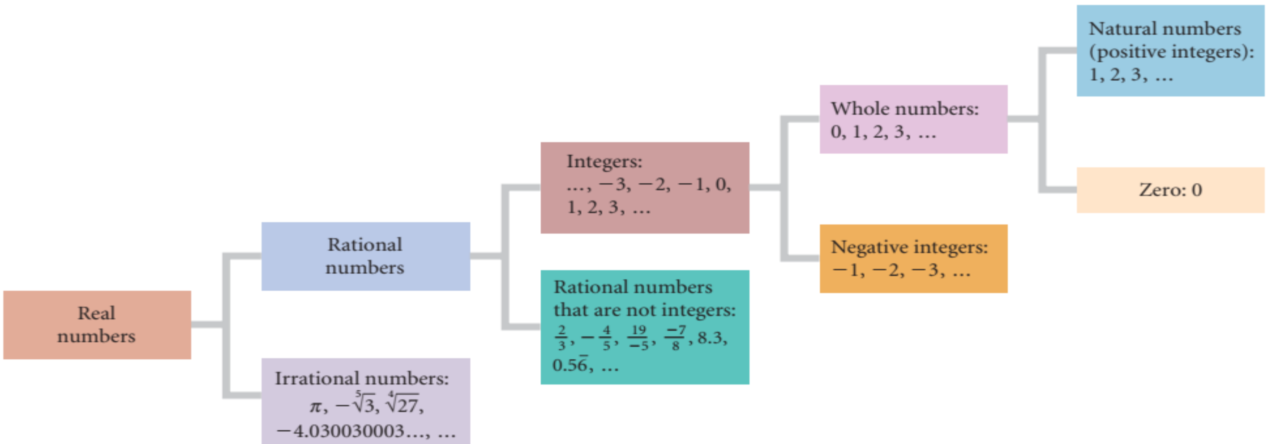
Contents

THEME 1. Basic Concepts of Algebra

Section 1.1 – Real Numbers

- Real numbers are any numbers that can be found on the number line.

- Real numbers are made up of smaller subsets of numbers.



Properties of Fractions

Formula	Example
1. $a = \frac{a}{1}$	$2 = \frac{2}{1}$
2. $\frac{a}{b} = \frac{a}{1}.\frac{1}{b}$	$\frac{-6}{2} = -6.\frac{1}{2} = -3$
3. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$	$\frac{2}{3} \cdot \frac{4}{5} = \frac{2 \cdot 4}{3 \cdot 5} = \frac{8}{15}$
4. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$	$\frac{3}{5} \div \frac{1}{2} = \frac{3}{5} \cdot \frac{2}{1} =$ $\frac{3 \cdot 2}{5 \cdot 1} = \frac{6}{5}$
5. $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$	$\frac{3}{5} + \frac{4}{5} = \frac{3+4}{5} = \frac{7}{5}$
6. $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$	$\frac{2}{3} + \frac{1}{2} = \frac{2(2)}{3(2)} + \frac{1(3)}{2(3)} =$ $\frac{4}{6} + \frac{3}{6} = \frac{7}{6}$
7. $\frac{ac}{bc} = \frac{a}{b}$	$\frac{12}{15} = \frac{\cancel{3}(4)}{\cancel{3}(5)} = \frac{4}{5}$

8. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$	Since $\frac{1}{3} = \frac{2}{6}$, $1(6) = 2(3)$

Practice: Perform the operations without a calculator.

1. $-\frac{6}{2} + 3$

2. $\frac{4}{3} + \frac{1}{5} =$

3. $\frac{3}{8} \cdot \frac{2}{4} =$

4. $\frac{3}{7} - \frac{1}{2} =$

5. $\frac{5}{6} \cdot \frac{2}{3} =$

6. $\frac{4}{3} \div \frac{4}{7} =$

7. $\frac{\frac{4}{2}}{5} + 3 =$

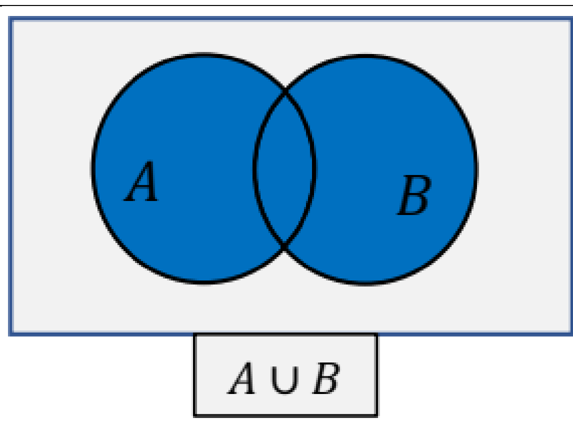
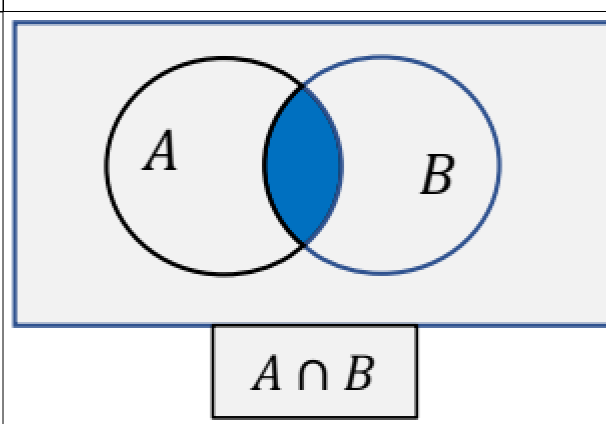
8. $\frac{-\frac{2}{3} + \frac{1}{3}}{\frac{2}{3}} =$

9. $2-\frac{3}{5}+\frac{5(1-\frac{3}{4})}{3}$

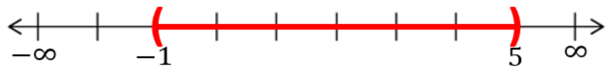
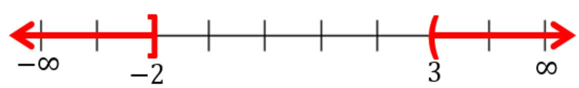
10. $\frac{a}{b}-c$, where $a=\frac{2}{\frac{3}{5}}$ $\quad b=1-\frac{2}{3}$ and $c=-\frac{1}{2}+2.5.$

Intervals: Types, Notation, and Graphs

Type	Interval Notation	Set Notation	Graph
Open	(a, b)	$\{x \mid a < x < b\}$	
Closed	$[a, b]$	$\{x \mid a \leq x \leq b\}$	
Half-open	$[a, b)$	$\{x \mid a \leq x < b\}$	
Half-open	$(a, b]$	$\{x \mid a < x \leq b\}$	
Open	(a, ∞)	$\{x \mid x > a\}$	
Half-open	$[a, \infty)$	$\{x \mid x \geq a\}$	
Open	$(-\infty, b)$	$\{x \mid x < b\}$	
Half-open	$(-\infty, b]$	$\{x \mid x \leq b\}$	

_A_ union _B_: The elements in A or B or both	**_A_ intersection _B_:** The elements common to A and B
	

Practice: Represent the following inequalities in interval notation.

Inequality	Graph	Interval Notation
$-1 < x < 5$		$(-1,5)$
$3 < x \leq 6$	....	....
$0 \leq x \leq 4$	....	....
$x > -3$	....	....
$x \leq 4$	....	....
$x \leq -2$ or $x > 3$		$(-\infty, -2] \cup (3, \infty)$

Practice: Find the union or intersection of sets as indicated

$$A = \{-5, -4, 1, 7\}, \quad B = \{-5, 0, 2\}, C = \{-3, 2, 0, 6\}$$

a) $A \cup B$

b) $A \cap B$

c) $A \cap (B \cup C)$

d) $(A \cap B) \cup (B \cap C)$

Practice: Find the union or intersection of sets as indicated

$$D = \{x|x < 3\}, \qquad E = (-\infty, -1] \quad and\ F = \{x|x \geq -2\ and\ x < 2\}$$

a) $D \cap E$

b) $D \cup E$

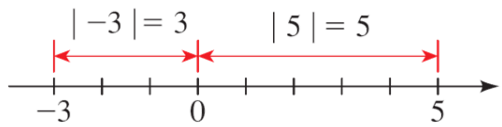
c) $D \cup F$

d) $E \cap F$

e) $F \cup E$

Absolute Value and Distance

- The absolute value of a number is its distance to zero.



- The definition of the absolute value of a is:

$$|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0 \end{cases}$$

- Some properties of absolute value are:

Property	**Example**
1. $\lvert a\rvert \geq 0$	$\lvert -4\rvert = 4 \geq 0$
2. $\lvert a\rvert = 0$ *if and only if* $a = 0$	
3. $\lvert a\rvert = \lvert -a\rvert$	$\lvert 2\rvert = \lvert -2\rvert = 2$
4. $\lvert ab\rvert = \lvert a\rvert\lvert b\rvert$	$\lvert -3 \cdot 4\rvert = \lvert -3\rvert\lvert 4\rvert = 12$
5. $\left\lvert\frac{a}{b}\right\rvert = \frac{\lvert a\rvert}{\lvert b\rvert}$	$\left\lvert\frac{-2}{3}\right\rvert = \frac{\lvert -2\rvert}{\lvert 3\rvert} = \frac{2}{3}$
6. $\lvert a + b\rvert \leq \lvert a\rvert + \lvert b\rvert$	$\lvert -2 + 3\rvert \leq \lvert -2\rvert + \lvert 3\rvert$

Practice: Evaluate each expression.

1. $\left|-\frac{2}{3}\right|$

2. $||-5| - |-8||$

3. $\frac{-4}{|-8|}$

4. $-4 - |-3 + 2|$

5. Show that if $|x| \leq a, for\ a \geq 0$ then $-a \leq x \leq a$.

6. Show that if $|x| \geq a, for\ a \geq 0$ then $x \in (-\infty, -a] \cup [a, \infty)$.

- If a and b are real numbers, then the **distance** between the points a and b on the number line is:

$$d(a, b) = |b - a|$$

Practice: Find the distance between the numbers -5 and 3 on the number line.
Answer:

Practice: Find the distance between the numbers $\frac{-2}{3}$ and $\frac{3}{4}$ on the number line.
Answer:

Section 1.2 – Exponents and Radicals

Expression	Base	Exponent	Meaning
$\mathbf{5^3}$	5	3	$5 \cdot 5 \cdot 5$
$\mathbf{6^1}$	6	1	6
$\boldsymbol{x^4}$	x	4	$x \cdot x \cdot x \cdot x$

Basic Properties of Integer Exponents

- $a^n = a \cdot a \cdot a \cdot \ldots a$ (n factors of a).
- $a^0 = 1$ (with the exception of 0^0 is undefined).
- $a^{-n} = \frac{1}{a^n}$ and $a^n = \frac{1}{a^{-n}}$

Laws for Working with Exponents

*In the table below, a and b represent real numbers, and m and n are integers.

Laws	Example
1. $a^m a^n = a^{m+n}$	$2^3 \cdot 2^2 = 2^{3+2} = 2^5$
2. $\frac{a^m}{a^n} = a^{m-n}$	$\frac{3^5}{3^2} = 3^{5-2} = 3^3$
3. $(a^m)^m$	$(4^3)^2 = 4^{3 \cdot 2} = 4^6$
4. $(ab)^m = a^m \cdot b^m$	$(3 \cdot 5)^2 = 3^2 \cdot 5^2$
5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{1}{3}\right)^2 = \frac{1^2}{3^2} = \frac{1}{9}$
6. $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$	$\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2$
7. $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$	$\frac{2^{-3}}{5^{-2}} = \frac{5^2}{2^3}$

Practice: Use the laws of exponents to simplify.

1. $\left(\frac{1}{2}xy^3\right)^2\left(\frac{4}{x^3y^5}\right)$	2. $\frac{a^2b^5}{a^6b^4}$	3. $(-3x^2y^5)^2(6x^4y^5)$
4. $(4xy^5)^2$	5. $\left(\frac{3x}{y^2}\right)^{-3}$	6. $(3x^4y^3x^6)^0$

Scientific Notation

A positive number x is in scientific notation if it is written: $x = a \times 10^n$, where $1 \leq a < 10$ and n is an integer.

Example: For question 1, express the number in scientific notation. For question 2, express the number in decimal notation.

1. 0.000043291 4.3291×10^{-5}	2. 3.216×10^4 32,160

Practice: For questions 1 and 2, express the number in scientific notation. For questions 3 and 4, express the numbers in decimal notation.

1. 14,293,000

2. 0.0000030103

3. 2.132×10^7

4. 2.132×10^{-7}

Radicals (roots)

- If n is any positive integer, then the **principle nth root** of a is defined as:

$$\sqrt[n]{a} = b \text{ means } b^n = a \qquad \text{(when it is defined)}$$

Example:

$$\sqrt[3]{27} = 3 \text{ because } 3^3 = 27$$

- The following are properties of roots.

Property	Example
1. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$	$\sqrt[3]{8 \cdot 27} = \sqrt[3]{8} \cdot \sqrt[3]{27} = 2 \cdot 3 = 6$
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt{\frac{4}{25}} = \frac{\sqrt{4}}{\sqrt{25}} = \frac{2}{5}$
3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt[3]{\sqrt{64}} = \sqrt[6]{64} = 2$
4. $\sqrt[n]{a^n} = a$ if n is odd.	$\sqrt[3]{-27} = \sqrt[3]{(-3)^3} = -3$
5. $\sqrt[n]{a^n} = \lvert a \rvert$ if n is even.	$\sqrt{25} = \sqrt{5^2} = 5$

- A **rational** exponent, $\frac{m}{n}$, where $\frac{m}{n}$ is in lowest terms and m and n are integers, with $n > 0$ can be written as:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

Practice: Simplify the expression and eliminate any negative exponents. Assume all variables represents positive numbers.

1. $\sqrt[6]{x^{18}}$	2. $\sqrt[7]{x^6} \cdot \sqrt[3]{x^5}$	3. $\left(\frac{8x^6}{y^3}\right)^{-\frac{2}{3}}$
4. $\sqrt[3]{27x^6y^9}$	5. $\left(\frac{4x^{-2}y^3}{12x^3y^{-2}}\right)^{-2}$	6. $(32x^{-10}y^{15})^{-\frac{1}{5}}$

Section 1.3 – Algebraic Expressions

- A **monomial** is an expression of the form ax^n, where a is a real number and n is a nonnegative integer.
  - A **binomial** is the sum of two monomials.
  - A **trinomial** is the sum of three monomials.
- The sum of monomials is called a **polynomial.**
  - The **degree** of a polynomial is equal to the highest power of the variable in the polynomial.

Adding and Subtracting Polynomials

Example: Perform the operation.

1.	2.	3.
$3x^2 + 2x - 5x^2$ $-2x^2 + 2x$	$(3x^3 + 2x^2) - (4x^3 - x^2 + 1)$ $3x^3 + 2x^2 - 4x^3 + x^2 - 1$ $-x^3 + 3x^2 - 1$	$(4x - 1) - (4x + 1)$ $4x - 1 - 4x - 1$ -2

Practice: Perform the operation.

1. $5x - (4x - 1)$

2. $1 - (2x^2 + 3x) + (4x^2 + 5)$

3. $(-x^3 + 2x^2 + 5x) - (x^2 + 6x)$

Multiplying Algebraic Expression

Example: Find the product.

1.	2.
$(3x-1)(4x+2)$ $(3x)(4x)+(3x)(2)+(-1)(4x)+(-1)(2)$ $12x^2+6x-4x-2$ $12x^2+2x-2$	$(x-3)(x^2+x-4)$ $x(x^2)+x(x)+x(-4)-3(x^2)-3(x)-3(-4)$ $x^3+x^2-4x-3x^2-3x+12$ $x^3-2x^2-7x+12$

Practice: Find the product.

1. $(4x+3)(x-2)$

2. $(x-5)^2$

3. $(x-1)(2x^2+3x-1)$

4. $(x-\sqrt{2})(x+\sqrt{2})$

Factoring

The first step of factoring to take out the **greatest common factor.**

Example: Factor the expression.

1.	**2.**	**3.**
$4x + 8$ $4(x + 2)$	$3x^3 + 6x^2 + 9x$ $3x(x^2 + 2x + 3)$	$4x^2y + 6xy^2 + 2xy$ $2xy(2x + 3y + 1)$

Practice: Factor.

1. $3x - 9$

2. $2x^2 - 10x$

3. $3x^2 + 12x + 3$

4. $6x^2 + 8x$

5. $4x^5 - 2x^3 + 8x^2$

6. $3(x + 1) - x(x + 1)$

Factoring Trinomials of the Form $\boldsymbol{x^2 + bx + c}$

- A trinomial of the form $x^2 + bx + c$ will factor as: $(x + h)(x + k)$, where h and k are two integers whose product is c and sum is b.
  - If c is positive, h and k will have the same sign, which will be the sign of b.
  - If c is negative, h and k will have opposite signs.

Examples: Factor the trinomial.

1. $x^2 + 11x + 30$ <u>Factors of 30</u>: 1, 30 2, 15 3, 10 5, 6 $(x + 5)(x + 6)$	2. $x^2 - 11x + 10$ <u>Factors of 10</u>: 1, 10 2, 5 $(x - 1)(x - 10)$	3. $x^2 + 4x - 12$ <u>Factors of 12</u>: 1, 12 2, 6 3, 4 $(x + 6)(x - 2)$

Practice: Factor.

1. $x^2 + 8x + 12$

2. $x^2 - 8x + 15$

3. $x^2 + 8x + 7$

4. $x^2 - 11x + 18$

5. $x^2 + 5x - 14$

6. $x^2 + 18x - 19$

7. $x^2 - 2x - 35$

8. $x^2 - 23x - 50$

Factoring Trinomials of the Form $\boldsymbol{ax^2 + bx + c}$

Examples: Factor the trinomial $3x^2 - 8x + 4$ using all three options.

Guess and Check	**Re-write With a Coefficient of 1**	**AC Method**
Note: Signs will both be negative. Try: $(3x-1)(x-4)$ Product: $3x^2 - 13x + 4 \leftarrow$ No! Try: $(3x-4)(x-1)$ Product: $3x^2 - 7x + 4 \leftarrow$ No! Try: $(3x-2)(x-2)$ Product: $3x^2 - 8x + 4 \leftarrow$Yes! $(3x-2)(x-2)$	$x^2 - 8x + (3 \cdot 4)$ $x^2 - 8x + 12$ $(x-6)(x-2)$ $\left(x - \frac{6}{3}\right)\left(x - \frac{2}{3}\right)$ $(x-2)(3x-2)$	$a = 3, b = -8,$ and $c = 4$ $a \times c$ | b $3 \times 4 = 12$ | -8 $-1, -12$ | -13 $\boldsymbol{-2, -6}$ | $\boldsymbol{-8}$ $-3, -4$ | -7 $3x^2 - 2x - 6x + 4$ $(3x^2 - 2x) + (-6x + 4)$ $x(3x-2) - 2(3x-2)$ $(3x-2)(x-2)$

Example: Factor the trinomial: $3x^2 + x - 10$

Step 1	$x^2 + x - 30$	Multiply $-10 \cdot 3$, and make this number your constant.
Step 2	$(x+6)(x-5)$	Factor by finding two numbers whose product is -30 and whose sum is 1.
Step 3	$\left(x + \frac{6}{3}\right)\left(x - \frac{5}{3}\right)$	Divide both numbers by 3 (the original leading coefficient).
Step 4	$(x+2)\left(x - \frac{5}{3}\right)$	Simplify the fractions.
Step 5	$(x+2)(3x-5)$	For any fraction that remains, bring the denominator up to become the coefficient of x.

Example: Factor.

$2x^2 + 7x + 6$

$x^2 + 7x + 12$

$(x+4)(x+3)$

$\left(x + \frac{4}{2}\right)\left(x + \frac{3}{2}\right)$

$(x+2)(2x+3)$

Practice: Factor.

1. $3x^2 + 19x + 20$

2. $2x^2 - 8x - 10$

3. $4x^2 - 7x - 2$

4. $3x^2 - 20x - 7$

5. $6x^2 + 19x + 10$

6. $5x^3 - 4x^2 - 12x$

7. $x^2 + 4x + 4$

8. $x^2 - 4x + 4$

9. $x^2 - 9$

Special Factoring Formulas

Name	Formula	Example
Difference of Perfect Squares	$A^2 - B^2 = (A - B)(A + B)$	$x^2 - 16$ $x^2 - 4^2$ $(x - 4)(x + 4)$
Perfect Square	$A^2 + 2AB + B^2 = (A + B)^2$	$x^2 + 8x + 16$ $(x + 4)(x + 4)$ $(x + 4)^2$
Perfect Square	$A^2 - 2AB + B^2 = (A - B)^2$	$x^2 - 6x + 9$ $(x - 3)(x - 3)$ $(x - 3)^2$
Difference of Cubes	$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$	$x^3 - 27$ $x^3 - 3^3$ $(x - 3)(x^2 + 3x + 9)$
Sum of Cubes	$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$	$x^3 + 64$ $x^3 + 4^3$ $(x + 4)(x^2 - 4x + 16)$

Practice: Factor.

1. $x^2 - 25$

2. $x^2 + 10x + 25$

3. $x^3 - 8$

4. $36 - x^2$

5. $\mathbf{2(a - 1)^2 - 3}$

Factor by Grouping

Example: Factor by grouping.

$12x^3 + 2x^2 - 30x - 5$	
$2x^2(6x + 1) - 5(6x + 1)$	Factor out $2x^2$ from the first two terms and – 5 from the last two terms.
$(6x + 1)(2x^2 - 5)$	Since $6x + 1$ is in each set of parenthesis, factor that out. The coefficients will combine to be the other factor.

Example: Factor.

1. $x^3 - 4x^2 + 3x - 12$ $x^2(x - 4) + 3(x - 4)$ $(x - 4)(x^2 + 3)$	2. $3x^3 - 9x - x^2 + 3$ $3x(x^2 - 3) - 1(x^2 - 3)$ $(x^2 - 3)(3x - 1)$

Practice: Factor.

1. $2x^3 + 5x^2 - 6x - 15$

2. $2x^3 - 5x - 2x^2 + 5$

Section 1.4 – Rational Expressions

- A **rational** expression is a fractional expression in which both the numerator and denominator are polynomials.
- The **domain** of an algebraic expression will be all values of the variable except those that would cause it to be undefined or imaginary.
  - For now, the domain will be all real numbers, unless the expression involves an even root or is a rational expression.
  - The number under an even root must be greater than or equal to zero.
  - The denominator of a rational expression cannot equal zero.

Example: Find the domain of the expression.

1. $4x^2 + \frac{1}{2}x - 3$ Domain: $\{x \mid x$ is a real number$\}$	2. $\frac{x}{(x+1)(x-2)}$ Domain: $\{x \mid x \neq -1$ and $x \neq 2\}$	1. $\sqrt{x+1}$ $x + 1 > 0$ $x > -1$ Domain: $\{x \mid x \geq -1\}$

Practice: Find the domain of the expression.

1. $\frac{4x-1}{x^2+3x-4}$

2. $\frac{1}{2}x^4 - 3$

3. $\sqrt{2x-3}$

4. $\frac{\sqrt{x}}{x-5}$

Simplifying Rational Expressions

- To simplify a rational expression:
  - Factor the numerator and denominator.
  - Cancel out any common factors from the numerator and denominator.
- To multiply rational expressions, multiply the numerators and denominators together, then simplify.
- To divide rational expressions, multiply the first expression by the reciprocal of the second, then simplify.

Example: Perform the operation, then simplify.

1. $\frac{x}{x-3}\cdot\frac{x^2-4x+3}{x^2+2x}$ $\frac{x}{x-3}\cdot\frac{(x-3)(x-1)}{x(x+2)}$ $\frac{\cancel{x(x-3)}(x-1)}{\cancel{x(x-3)}(x+2)}$ $\frac{x-1}{x+2}$	1. $\frac{x^2-16}{2x-10}\cdot\frac{2}{x+4}$ $\frac{(x-4)(x+4)}{2(x-5)}\cdot\frac{2}{(x+4)}$ $\frac{\cancel{2}(x-4)\cancel{(x+4)}}{\cancel{2}(x-5)\cancel{(x+4)}}$ $\frac{x-4}{x-5}$	2. $\frac{x-4}{x^2-1}\div\frac{3x-12}{x+1}$ $\frac{x-4}{(x-1)(x+1)}\cdot\frac{x+1}{3(x-4)}$ $\frac{\cancel{(x-4)(x+1)}}{3(x-1)\cancel{(x+1)(x-4)}}$ $\frac{1}{3(x-1)}$

Practice: Perform the operation, then simplify.

1. $\frac{5x}{x-4}\cdot\frac{2x-8}{25}$

2. . $\frac{x^2+5x+6}{x^2-9}\cdot\frac{x^2-2x-3}{x^2+3x+2}$

3. $\frac{2-x}{x+5} \div \frac{x-2}{x^2-25}$

4. $\frac{x^2+6x+8}{x-2} \div \frac{x^2-4}{x-2}$

Adding and Subtracting Rational Expressions

- To add or subtract rational expressions:
  - Find the **least common denominator** and rewrite each expression using the common denominator.
  - Add the numerators together, but keep the common denominator.

Example: Perform the operation and simplify.

1. $\frac{3}{x}+\frac{2}{x-1}$	2. $\frac{-2}{x+1}+\frac{4}{x-1}$	3. $\frac{4}{(x-1)^2}-\frac{3}{(x-1)(x+2)}$
$\frac{3(x-1)}{x(x-1)}+\frac{2x}{x(x-1)}$	$\frac{-2(x-1)}{(x+1)(x-1)}+\frac{4(x+1)}{(x+1)(x-1)}$	$\frac{4(x+2)}{(x-1)^2(x+2)}-\frac{3(x-1)}{(x-1)^2(x+2)}$
$\frac{3(x-1)+2x}{x(x-1)}$	$\frac{-2(x-1)+4(x+1)}{(x+1)(x-1)}$	$\frac{4(x+2)-3(x-1)}{(x-1)^2(x+2)}$
$\frac{3x-3+2x}{x(x-1)}$	$\frac{-2x+2+4x+4}{(x+1)(x-1)}$	$\frac{4x+8-3x+3}{(x-1)^2(x+2)}$
$\frac{5x-3}{x(x-1)}$	$\frac{2x+6}{(x+1)(x-1)}$	$\frac{x+11}{(x-1)^2(x+2)}$

Practice: Perform the operation and simplify.

1. $\frac{3}{x+4}+\frac{2}{x-5}$

2. $\frac{4}{x}-\frac{3x}{x-2}$

3. $\frac{-2}{x+1} - \frac{2x}{x(x+1)}$

4. $\frac{4}{2x} + \frac{5x}{4}$

Compound Fractions

- A **compound fraction** is a fraction that has a fraction in either the numerator, denominator, or both.

Example: Simplify.

1. $$\frac{\frac{3}{x}+2}{\frac{1}{x}+5}=\frac{\frac{3}{x}+\frac{2x}{x}}{\frac{1}{x}+\frac{5x}{x}}=\frac{\frac{3+2x}{x}}{\frac{1+5x}{x}}$$ $$\frac{3+2x}{x}\div\frac{1+5x}{x}$$ $$\frac{3+2x}{x}\cdot\frac{x}{1+5x}$$ $$\frac{\cancel{x}(3+2x)}{\cancel{x}(1+5x)}$$ $$\frac{3+2x}{1+5x}$$	5. $$\frac{\frac{-x}{y}+2x}{2-\frac{1}{y}}=\frac{\frac{-x}{y}+\frac{2xy}{y}}{\frac{2y}{y}-\frac{1}{y}}=\frac{\frac{-x+2xy}{y}}{\frac{2y-1}{y}}$$ $$\frac{-x+2xy}{y}\div\frac{2y-1}{y}$$ $$\frac{-x+2xy}{y}\cdot\frac{y}{2y-1}$$ $$\frac{y(-x+2xy)}{y(2y-1)}$$ $$\frac{x\cancel{y(2y-1)}}{\cancel{y(2y-1)}}$$ $$x$$

Practice: Simplify.

1. $\dfrac{\frac{2}{x}-4}{\frac{3}{x}+1}$

2. $\dfrac{\frac{3}{x+1}-2}{\frac{1}{x+1}}$

3. $\frac{2+\frac{3}{x+5}}{\frac{1}{x+5}-1}$

Rationalizing the Denominator or the Numerator

Multiply the numerator and denominator by the **conjugate** of the expression that has the square root.

The **conjugate** of $a + \sqrt{b}$ is $a - \sqrt{b}$

Example: Rationalize the denominator.

1.	4.
$\frac{2}{1-\sqrt{3}}$	$\frac{3}{\sqrt{x}+1}$
$\frac{2(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})}$	$\frac{3(\sqrt{x}-1)}{(\sqrt{x}+1)(\sqrt{x}-1)}$
$\frac{2(1+\sqrt{3})}{1\cancel{+\sqrt{3}-\sqrt{3}}-(\sqrt{3})^2}$	$\frac{3(\sqrt{x}-1)}{(\sqrt{x})^2\cancel{-\sqrt{x}+\sqrt{x}}-1}$
$\frac{2(1+\sqrt{3})}{1-3}$	$\frac{3(\sqrt{x}-1)}{x-1}$
$\frac{2(1+\sqrt{3})}{-2}$	
$-(1+\sqrt{3})$	

Practice: Rationalize the denominator.

1. $\frac{3}{1+\sqrt{3}}$

2. $\frac{x}{1-\sqrt{2}}$

3. $\frac{x+2}{\sqrt{x}+\sqrt{2}}$

Partial Fractions

- The sum of a set of rational expressions is found by combining two or more such expressions into one expression.
- The reverse process is taking one rational expression and writing it as the sum of two or more rational expressions.
    - This process is called the partial fraction decomposition.

Add rational expressions →

$$\frac{1}{x-1}+\frac{2}{x+2}=\frac{3x}{(x-1)(x+2)}$$

← Partial Fraction Decomposition

Steps for Finding the Partial Fraction Decomposition of a Rational Expression

- If the rational expression has a higher degree in the numerator than the denominator, divide the denominator into the numerator.
- The following steps would then apply to the remainder, which would be a proper fraction (highest degree in the denominator).
    - Factor the denominator completely.
    - Set the rational expression equal to the sum of two or more rational expressions based on the factors of the denominator.
        - If the factor is linear, write the rational expression as: $\frac{A}{mx+b}$.
        - If the factor is quadratic, write the rational expression as: $\frac{Ax+B}{ax^2+bx+c}$
        - If a factor repeats, add rational expressions with denominator having powers of the factor from 1 until the power:
        $$\frac{A}{(mx+b)}+\frac{B}{(mx+b)^2}+\frac{C}{(mx+b)^3}\dots$$
        - Note: Use each capital letter only once.
    - Multiply the entire equation by the least common denominator.
    - Use algebraic techniques and substitution to solve for the variables $A, B, C, \dots$

Example: Rewrite the following rational expression as the sum of two or more rational expressions with $A, B, C, \dots$ as the numerator of the fractions as indicated above.

1. $\frac{2x}{(x-2)(x+5)^2}=\frac{A}{x-2}+\frac{B}{x+5}+\frac{C}{(x+5)^2}$	2. $\frac{5x}{(x-1)(x^2+2x+5)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+2x+5}$

Example: Find the partial fraction decomposition.

	$\frac{9x+21}{x^2+2x-15}$
1. Factor the denominator.	$\frac{9x+21}{(x+5)(x-3)}$
2. Break up the rational expression using the rules for partial fractions.	$\frac{9x+21}{(x+5)(x-3)} = \frac{A}{x+5} + \frac{B}{x-3}$
3. Multiply each term by the least common denominator: $(x+5)(x-3)$.	$\frac{(9x+21)(x+5)(x-3)}{(x+5)(x-3)} = \frac{A(x+5)(x-3)}{x+5} + \frac{B(x+5)(x-3)}{x-3}$ $9x+21 = A(x-3) + B(x+5)$
4. Plug in -5 for x in the equation above to solve for A.	$x=-5$: $9(-5)+21 = A(-5-3) + B(-5+5)$ $-45+21 = A(-8) + B(0)$ $-24 = -8A$ $A = 3$
5. Plug in 3 for x in the equation above to solve for B.	$x=3$: $9(3)+21 = A(3-3) + B(3+5)$ $27+21 = A(0) + B(8)$ $48 = 8B$ $B = 6$
6. Re-write the equation with the values of A and B.	$\frac{3}{x+5} + \frac{6}{x-3}$

Practice: Find the partial fraction decomposition for each rational expression.

1. $\frac{x}{x^2+4x-5}$

2. $\frac{3x}{x^2+x-2}$

3. $\frac{13}{(2x+3)(x^2+1)}$

Section 1.5 –Equations

Linear equations	Nonlinear equations	
$4x - 5 = 3$	$x^2 + 2x = 8$	Not linear; contains the square of the variable
$2x = \frac{1}{2}x - 7$	$\sqrt{x} - 6x = 0$	Not linear; contains the square root of the variable
$x - 6 = \frac{x}{3}$	$\frac{3}{x} - 2x = 1$	Not linear; contains the reciprocal of the variable

Example: Solve.

1.	2.
$3(1-x) = 2(4x+1)$ $3 - 3x = 8x + 2$ $3 - 11x = 2$ $-11x = -1$ $x = \frac{1}{11}$	$\frac{2}{x-1} + \frac{1}{x+1} = \frac{4x}{x^2-1}$ $\frac{2(x+1)\cancel{(x-1)}}{\cancel{(x-1)}} + \frac{1\cancel{(x+1)}(x-1)}{\cancel{(x+1)}} = \frac{4x\cancel{(x+1)(x-1)}}{\cancel{(x+1)(x-1)}}$ $2(x+1) + 1(x-1) = 4x$ $2x + 2 + x - 1 = 4x$ $3x + 1 = 4x$ $1 = x$

Practice: Solve.

1. $3(x-1) = 5(2-x)$

2. $\frac{1}{2}(4x-2) = \frac{1}{3}(3x+12)$

3. $\frac{3}{x-1}+2=\frac{x}{x-1}$

The process for solving an equation depends on the type of equation it is.

Type Of Equation	Rules To Solve	Example
Absolute Value	• Solve for the absolute value expression. • Re-write as two equations. $\lvert ax+b\rvert = k,\ k>0$ becomes: $ax+b=k$ and $ax+b=-k$	$\lvert 2x-3\rvert + 5 = 7$ $\lvert 2x-3\rvert = 2$ $2x-3=2$ and $2x-3=-2$ $2x=5$ and $2x=1$ $x=\frac{5}{2}$ and $x=\frac{1}{2}$
Quadratic	**Zero product property** • Move all terms to one side of equation. • Factor the expression*. • Set each factor equal to zero and solve.	$x^2+5=-6x$ $x^2+6x+5=0$ $(x+5)(x+1)=0$ $x+5=0$ and $x+1=0$ $x=-5$ and $x=-1$
	• **Completing the square**: ▪ Get the terms with x on one side of the equation and the constant on the other. ▪ Add $\left(\frac{b}{2}\right)^2$ to both sides of the equation. ▪ Factor the trinomial. (It will be a perfect square!) ▪ Take the square root of both sides ▪ Solve the remaining linear equation.	$x^2+4x-3=0$ $x^2+4x=3$ $x^2+4x+\left(\frac{4}{2}\right)^2=3+\left(\frac{4}{2}\right)^2$ $x^2+4x+4=3+4$ $(x+2)(x+2)=7$ $(x+2)^2=7$ $\sqrt{(x+2)^2}=\pm\sqrt{7}$ $x+2=\pm\sqrt{7}$ $x=-2\pm\sqrt{7}$
	*If not factorable, use the **quadratic formula**. $f(x)=ax^2+bx+c$: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	$x^2-3x=1$ $x^2-3x-1=0$ $a=1, b=-3, c=-1$ $x=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(-1)}}{2(1)}$ $x=\frac{3\pm\sqrt{9+4}}{2}$ $x=\frac{3\pm\sqrt{13}}{2}$

Polynomial	• If not given any additional information with the equation, it must be factorable. • Bring all terms to one side of the equation. • Factor the expression*. • Set each factor equal to zero and solve. *If it is not factorable, one factor may be given. Divide this factor into the expression to find the other factor.	$12x^3 + 15x = -20x^2 - 25$ $12x^3 + 20x^2 + 15x + 25 = 0$ $4x^2(3x+5) + 5(3x+5) = 0$ $(4x^2+5)(3x+5) = 0$ $4x^2 + 5 = 0$ and $3x + 5 = 0$ $4x^2 = -5$ and $3x = -5$ $x^2 = -\frac{5}{4}$ and $x = -\frac{5}{3}$ $x = \pm\sqrt{-\frac{5}{4}}$ $x = \pm i\frac{\sqrt{5}}{2}$
Rational	• Factor each denominator. • Multiply each term by the LCD. • Solve the remaining equation. • Verify that none of the solutions would result in 0 when plugged into the denominators of the original fractions.	$\frac{6}{x} - \frac{1}{x^2+6x} = \frac{1}{x}$ $\frac{6}{x} - \frac{1}{x(x+6)} = \frac{1}{x}$ $\frac{6x(x+6)}{x} - \frac{1x(x+6)}{x(x+6)} = \frac{1x(x+6)}{x}$ $6(x+6) - 1 = 1(x+6)$ $6x + 36 - 1 = x + 6$ $6x + 35 = x + 6$ $5x + 35 = 6$ $5x = -29$ $x = -\frac{29}{5}$
Root	• Solve for the root expression. • Raise both sides of the equation to the value of the root. • Solve the remaining equation. • Check each proposed solution in the original equation to see which value(s) are solutions.	$\sqrt{x+4} = x - 2$ $\left(\sqrt{x+4}\right)^2 = (x-2)^2$ $x + 4 = x^2 - 4x + 4$ $0 = x^2 - 5x$ $0 = x(x-5)$ $x = 0$ and $x - 5 = 0$ $x = 0$ and $x = 5$ Check $x = 0$: $\sqrt{0+4} = 0 - 2$ $2 = -2$ No! $x = 5$: $\sqrt{5+4} = 5 - 2$ $3 = 3$ Yes! Solution: {5}

Example: Solve. Find first the domain and then give exact answers.

1. $\frac{4}{x+1} + \frac{1}{x^2-5x-6} = \frac{1}{x-6}$

2. $2 + \sqrt{x+10} = x$

3. $|4 - 3x| + 2 = 6$

4. $5x^2 - 10x = 0$

5. $\frac{5}{x+1} = \frac{6}{x^2-2x-3} + \frac{1}{x-3}$

6. $.(x+6)^{\frac{1}{2}} - (2x-4)^{\frac{1}{2}} = 0$

7. $|3x+1| = \left|\frac{1}{2}x - 5\right|$

8. $x^2 - 3x = -5$

9. $\sqrt[3]{x+1} = 2$

Section 1.6 – Complex Numbers

- A **complex number** is an expression of the form $a + bi$, where:
  - a is the real part.
  - b is the imaginary part.
- i is defined as follows:

$$i = \sqrt{-1}$$
$$i^2 = -1$$

- Complex numbers are equal if and only if their real parts and imaginary parts are equal.
- Perform operations with complex numbers by treating i like a variable.
  - Remember to replace i^2 with -1 and simplify.
  - Simplify i^n when $n > 1$ (use properties of exponents as necessary).
  - To divide, multiply the numerator and denominator by the complex conjugate of the denominator.

Example: Express in the form $a + bi$.

1.	2.	3.
$3 - 2i(1 + i)$ $3 - 2i - 2i^2$ $3 - 2i - 2(-1)$ $3 - 2\mathrm{i} + 2$ $5 - 2i$	$\frac{3 - 2i}{1 + 5i}$ $\frac{3 - 2i}{1 + 5i} \cdot \frac{1 - 5i}{1 - 5i}$ $\frac{3 - 15i - 2i + 10i^2}{1 - 5i + 5i - 25i^2}$ $\frac{3 - 17i + 10(-1)}{1 - 25(-1)}$ $\frac{-7 - 17i}{26}$	i^{41} i^{40+1} $i^{40} \cdot i^1$ $(i^2)^{20} \cdot i$ $(-1)^{20} \cdot i$ $1 \cdot i$ i

Practice: Express in the form $a + bi$.

1. $(3 - 2i)^2$

2. $3i + 2i(4 - 5i)$

3. $\frac{4i+1}{3i-2}$

4. i^{24}

5. i^{27}

6. $\frac{1}{i^{-5}}$

Square Roots of Negative Numbers

- Use the fact that $i = \sqrt{-1}$ to simplify the square root of a negative number.
  - Simplify negative square roots **before** performing operations on them.

Example: Evaluate the radical expression and express the result in $a + bi$ form.

1.	2. $\sqrt{-6}\sqrt{-3}$	3. $\left(2-\sqrt{-2}\right)\left(\sqrt{-1}+1\right)$
$\sqrt{-16}$	$\sqrt{-1\cdot 6}\sqrt{-1\cdot 3}$	$\left(2-i\sqrt{2}\right)(i+1)$
$\sqrt{-1\cdot 16}$	$\sqrt{-1}\cdot\sqrt{6}\cdot\sqrt{-1}\cdot\sqrt{3}$	$2i+2-i^2\sqrt{2}-i\sqrt{2}$
$\sqrt{-1}\cdot\sqrt{16}$	$i\sqrt{6}\cdot i\sqrt{3}$	$2i+2-(-1)\sqrt{2}-i\sqrt{2}$
$i\cdot 4$	$i^2\sqrt{6\cdot 3}$	$2i+2+\sqrt{2}-i\sqrt{2}$
$4i$	$-1\sqrt{18}$	$\left(2+\sqrt{2}\right)+\left(2i-i\sqrt{2}\right)$
	$-1\sqrt{9\cdot 2}$	$\left(2+\sqrt{2}\right)+\left(2-\sqrt{2}\right)i$
	$-1\sqrt{9}\sqrt{2}$	
	$-3\sqrt{2}$	

Practice: Evaluate the radical expression and express the result in $a + bi$ form.

1. $\sqrt{-12}$

2. $\sqrt{-5}\sqrt{-10}$

3. $\sqrt{\frac{-25}{16}}$

4. $\left(3+\sqrt{-2}\right)\left(2+\sqrt{-4}\right)$

Complex Solutions of Quadratic Equations

Example: Find all solutions of the equation and express them in the form $a + bi$.

$$3x^2 + 2x = -1$$
$$3x^2 + 2x + 1 = 0$$
$$a = 3; b = 2; \text{ and } c = 1$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(3)(1)}}{2(3)} = \frac{-2 \pm \sqrt{4 - 12}}{6} = \frac{-2 \pm \sqrt{-8}}{6}$$
$$x = \frac{-2 \pm \sqrt{-4 \cdot 2}}{6} = \frac{-2 \pm 2i\sqrt{2}}{6} = \frac{-2}{6} \pm \frac{2i\sqrt{2}}{6}$$
$$x = \frac{-1}{3} \pm \frac{i\sqrt{2}}{3}$$

Practice: Find all solutions of the equation and express them in the form $a + bi$.

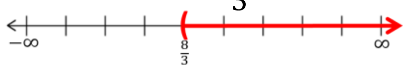
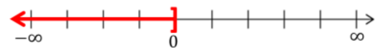
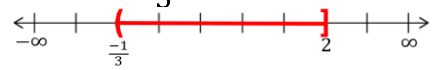
1. $x^2 + 25 = 0$

2. $x^2 + x + 5 = 0$

3. $2x^2 = 3x - 6$

Section 1.7 – Inequalities

Example: Solve linear inequalities. Represent the final answer in interval notation.

1.	2.	3.
$3x - 1 > 7$ $3x > 8$ $x > \frac{8}{3}$  $\left(\frac{8}{3}, \infty\right)$	$2 - x \geq 2$ $-x \geq 0$ $x \leq 0$  $(-\infty, 0]$	$-2 < 3x - 1 \leq 5$ $-1 < 3x \leq 6$ $\frac{-1}{3} < x \leq 2$  $\left(\frac{-1}{3}, 2\right]$

Practice: Solve. Represent the final answer in interval notation.

1. $3x + 2 \geq 5$

2. $3 - 5x > 1$

3. $14x + 3$

3. $-1 \leq \frac{1}{2}x - 4 < 3$

5. $-10 < 2x + 1 \leq 3$

6. $4 \leq 1 - x \leq 8$

7. $-2x - 3 < 5$.

8. $2 - 3x \geq 3$.

Solving nonlinear inequalities. Express the solution in interval notation.

1.

$$x^2 - 4x < 5$$
$$x^2 - 4x - 5 < 0$$
$$(x-5)(x+1) < 0$$

$$x - 5 = 0 \text{ or } x + 1 = 0$$
$$x = 5 \text{ or } x = -1$$

Split the number line at $x = -1$ and $x = 5$. This will create three intervals: $(-\infty, -1), (-1,5)$, and $(5, \infty)$.

Plug in a value from each interval into both factors and determine the sign of the product.

Interval	$(-\infty, -1)$	$(-1,5)$	$(5, \infty)$
Test value in Interval	$x = -2$	$x = 0$	$x = 6$
Plug into factor: $(x-5)$	$-2-5$ -7	$0-5$ -5	$6-5$ 1
Plug into factor: $(x+1)$	$-2+1$ -1	$0+1$ 1	$6+1$ 7
Sign of $(x-5) \cdot (x+1)$	$-7(-1)$ 7	$-5(1)$ -5	$1(7)$ 7

Since the original inequality was: $(x-5)(x+1) < 0$, the solution will be the interval(s) where the test value resulted in the product being negative:

$(-1,5)$

Note: Use parenthesis since the inequality symbol was $<$, not $\leq$.

2.

$$\frac{2}{x+4} \geq -1$$
$$\frac{2}{x+4} + 1 \geq 0$$
$$\frac{2 + 1(x+4)}{x+4} \geq 0$$
$$\frac{x+6}{x+4} \geq 0$$

Find the zeros and vertical asymptotes:

Zero: $x + 6 = 0$ V.A: $x + 4 = 0$

$x = -6$ $x = -4$

Split the number line at $x = -6$ and $x = -4$.

Note: Use a bracket at -6, since the inequality includes the equal sign. Use parenthesis at -4, since the rational expression is undefined there.

$(-\infty, -6] \cup [-6, -4) \cup (-4, \infty)$

Plug in a value from each interval into the numerator and denominator and determine the sign of the quotient.

Interval	$(-\infty, -6]$	$[-6, -4)$	$(-4, \infty)$
Test value in Interval	$x = -8$	$x = -5$	$x = 0$
Plug into factor: $(x+6)$	$-8+6$ -2	$-5+6$ 1	$0+6$ 6
Plug into factor: $(x+4)$	$-8+4$ -4	$-5+4$ -1	$0+4$ 4
Sign of $\frac{x+6}{x+4}$	$\frac{-2}{-4}$ $\frac{1}{2}$	$\frac{1}{-1}$ -1	$\frac{6}{4}$ 1.5

Since the inequality is ≥ 0, the solution will be the interval(s) where the test value resulted in a positive quotient:

$(-\infty, -6] \cup (-4, \infty)$

Practice: Solve. Express the solution in interval notation.

1. $x^2 - 2x - 24 > 0$

2. $x^2 \leq 25$

3. $3x^2 < 6x$

4. $\frac{2x-5}{x+7} > 0$

5. $\frac{1}{x+5} \leq 4$

- Solving absolute value inequality, look at the inequality symbol in the problem:

	Case 1 $\|ax+b\| > k$	Case 2 $\|ax+b\| < k$
Setup	$ax+b > k$ or $ax+b < -k$	$-k < ax+b < k$
Solution	$(-\infty, m) \cup (n, \infty)$	(m, n)
Example	$\|2x-3\| > 5$ $2x-3 > 5$ or $2x-3 < -5$ $2x > 8$ or $2x < -2$ $x > 4$ or $x < -1$ Solution: $(-\infty, -1) \cup (4, \infty)$	$\|2x-3\| < 5$ $-5 < 2x-3 < 5$ $-2 < 2x < 8$ $-1 < x < 4$ Solution: $(-1,4)$

**Note: In above cases, $k > 0$ and $m < n$.*

- Pay special attention when $k < 0$.

Example: Use the graph of $f(x) = |2x-4|$ to solve.

a) $|2x-4| > -1$
$(-\infty, \infty)$

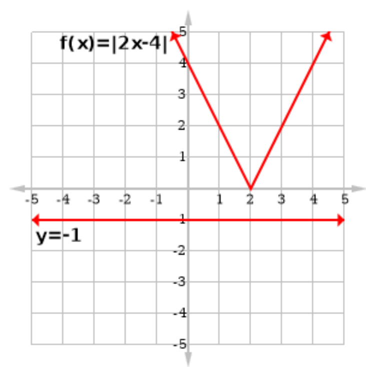
b) $|2x-4| < -1$
$\emptyset$

c) $|2x-4| > 0$
$(-\infty, 2) \cup (2, \infty)$

d) $|2x-4| \geq 0$
$(-\infty, \infty)$

e) $|2x-4| < 0$
$\emptyset$

f) $|2x-4| \leq 0$
$\{2\}$



Practice: Solve. Represent the final answer in interval notation.

1. $|4x+1| < 5$

2. $|2+x| - 3 \leq 6$

3. $|3x-4| < -2$

4. $\left|\frac{1}{2}x + 3\right| > 2$

5. $|3 - x| \geq 3$

6. $|5x + 7| \geq -6$

Section 1.8 – Graphs of Equations; Circles

- To find the distance between any two points $A(x_1, y_1)$ and $B(x_2, y_2)$, use the distance formula:

$$d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- To find the midpoint of the line segment between any two points $A(x_1, y_1)$ and $B(x_2, y_2)$, use the formula:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example: Find the distance between the points $(-2,1)$ and $(5,-2)$. Then find the midpoint of the line segment between the two points.

Distance:	Mid – Point
$D = \sqrt{\left(5 - (-2)\right)^2 + (-2 - 1)^2}$ $\sqrt{(7)^2 + (-3)^2}$ $\sqrt{49 + 9}$ $\sqrt{58}$	$\left(\frac{-2 + 5}{2}, \frac{1 + (-2)}{2}\right)$ $\left(\frac{3}{2}, \frac{-1}{2}\right)$

Practice: Find the distance and midpoint between the points $(4,-3)$ and $(-2,0)$.

Graphs of Equations in Two Variables

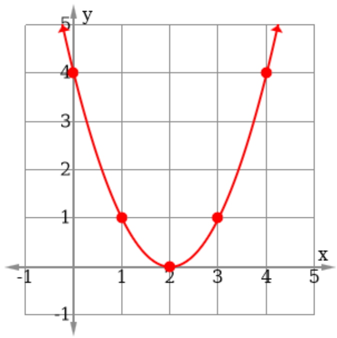
Example: Graph using the method indicated.

1. $\underline{y = (x-2)^2}$; Table

Choose values to plug in for x:

x	$y = (x-2)^2$	(x, y)
0	$(0-2)^2 = (-2)^2 = 4$	(0,4)
1	$(1-2)^2 = (-1)^2 = 1$	(1,1)
2	$(2-2)^2 = 0^2 = 0$	(2,0)
3	$(3-2)^2 = 1^2 = 1$	(3,1)
4	$(4-2)^2 = 2^2 = 4$	(4,4)

Plot the points and connect them with a smooth line:

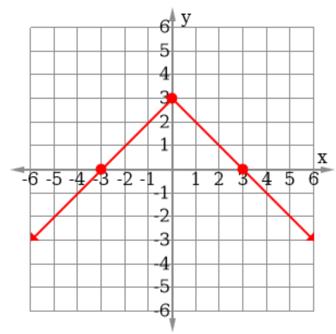


5. $\underline{y = -|x| + 3}$; Intercepts

Find the intercepts by:

y −intercept: Plug in 0 for x	x −intercept: Plug in 0 for y
$y = -\|0\| + 3$ $y = -0 + 3$ $y = 3$ y −intercept: (0,3)	$0 = -\|x\| + 3$ $-3 = -\|x\|$ $3 = \|x\|$ $x = 3$ or $x = -3$ x −intercepts: (3,0) and (−3,0)

Plot the intercepts and connect them with a smooth curve.



Circles

- The equation of a circle with center at (h, k) and radius r is:

$$(x-h)^2 + (y-k)^2 = r^2$$

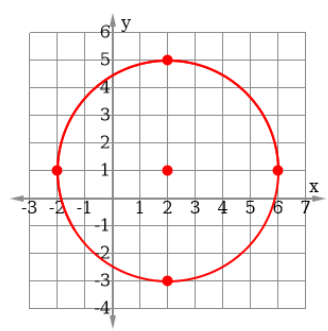
Example: Graph the circle.

$$(x-2)^2 + (y-1)^2 = 16$$

Center: (2,1)

Radius: 4

To graph, plot the center and then plot the point four units above, below, left, and right of the center. Connect these four points to form the circle.



Practice. Graph using the method indicated.

1. $y = \sqrt{x-1}$; Table

2. $4x + 6y = 12$; Intercepts

3. $(x-3)^2 + y^2 = 9$

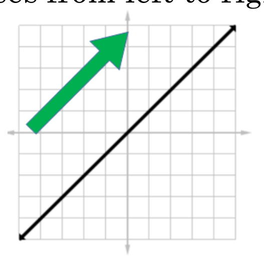
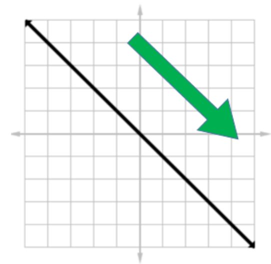
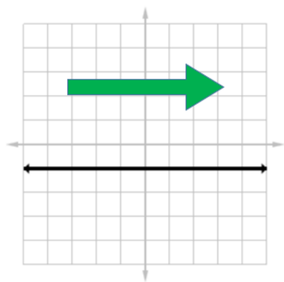
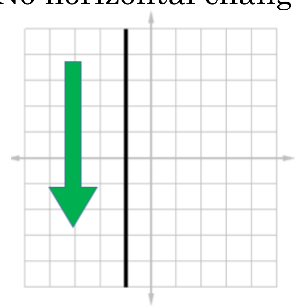
Practice: Write the equation of the circle described.

1. Center: $(-3,2)$; radius of 5

2. Center: $(0,4)$ Through point: $(1,-1)$

Section 1.9 – Lines

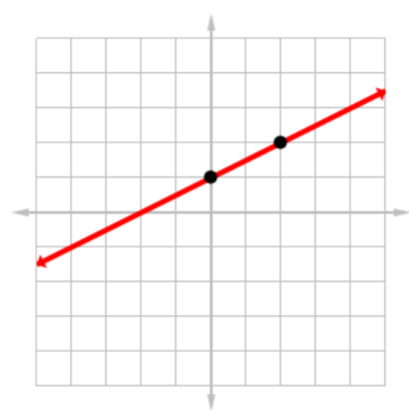
- The **slope** of a line is the rate of change of the rise (vertical) over run (horizontal).
  - There are four types of slope:

1. Positive	2. Negative
Rises from left to right. 	Falls from left to right. 
3. Zero: Horizontal Line	**4. Undefined: Vertical Line**
No vertical change. 	No horizontal change. 

Example: Find the slope of the line graphed.

The change in the y values (heights) from the point (0,1) to (2,2) is 1.
The change in the x values (horizontal change) from the point (0,1) to (2,2)is 2.
Therefore, the slope is:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{1}{2}$$



Equations of Lines

- There are three ways to represent the equation of any line:

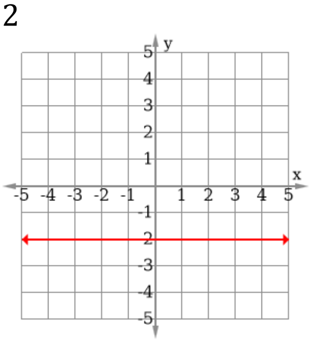
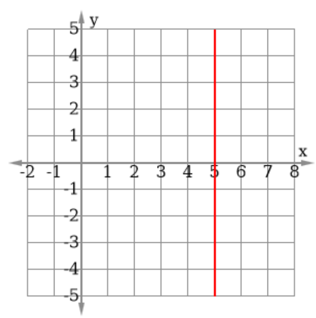
Form	Equation	Description
Point – Slope	$y - y_1 = m(x - x_1)$	Equation of the line though the point (x_1, y_1) with slope m.
Slope – Intercept	$y = mx + b$	Equation of the line with y −intercept at $(0, b)$ and slope m.
General Equation	$Ax + By + C = 0$	Highest power of x and y is one, and A, B both are not zero.

- Lines are **parallel** if they have the same slope, but different $y -$ intercepts.
- Lines are **perpendicular** if their slopes are opposite reciprocals of each other.
    - The product of the slopes of perpendicular lines is equal to -1.
    - Horizontal and vertical lines are perpendicular to each other.

Horizontal and Vertical Lines

- A horizontal line is of the form $y = b$, where b is the height of the line.
    - A horizontal line has a slope of zero.
- A vertical line is of the form $x = a$, where a is the location of the line along the x axis.
    - A vertical line has undefined slope.

Example: Graph the line.

1. $y = -2$ 	2. $x = 5$ 

Example: Find the equation of the line in standard form.

1. Slope -2; Through the point $(-4, -1)$ $y - (-1) = -2(x - (-4))$ $y + 1 = -2(x + 4)$ $y + 1 = -2x - 8$ $y = -2x - 9$	2. Through $\left(-2, \frac{1}{2}\right)$; Perpendicular to $x = 2$ The line $x = 2$ is a vertical line, and horizontal lines are perpendicular to vertical lines. The equation of any horizontal line is $y = b$. In this case $b = \frac{1}{2}$: $y = \frac{1}{2}$

Practice: Give the slope and $y -$ intercept of each line. Then, graph the line and label at least two points on the line.

1. $y = -x + 2$

2. $y = \frac{2}{3}x$

3. $x = -2$

Practice: Write the equation of the line described in slope – intercept form.

1. Through (3,2) and $(-1, 5)$.

2. Through (2,0); Parallel to $2x + 3y = -6$.

3. $x -$ intercept at $(-2,0)$; $y -$ intercept at $(0,2)$.

4. Through $\left(\frac{1}{2}, \frac{1}{2}\right)$, perpendicular to $y-$ axis.

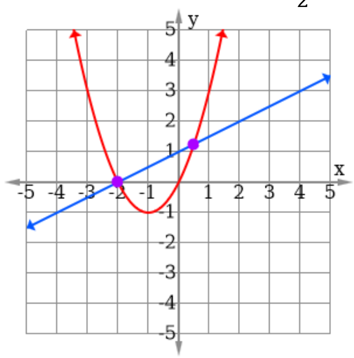
1.10– Solving Equations and Inequalities Graphically

Example: Solve the equation graphically.

1.

$$x^2 + 2x = \frac{1}{2}x + 1$$

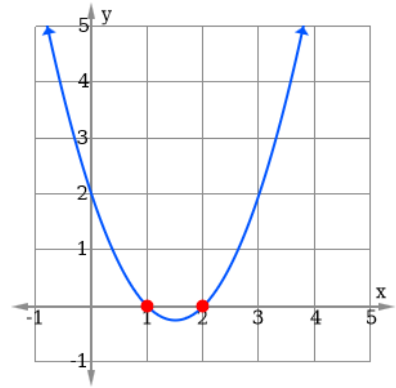
Graph $y_1 = x^2 + 2x$ and $y_2 = \frac{1}{2}x + 1$



Two real solutions: $x = -2$ and $x = \frac{1}{2}$

3.

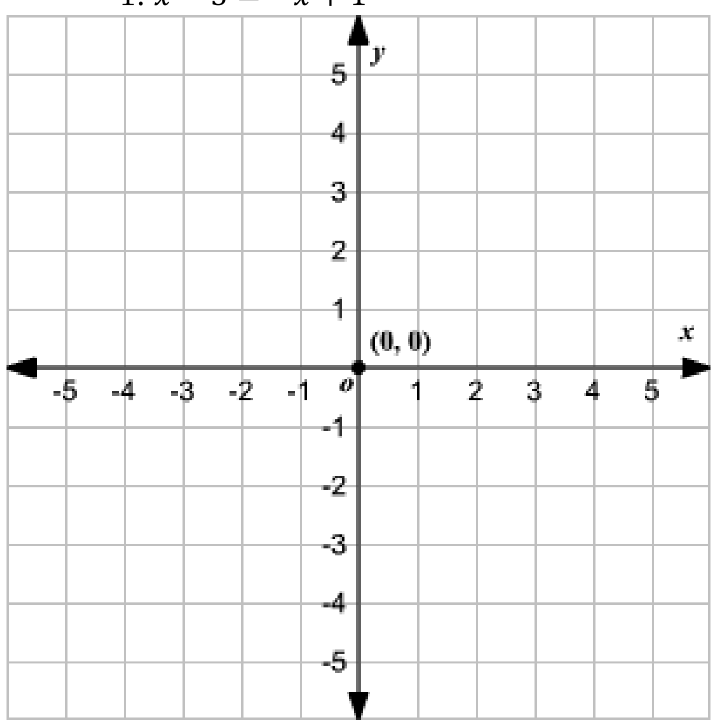
$$x^2 + 2 = 3x$$
$$x^2 - 3x + 2 = 0$$



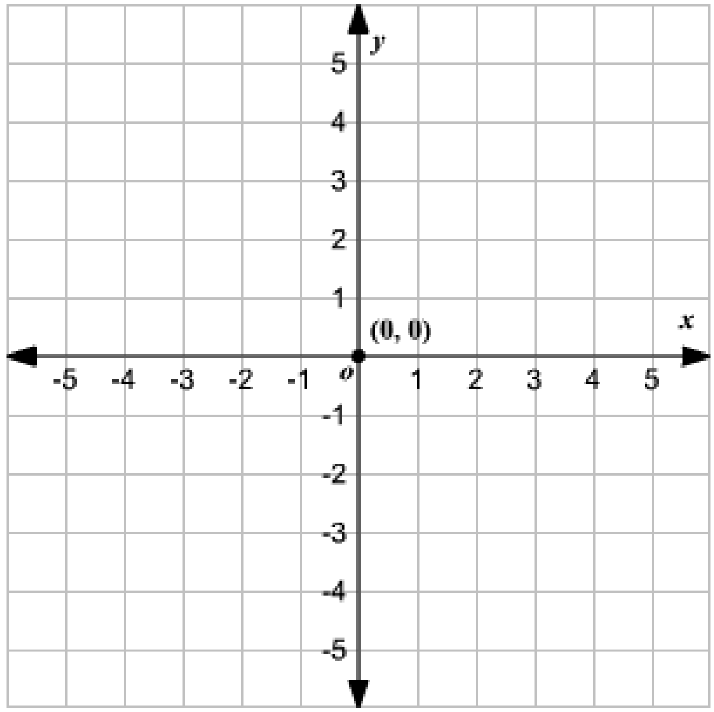
Two real solutions: $x = 1$ and $x = 2$

Practice: Solve the equation graphically. Round answers to 2 decimal places as needed.

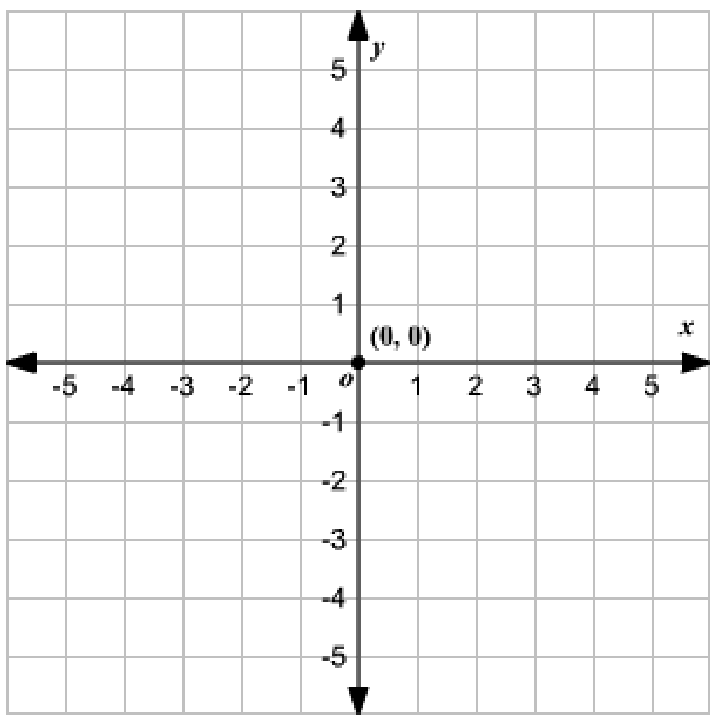
1. $x - 3 = -x + 1$



2. $x^2 - 3 = \sqrt{x}$



3. $x^2 - 3x = x$



Example: Solve graphically.

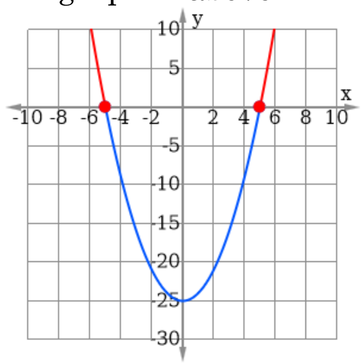
1.

$$x^2 \geq 25$$
$$x^2 - 25 \geq 0$$

Find the $x -$ intercepts:

$$x^2 - 25 = 0$$
$$(x-5)(x+5) = 0$$
$$x - 5 = 0 \text{ or } x + 5 = 0$$
$$x = 5 \text{ or } x = -5$$

Graph, the solution will be the interval(s) where the graph is **above** the $x -$ axis:



Solution: $(-\infty, -5] \cup [5, \infty)$

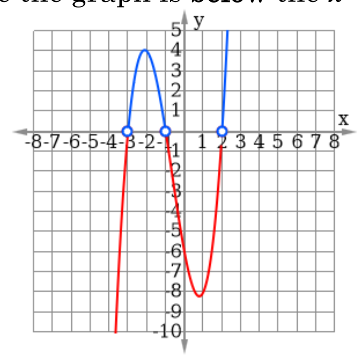
2.

$$x^3 - 6 < -2x^2 + 5x$$
$$x^3 + 2x^2 - 5x - 6 < 0$$

Find the $x -$ intercepts using technology, if necessary:

$(-3,0), (-1,0)$, and $(2,0)$

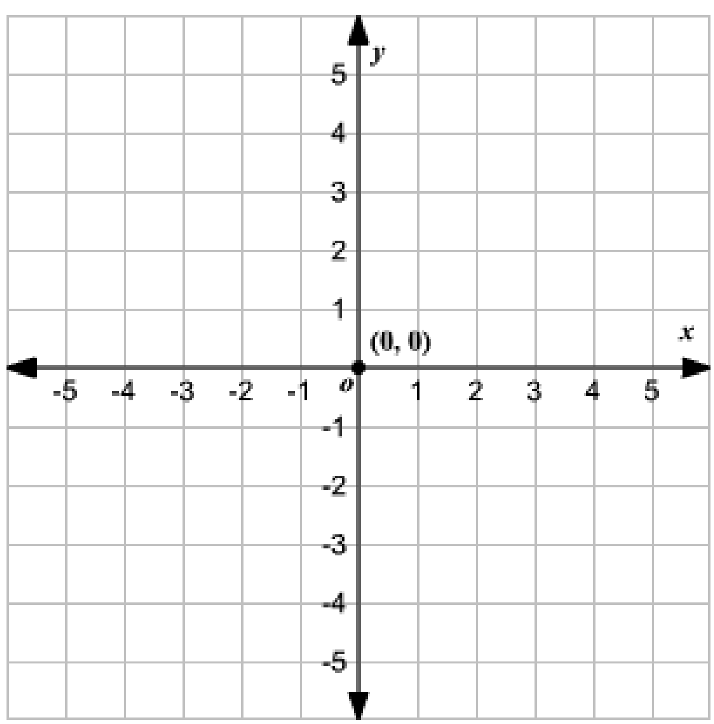
Graph, the solution will be the interval(s) where the graph is **below** the $x -$ axis:



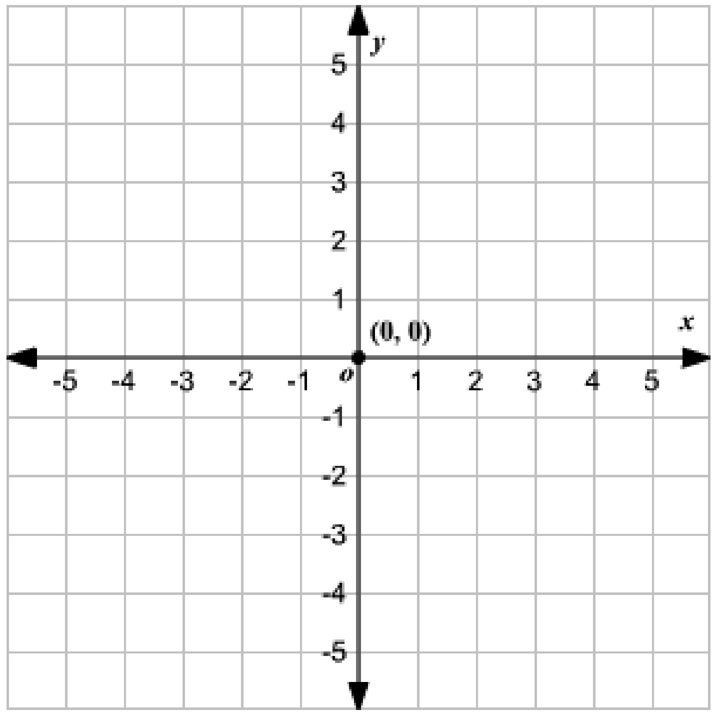
Solution: $(-\infty, -3) \cup (-1, 2)$

Practice: Solve the inequality graphically.

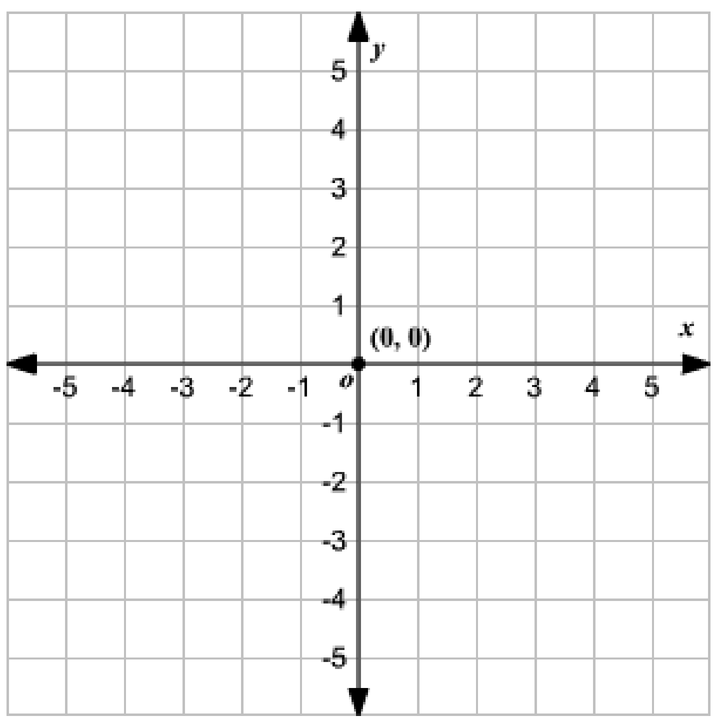
1. $3x^2 + 4x < 0$



2. $|x - 2| \geq 1$



3. $\sqrt{x+3} > 1$



THEME 2. Functions & Graphs

Section 2.1 – Functions

- A **function** is a rule that assigns to each element x exactly one element, called $f(x)$.
- A function will attach one output to each input.
- A piecewise function is defined differently for different intervals of the domain.

Example: A function is defined by $f(x) = x + 1$.

1. <u>Describe the change of the input to find the output. Then evaluate $f(-2)$.</u> *To find the output, add one to the input.* $f(-2) = -2 + 1 = -1$	2. <u>Find the domain and range of f.</u> It is possible to add one to any input, so the **domain** is all real numbers:$(-\infty, \infty)$. Adding 1 to the input, which can be any real number, will result in an output, or **range,** that is also all real numbers: $(-\infty, \infty)$.

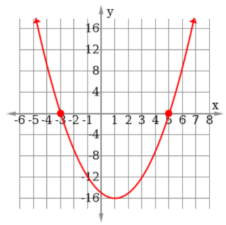
Example: Evaluate.

1. Given $f(x) = -x^2 + 3x$, find $f(-1)$. $f(-1) = -(-1)^2 + 3(-1)$ $-1 - 3$ -4 Note: Since $f(-1) = -4$, the graph of $f(x)$ goes through the point $(-1, -4)$.	2. Given $f(x) = \begin{cases} 3x - 1 & \text{if } x \le 2 \\ x^2 & \text{if } x > 2 \end{cases}$, find $f(1)$. Since $1 \le 2$, plug 1 in for x in the top expression: $f(1) = 3(1) - 1 = 3 - 1 = 2$ Note: Since $f(1) = 2$, the graph of $f(x)$ goes through the point (1,2).

Domain of A Function

- The **domain** of a function is the set of all possible inputs for which it is defined.
  - A rational expression is defined for all values except those which make the denominator equal to zero.
  - An even root is defined whenever the radicand (the expression under the root) is greater than or equal to zero.

Example: Find the domain of each function.

1. $f(x) = 3x^2 + 2x - 1$ $(-\infty, \infty)$	2. $f(x) = 4^{x+1}$ $(-\infty, \infty)$	3. $f(x) = \dfrac{2x+1}{2x-6}$ $2x - 6 \neq 0$ $2x \neq 6$ $x \neq 3$ $(-\infty, 3) \cup (3, \infty)$
4. $f(x) = \ln(4 - 3x)$ $4 - 3x > 0$ $-3x > -4$ $x < \dfrac{4}{3}$ $\left(-\infty, \dfrac{4}{3}\right)$	5. $f(x) = \sqrt{x^2 - 2x - 15}$ $x^2 - 2x - 15 \geq 0$ $x^2 - 2x - 15 = 0$ $(x - 5)(x + 3) = 0$ $x - 5 = 0$ and $x + 3 = 0$ $x = 5$ and $x = -3$  $(-\infty, -3] \cup [5, \infty)$	6. $f(x) = \sqrt[3]{x^2 - 2x - 15}$ $(-\infty, \infty)$

Practice: Find the domain of each function.

1. $f(x) = \frac{3x-1}{5x+7}$

3. $f(x) = \sqrt{5 - 10x}$

4. $f(x) = 3x^3 + 2x^2 - 7$

5. $f(x) = \sqrt[3]{x + 7}$

6. $f(x) = \sqrt{x^2 + 3}$

8. $f(x) = \frac{2x-5}{x^2+3x-4}$

10. $f(x) = \sqrt{2x^2 + 5x - 3}$

12. $f(x) = x^2 - \frac{1}{2}x$

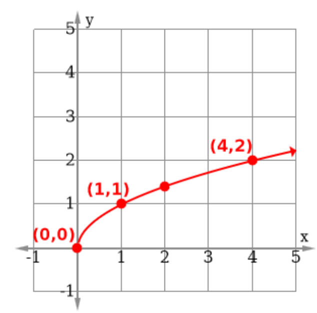
Section 2.2 – Graphs of Functions

- To graph a function:
  - Plug in different values of x to find the corresponding values of y.
  - Plot the points, (x, y).
  - Connect the points.
- It is also possible to graph functions using a graphing calculator.
- Some basic types of functions:
  - **Constant Function:** $f(x) = b$ (A horizontal line with a height of b)
  - **Linear Function:** $f(x) = mx + b$ (A line with a y −intercept of b and a slope of m)
  - **Power Function:** $f(x) = x^n$ (The general shape depends on whether n is odd or even)
  - **Root Function:** $f(x) = x^{\frac{1}{n}}$ (The general shape depends on whether n is odd or even)

Example: Graph $f(x) = \sqrt{x}$ by making a table of values and plotting points.

Choose values of x in the domain of $f(x)$, $x \geq 0$:

$\boldsymbol{x}$	$\boldsymbol{f(x) = \sqrt{x}}$	$\boldsymbol{(x, y)}$
0	$\sqrt{0} = 0$	(0,0)
1	$\sqrt{1} = 1$	(1,1)
2	$\sqrt{2} \approx 1.41$	(2,1.41)
3	$\sqrt{3} \approx 1.73$	(3,1.73)
4	$\sqrt{4} = 2$	(4,2)



Practice: Sketch a graph of the function by making a table of values.

1. $f(x) = (x - 2)^2$

2. $g(x) = -\sqrt{x}$

Graphing Piecewise Defined Functions

Example: Graph the function $f(x) = \begin{cases} 3x - 1 & \text{if} \quad x \le 0 \\ x^2 & \text{if} \quad x > 0 \end{cases}$

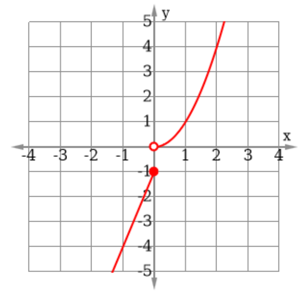
Make a table of values to graph each piece of the function.

To graph the first piece, plug in only values of x that are less than or equal to 2.

x	$f(x) = 3x - 1$	(x, y)
-2	$3(-2) - 1 = -7$	$(-2, -7)$
-1	$3(-) - 1 = -4$	$(-1, -4)$
0	$3(0) - 1 = -1$	$(0, -1)$

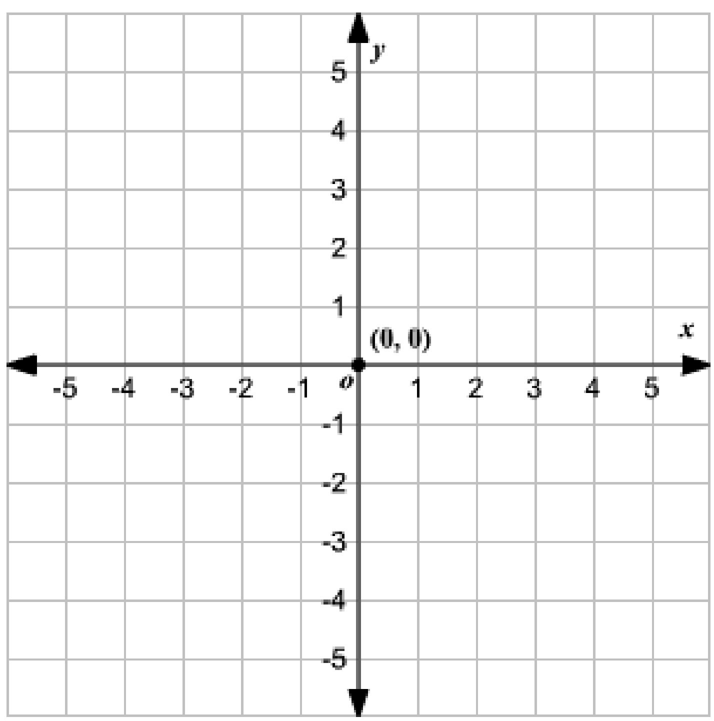
To graph the second piece, plug in values of x that are greater than or equal to 2. (Note: the point found by plugging in 2 will be an open circle.)

x	$f(x) = x^2$	(x, y)
0	$0^2 = 0$	(0,0)
1	$1^2 = 1$	(1,1)
2	$2^2 = 4$	(2,4)

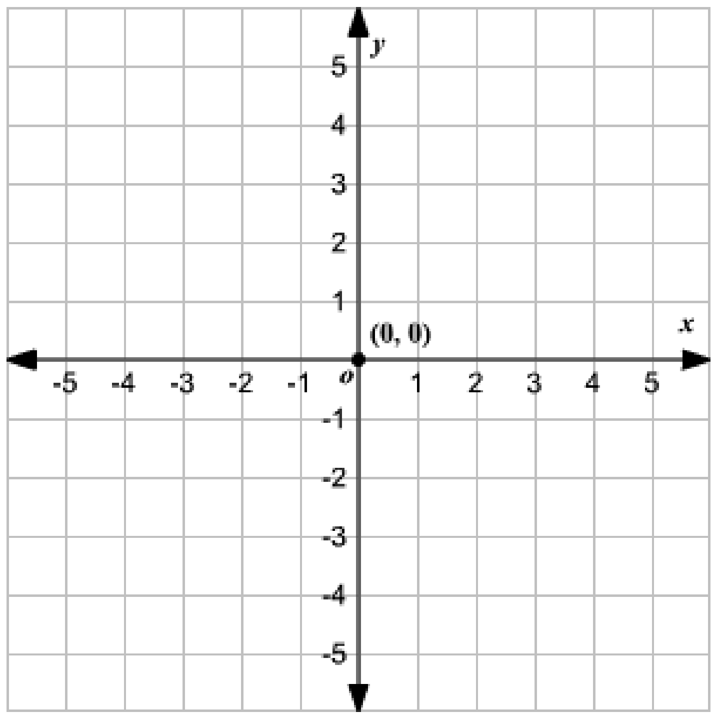


Practice: For questions 1 – 2, graph the piecewise function.

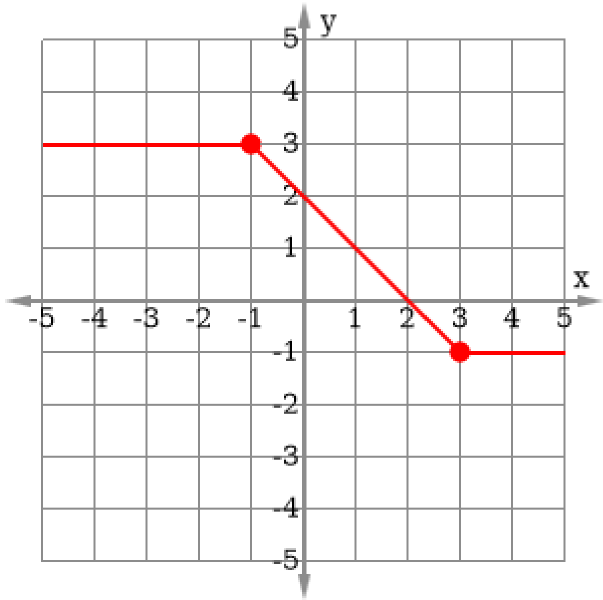
1. $f(x) = \begin{cases} 3 & \text{if} \quad x \le -1 \\ 2x - 1 & \text{if} \quad x > -1 \end{cases}$



2. $g(x) = \begin{cases} -x^2 & \text{if} \quad x \leq 1 \\ -1 & \text{if} \quad x > 1 \end{cases}$



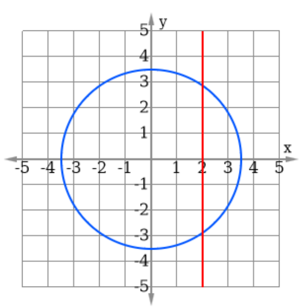
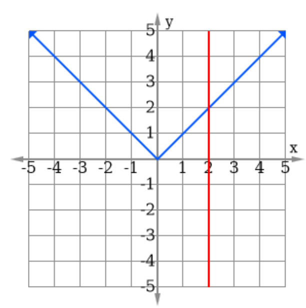
3. The graph of a piecewise function is given. Find a formula for the function in the indicated form.



Graphs of Functions

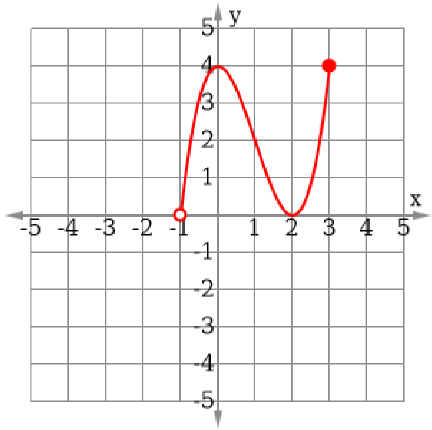
- A graph represents a function if it passes the **vertical line test**.
  - If every vertical line will only cross the graph in one place, the graph represents a function.
  - If it is possible to draw a vertical line that crosses the graph in more than one place, it is not a function.

Example: Determine if the graph represents a function.

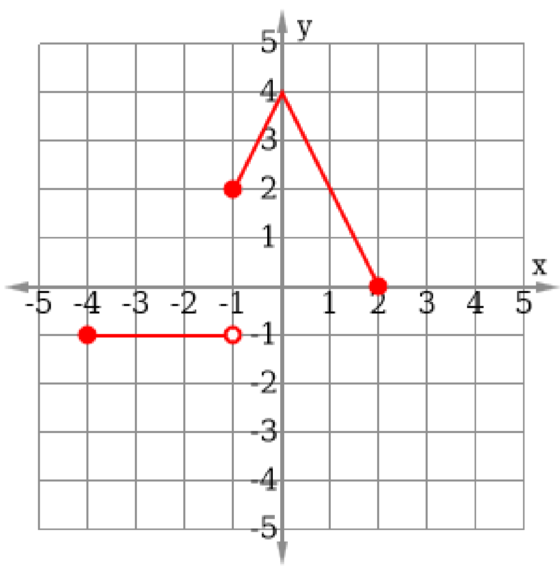
1.  Not a function, since a vertical line crosses the graph more than once.	2.  Is a function, since any vertical line will only cross the graph once.

Practice: For questions 1 – 2, determine if the graph represents a function. Then, give the domain and range.

1.

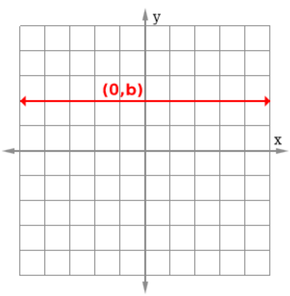
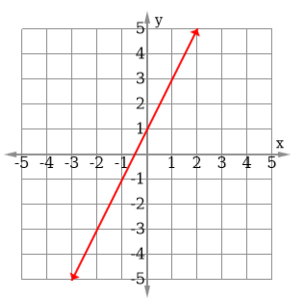
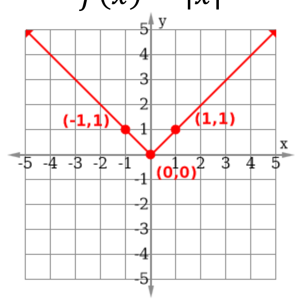
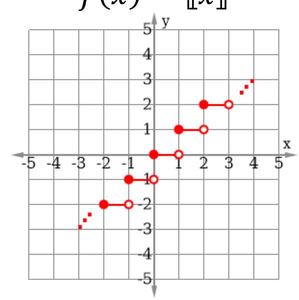


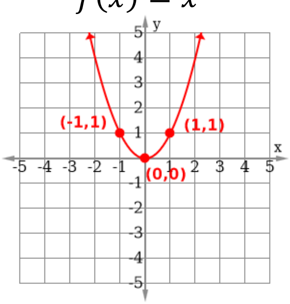
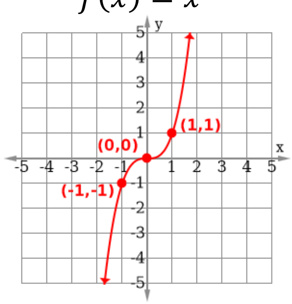
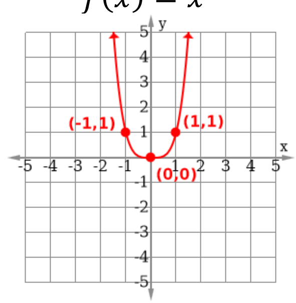
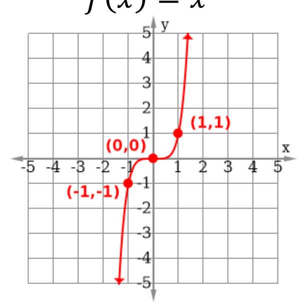
2



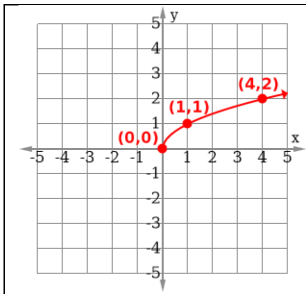
Key Functions

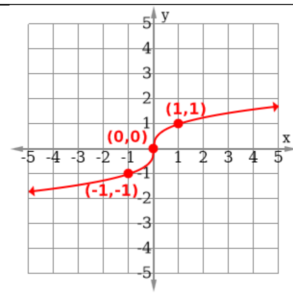
- The following are some key functions that are frequently referenced in Precalculus.

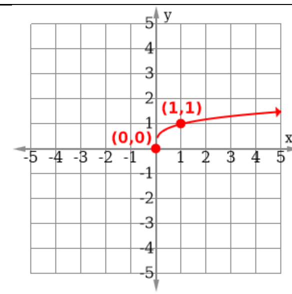
Linear Functions		**Other Functions**	
Constant: $f(x) = b$	$f(x) = mx + b$	Absolute Value: $f(x) = \lvert x \rvert$	Greatest Integer: $f(x) = [\![x]\!]$
			
Domain: $(-\infty, \infty)$ Range: $\{b\}$	Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$	Domain: $(-\infty, \infty)$ Range: $[0, \infty)$	Domain: $(-\infty, \infty)$ Range: $\{\ldots - 2, -1, 0, 1, 2, \ldots\}$

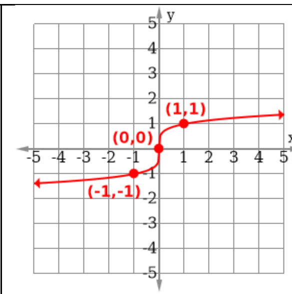
Power Functions			
$f(x) = x^2$	$f(x) = x^3$	$f(x) = x^4$	$f(x) = x^5$
			
Domain: $(-\infty, \infty)$ Range: $[0, \infty)$	Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$	Domain: $(-\infty, \infty)$ Range: $[0, \infty)$	Domain: $(-\infty, \infty)$ Range: $(-\infty, \infty)$

Root Functions			
$f(x) = \sqrt{x}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt[4]{x}$	$f(x) = \sqrt[5]{x}$

<table>
<tr>
<td>

Domain: $[0, \infty)$
Range: $[0, \infty)$</td>
<td>

Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$</td>
<td>

Domain: $[0, \infty)$
Range: $[0, \infty)$</td>
<td>

Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$</td>
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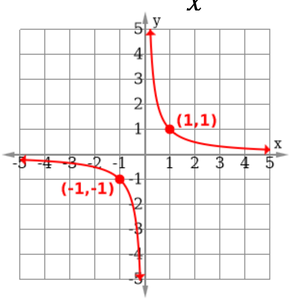
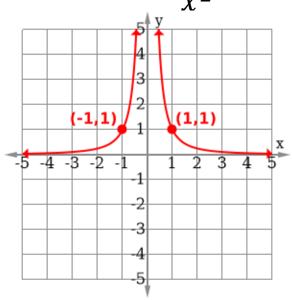
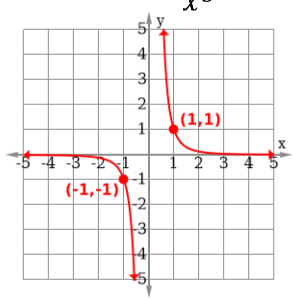
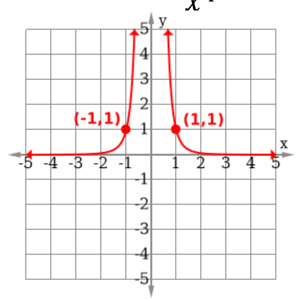
<table>
<tr><th colspan="4">Reciprocal Functions</th></tr>
<tr>
<td>$f(x) = \frac{1}{x}$


Domain:
$(-\infty, 0) \cup (0, \infty)$
Range:
$(-\infty, 0) \cup (0, \infty)$</td>
<td>$f(x) = \frac{1}{x^2}$


Domain:
$(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$</td>
<td>$f(x) = \frac{1}{x^3}$


Domain:
$(-\infty, 0) \cup (0, \infty)$
Range:
$(-\infty, 0) \cup (0, \infty)$</td>
<td>$f(x) = \frac{1}{x^4}$


Domain:
$(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$</td>
</tr>
</table>

Section 2.3 – Getting Information from the Graph of a Function

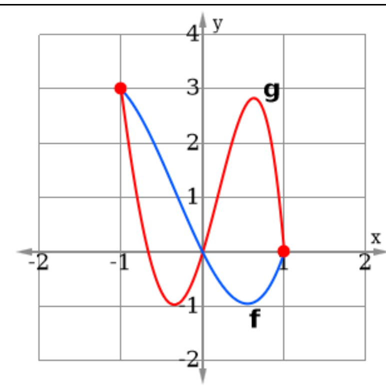
Example: Use the graph to answer the questions.

1. Find the domain and range of g.
   Domain: $[-1,1]$ Range: $[-1,3]$

2. Find $f(-1)$ and $f(1)$
   $f(-1) = 3$ $f(1) = 0$

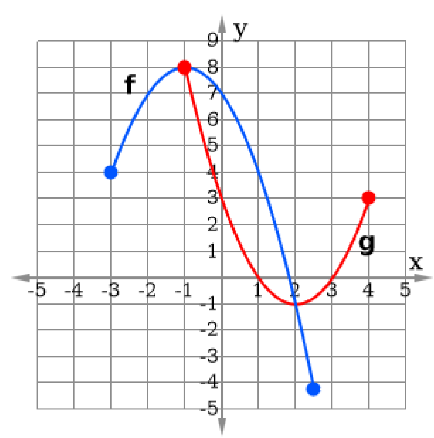
For questions 3 – 5, find the values of x for which:

3. $f(x) = g(x)$ — $\{-1,0,1\}$
4. $f(x) > g(x)$ — $(-1,0)$
5. $f(x) \le g(x)$ — $[0,1]$



Practice: Use the graph of f and g to answer the questions 1 – 5.

1. Find the domain and range of g.
2. Find $f(-3)$ and $f(2)$



For questions 3 – 5, find the values of x for which

3. $f(x) = g(x)$

4. $f(x) > g(x)$

5. $f(x) \le g(x)$

Practice: Use the graph of $f(x)$, given, to answer questions 1 – 5.

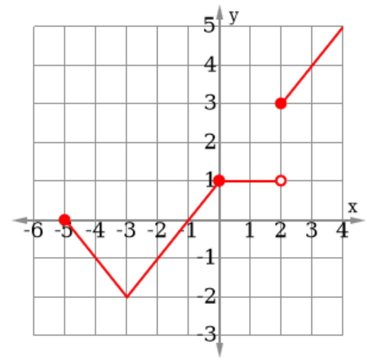
1. Domain

2. Range

3. Evaluate $f(-4)$

4. Solve: $f(x) = 0$

5. Solve: $f(x) < 0$



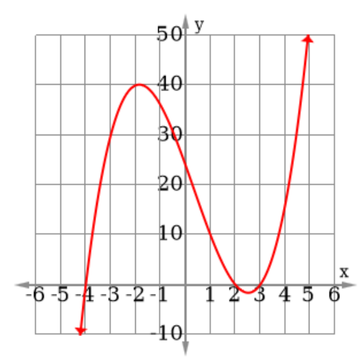
Example: Solve graphically.

1. $x^3 - 14x = x^2 - 24$
2. $x^3 - 14x > x^2 - 24$
3. $x^3 - 14x \leq x^2 - 24$

Bring all terms to one side and graph the function:

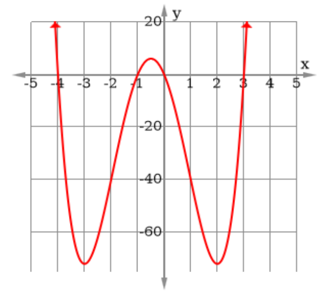
$$x^3 - x^2 - 14x + 24 = 0$$

1. The solution corresponds to the $x-$ intercepts: $\{-4,2,3\}$.
2. The solution corresponds to the values of x for which the graph is **above** the $x-$ axis: $(-4,2) \cup (3, \infty)$.
3. The solution corresponds to the values of x for which the graph is **below** the $x-$ axis: $(-\infty, -4] \cup [2,3]$.



Practice: Use the graph of $f(x) = 2x^4 + 4x^3 - 22x^2 - 24x$, given, to answer questions 1 – 3.

1. $2x^4 - 22x^2 = -4x^3 + 24x$



2. $2x^4 - 22x^2 \geq -4x^3 + 24x$

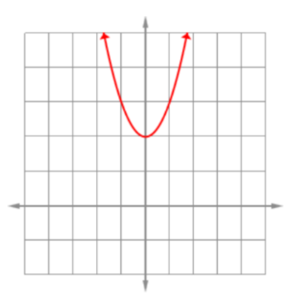
3. $2x^4 - 22x^2 < -4x^3 + 24x$

Even and Odd Functions

- A function is classified as an even function if:
  - It is symmetric with respect to the y-axis.
  - $f(-x) = f(x)$
  - If, for every point (a, b) on $f(x)$, there exists the point $(-a, b)$.
- A function is classified as an odd function if:
  - It is symmetric with respect to the origin.
  - $f(-x) = -f(x)$
  - If, for every point (a, b) on $f(x)$, there exists the point $(-a, -b)$.

Example: Show that the function $y = x^2 + 2$ is even, based on: a) its definition, b) graph, and c) list of ordered pairs.

<table>
<tr>
<td>a)
$y = x^2 + 2$
$f(-x) = (-x)^2 + 2$
$f(-x) = x^2 + 2$

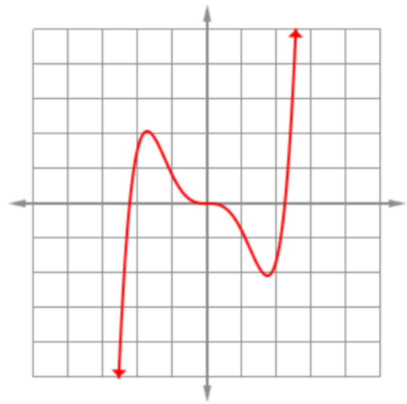
Since $f(-x) = f(x)$, $y = x^2 + 2$ it is an even function.</td>
<td>b)

Since the graph of $y = x^2 + 2$ is symmetric with respect</td>
<td>c)
The following points are on the graph of $y = x^2 + 2$
<table><tr><td>x</td><td>-2</td><td>-1</td><td>0</td><td>1</td><td>2</td></tr><tr><td>y</td><td>6</td><td>3</td><td>2</td><td>3</td><td>6</td></tr></table>
Since the height is the same when $x = 2$ and $x = -2$, and the height is the same when</td>
</tr>
</table>

	to the y −axis, it is an even function.	$x = -1$ and $x = 1$, it is an even function.

Practice: For questions 1 – 3, determine if the function is even, odd, or neither.

1. $y = x^3 - 3x + 1$

2.

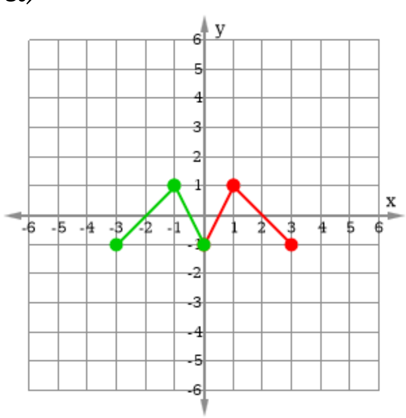


3.

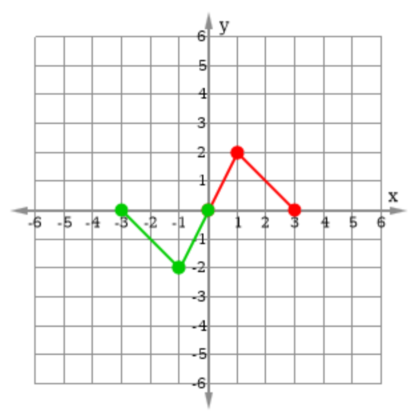
x	-2	-1	0	1	2
y	-5	-3	1	-3	-5

4. Complete the graph of $f(x)$ if it is an a) even function, b) odd function.

a)



b)



2.4– Combining Functions

Sums, Differences, Products, and Quotients

In the examples below, let $f(x) = 3x + 1$ and $g(x) = x + 2$.

	Algebra of Functions		
	Rule	**Domain**	**Example**
Addition $\boldsymbol{f + g}$	$(f+g)(x)$ $f(x) + g(x)$	Values of x common to the domain of both $f(x)$ and $g(x)$.	$(f+g)(x)$ $(3x+1)+(x+2)$ $4x+3$ Domain: $(-\infty, \infty)$
Subtraction $\boldsymbol{f - g}$	$(f-g)(x)$ $f(x) - g(x)$	Values of x common to the domain of both $f(x)$ and $g(x)$.	$(f-g)(x)$ $3x+1-(x+2)$ $3x+1-x-2$ $2x-1$ Domain: $(-\infty, \infty)$
Multiplication $\boldsymbol{fg}$	$(fg)(x)$ $f(x) \cdot g(x)$	Values of x common to the domain of both $f(x)$ and $g(x)$.	$(fg)(x)$ $(3x+1)(x+2)$ $3x^2+6x+x+2$ $3x^2+7x+2$ Domain: $(-\infty, \infty)$
Division $\boldsymbol{\frac{f}{g}}$	$\left(\frac{f}{g}\right)(x)$ $\frac{f(x)}{g(x)}$	Values of x in the domain of both $f(x)$ and $g(x)$. Also exclude value(s) of x for which $g(x) = 0$	$\left(\frac{f}{g}\right)(x)$ $\frac{3x+1}{x+2}$ Domain: $x \neq -2$ $(-\infty - 2) \cup (-2, \infty)$

Composition of Functions

- The notation $(f \circ g)(x)$ represents the composition of f and g.
- To perform the composition, rewrite the problem:

$$(f \circ g)(x) = f(g(x))$$

- The domain will be any value of x that is in the domain of $g(x)$ and of $f(g(x))$.
- To evaluate the composition $f \circ g$ at a number:
  - Plug the number into $g(x)$ and simplify.
  - Plug the solution from the first step into $f(x)$ and simplify.

Example: Find $(g \circ f)(x)$, then give the domain.

1. $f(x) = 3x + 1$ and $g(x) = x^2$ $g(f(x))$ $(3x+1)^2$ $9x^2 + 3x + 3x + 1$ $9x^2 + 6x + 1$ Domain: $(-\infty, \infty)$	2. $f(x) = \sqrt{x+1}$; $g(x) = 2x$ $g(f(x))$ $2\sqrt{x+1}$ Domain: $x \geq -1$ $[-1, \infty)$

Practice: For problems 1 – 6, let $f(x) = 2x - 1$ and $g(x) = x^2 - 3x$. Perform the composition or operation indicated. Give the domain for 6.

1. $(f - g)(x)$

2. $(f + g)(-2)$

3. $(fg)(0)$

4. $(g \circ f)(x)$

5. $(g \circ f)(2)$

6. $\left(\frac{g}{f}\right)(x)$

7. Let $f(x) = \frac{1}{x}$ and $g(x) = 2 - x$. Find the given composition and its domain.

a) $f \circ g$

b) $g \circ f$

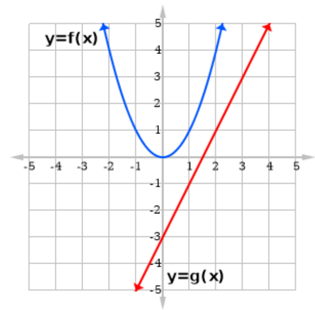
c) $f \circ f$

8. Given $F(x) = 3(x-5)^2$, define f and g such that $f \circ g = F(x)$

Example: Use the graph to evaluate each expression.

1. $(f+g)(0)$: $f(0) + g(0) = 0 + (-3) = -3$

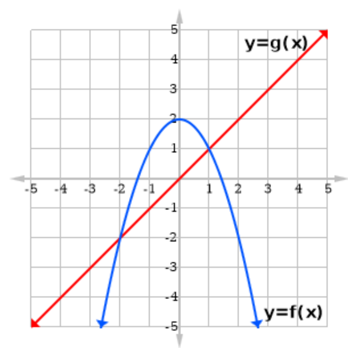
2. $(g \circ f)(-2) = g(f(-2)) = g(4) = 5$



Practice: Use the graph of $f(x)$ and $g(x)$ for problems 1 – 5.

For problems 1 – 3, evaluate.

1. $(f+g)(2)$ 2. $(fg)(-1)$ 3. $(f \circ g)(2)$



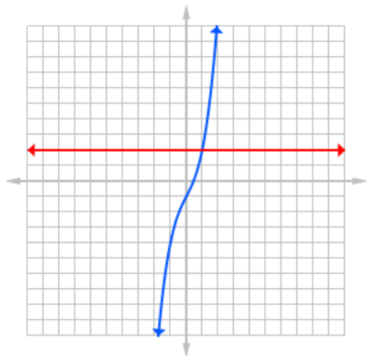
Review: For problems 4 – 5, solve the inequality.

4. $f(x) > g(x)$ 5. $f(x) \leq g(x)$

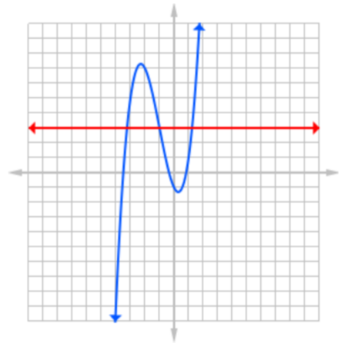
Section 2.5 – One-To-One Functions and Their Inverses

- A function $f(x)$ is one-to-one if: $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$
  - A one-to-one function will not have any repeating y – coordinates.
- To verify a function is one-to-one graphically, perform the horizontal line test.
  - If any horizontal line passes the graph at most once, it is one-to-one.

One – to – One

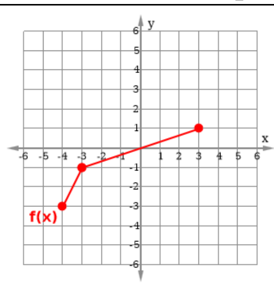


Not One – to – One



- The notation, f^{-1} represents the inverse of function f.
  - In order for a function to have an inverse, it must be one-to-one.
- For every point (x, y) on the graph of f, the point (y, x) will be included in f^{-1}.
- Since x and y are switched for every point on the graph of $f(x)$:
  - The domain of $f(x)$ will be the range of $f^{-1}(x)$.
  - The range of $f(x)$ will be the domain of $f^{-1}(x)$.
  - The graph of the f^{-1} is a reflection of the graph of f across the line $y = x$.

Example: Use the graph of $f(x)$, below, to list three points on the graph of $f(x)$ and $f^{-1}(x)$, then give the domain and range of each function. Finally, graph $f^{-1}(x)$ on the same coordinate plane as $f(x)$.

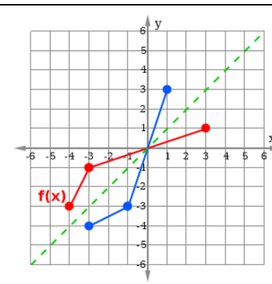


Points

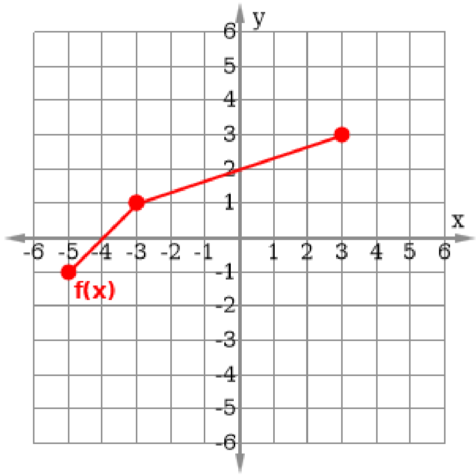
$f(x)$	$f^{-1}(x)$
$(-4,-3)$	$(-3,-4)$
$(-3,-1)$	$(-1,-3)$
$(3,1)$	$(1,3)$

$f(x)$:
Domain: $[-4,3]$
Range: $[-3,1]$

$f^{-1}(x)$:
Domain: $[-3,1]$
Range: $[-4,3]$



Practice: Use the graph of $f(x)$, below, to list three points on the graph of $f(x)$ and $f^{-1}(x)$, then give the domain and range of each function. Finally, graph $f^{-1}(x)$ on the same coordinate plane as $f(x)$.



Points

$f(x)$	$f^{-1}(x)$
$(-5,-1)$	
$(-3,1)$	
$(3,3)$	

$f(x)$:
Domain:

Range:

$f^{-1}(x)$:
Domain:

Range:

- Finding inverse of a one-to-one function.

  Strategy:
  - Replace $f(x)$ with y.
  - Switch each x and y.
  - Solve the new equation for y.
  - Replace y with $f^{-1}(x)$.

Example: Find the inverse of each function.

1.	2.
$f(x) = 3x + 1$ $y = 3x + 1$ $x = 3y + 1$ $x - 1 = 3y$ $\frac{x-1}{3} = y$ $f^{-1}(x) = \frac{x-1}{3}$ or $f^{-1}(x) = \frac{1}{3}x - \frac{1}{3}$	$f(x) = \frac{3}{x+4}$ $y = \frac{3}{x+4}$ $x = \frac{3}{y+4}$ $x(y+4) = \frac{3(y+4)}{y+4}$ $xy + 4x = 3$ $xy = 3 - 4x$ $f^{-1}(x) = \frac{3-4x}{x}$

Practice: Find the inverse of each function.

1. $f(x) = 4 - 5x$

2. $f(x) = \frac{2}{3x-1}$

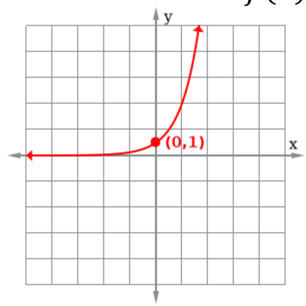
3. $f(x) = \frac{-2}{5+2x}$

4. $f(x) = \frac{3x}{1+2x}$

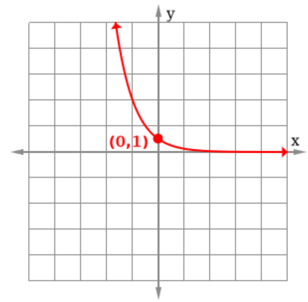
THEME 3. Exponential Functions and Logarithmic Functions

Section 3.1 – Exponential Functions

- An exponential function is a function of the form: $f(x) = a^x$, where $a > 0$ and $a \neq 1$.
  - Domain: $(-\infty, \infty)$
  - Range: $(0, \infty)$
  - Horizontal Asymptote: $y = 0$



$f(x) = a^x$ for $a > 1$



$f(x) = a^x$ for $0 < a < 1$

Practice: Graph $f(x) = 3^x$ by completing a table of values.

Answer:

x	$y = 3^x$	(x, y)
-2		
-1		
0		
1		
2		

Section 3.2– Logarithmic Functions

- Every exponential function is one-to-one and has an inverse.
- The inverse of the exponential function with base a ($a \neq 1$) is denoted by $\log_a(x)$:

$$\log_a(x) = y \Leftrightarrow a^y = x$$

- A logarithm is an exponent:
  - $\log_a(x)$ is the exponent to which a must be raised in order to obtain x.
- Use the definitions of logs to switch back and forth between log from and exponential form:
  - Both expressions represent the same thing, however the log form is solved for y.

Log form: $\log_b(x) = y$ **Exponential Form:** $x = b^y$

  - Notice that b is the base in both the exponential form and the log form.

Example: Re-write the equation.

1. Exponential $\log_4(x) = 2$ $4^2 = x$ $16 = x$	2. Logarithmic $3 = 5^x$ $\log_5(3) = x$

Practice: For questions 1 – 3, write the equation in the equivalent exponential form.

1. $\log_3(x) = 5$

2. $\text{Log}_{\frac{1}{2}}(x) = 3$

3. $\text{Log}_5(x) = 2$

For questions 4 – 6, write the equation in the equivalent logarithmic form.

4. $4^x = 7$

5. $e^3 = x$

6. $7^x = 49$

Example: Evaluate the following logarithmic expressions:

Logarithmic Expression	Question	Solution
$\log_3(81)$	What power must 3 be raised to in order to equal 81?	4
$\log_3\left(\frac{1}{3}\right)$	What power must 3 be raised to in order to equal $\frac{1}{3}$?	-1
$\log_5(25)$	What power must 5 be raised to in order to equal 25?	2
$\log_2(1)$	What power must 2 be raised to in order to equal 1?	0 Anything to the 0 power is 1.

Practice: Evaluate the expression without using a calculator.

1. $\log_6(36)$

2. $\log_2(16)$

3. $\log_3\left(\frac{1}{9}\right)$

4. $\log_3(1)$

Properties of Log Functions

- For $a > 0, a \neq 1$, and any real number k, the following hold:

Property	Reason
$\log_a(1) = 0$	Raise a to the zero power to get 1.
$\log_a(a) = 1$	Raise a to the first power to get a.
$\log_a(a^x) = x$	Raise a to the x power to get a^x.
$a^{\log_a(x)} = x$	$\log_a(x)$ is the power a must be raised to in order to get x (and it is being raised to that power).

Practice: Use the properties of logarithms to evaluate the expression.

1. $\log_4(4^5)$

2. $\log_e(e^5)$

3. $\log_2(8)$

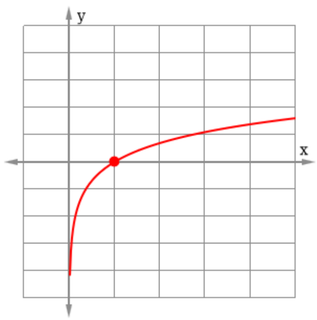
4. $\log_8(1)$

5. $\log(1)$

6. $\log_8\left(\frac{1}{8}\right)$

Graphs of Logarithmic Functions

- The graph of logarithmic functions are as follows:



$y = \log_a(x)$

$y = \log_a(x)$, $a > 1$

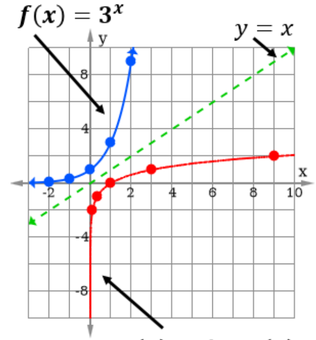
- Domain: $(0, \infty)$
- Range: $(-\infty, \infty)$
- $f(x) = \log_a(x)$ is increasing and continuous on its entire domain.
- The y-axis is a vertical asymptote as $x \to 0$ from the right.
- The graph passes through the points $(a^{-1}, -1), (1,0)$, and $(a, 1)$.

Example: Use the fact that $f(x) = 3^x$ and $g(x) = \log_3(x)$ are inverses of each other to sketch both functions.

Construct a table of values to graph $f(x) = 3^x$:

$\boldsymbol{x}$	-2	-1	0	1	2	3
$\boldsymbol{f(x) = 3^x}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
$\boldsymbol{(x, y)}$	$\left(-2, \frac{1}{9}\right)$	$\left(-1, \frac{1}{3}\right)$	$(0,1)$	$(1,3)$	$(2,9)$	$(3,27)$

Switch the values of x and y obtain points on the graph of $g(x) = \log_3(x)$.



- Since logarithmic functions are the inverses of exponential functions:
  - The range of an exponential function is $(0, \infty)$, therefore the domain of a log function is $(0, \infty)$.
- To find the domain of any log function:
  - Set the argument greater than zero.
  - Solve.
  - Write the final answer in interval notation.

Example: Find the domain of the function.

1.	2.
$f(x) = \log_3(x+5)$ $x + 5 > 0$ $x > -5$ Domain: $(-5, \infty)$	$f(x) = \ln(x^2 + 1)$ $x^2 + 1 > 0$ ←Always! Domain $(-\infty, \infty)$

Practice: Find the domain of the function.

1. $f(x) = \log(3x - 2)$

2. $f(x) = \log_3(x^2 + 7x + 10)$

3. $f(x) = \ln(x^2 + 2)$

4. . $f(x) = \log_3(x^2 + 7x + 10)$

5. $f(x) = \ln((-x - 2)^2)$

6. $f(x) = \ln(x^2 - 1)$

7. $f(x) = \ln(\ln(x) - 1)$

Special Log Functions

- A log function with base 10 is called the common log and is denoted: $\log(x)$.
- A log function with base e is called the natural log and is denoted: $\ln(x)$.

Practice: Evaluate without using a calculator.

1. $\log(1000)$

2. $\ln(e^5)$

3. $\log(\sqrt{10})$

Section 3.3 – Laws of Logarithms

- Let a be a positive number with $a \neq 1$. Let $A, B,$ and C be any real numbers with $A > 0$ and $B > 0$.

Name of Law	Law	Description of Law
Product	$log_a(AB) = log_a(A) + log_a(B)$	The logarithm of a product is the sum of the logarithms of each factor.
Quotient	$log_a\left(\frac{A}{B}\right) = log_a(A) - log_a(B)$	The logarithm of a quotient is the difference of the logarithm of the numerator and logarithm of the denominator.
Power	$\log_a(A^C) = C\ log_a(A)$	The logarithm of a term that is being raised to a power is the product of the power and the logarithm of the term.

Example: Assuming all variables represent positive real numbers, use the properties of logarithms to rewrite each expression.

1. $\log(5x)$ $\log(5) + \log(x)$	2. $\log_3\left(\frac{1}{3}\right)$ $\log_3(3^{-1})$ -1	3. $\ln(1)$ $\log_e(1)$ 0
4. $\log\left(\frac{10}{y}\right)$ $\log(10) - \log(y)$ $1 - \log(y)$	5. $\log_2(4y^3)$ $\log_2(4) + \log_2(y^3)$ $\log_2(2^2) + 3\log_2(y)$ $2 + 3\log_2(y)$	6. $\log_5(\sqrt{5})$ $\log_5\left(5^{\frac{1}{2}}\right)$ $\frac{1}{2}$

Practice: For questions 1 – 2, use the Laws of Logarithms to evaluate the expression.

1. $\log(25) + \log(4)$

2. $\log_3(135) - \log_3(5)$

For questions 3 – 6, use the Laws of Logarithms to expand the expression.

3. $\log_2(3x)$

4. $\log_4\left(\frac{4x^2}{y}\right)$

5. $\log_2(16x^2y^5)$

6. $\log_2\left(\sqrt{4(x+y)z^3}\right)$

For problems 7 – 8 use the Laws of Logarithms to combine the expressions.

7. $\frac{1}{2}\log(16) - 3\log(x+1)$

8. $\log_2(x) + 3\log_2(y) - 4\log_2(z)$

Change of Base Formula

- It is possible to change the base of a logarithm if needed using the following:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

  - Any positive value can be used for a, so 10 and e are often used.

Practice: Use the change of base formula and a calculator to evaluate the logarithm. Use either a common or natural log and round to 5 decimal places.

1. $\log_5(23)$

2. $\log_4(12)$

Section 3.4– Exponential and Logarithmic Equations

- To solve an exponential equation:
  - Solve for the exponential term.
  - Use properties of exponents to write the equation in the form: $a^m = a^n$.
    - If this is possible, set $m = n$.
    - If not possible, take the log of both sides and use the power rule to bring the exponent down and become the coefficient of the log function.
  - Solve the remaining equation for x.

Example: Solve for x, give exact answers.

1. $2^{x+1} = 8$ $2^{x+1} = 2^3$ $x + 1 = 3$ $x = 2$	2. $9^{2x} = 27^{x-1}$ $(3^2)^{2x} = (3^3)^{x-1}$ $3^{2\cdot 2x} = 3^{3(x-1)}$ $4x = 3(x-1)$ $4x = 3x - 3$ $x = -3$	
3. $3^{2x-1} = \frac{1}{27}$ $3^{2x-1} = 3^{-3}$ $2x - 1 = -3$ $2x = -2$ $x = -1$	4. $5^x = 7$ $\log(5^x) = \log(7)$ $x \cdot \log(5) = \log(7)$ $x = \frac{\log(7)}{\log(5)}$	5. $e^{4x} = 3$ $\ln(e^{4x}) = \ln(3)$ $4x = \ln(3)$ $x = \frac{\ln(3)}{4}$

Practice: Solve for x. .

1. $25 = 5^{3x+1}$

2. $4^{x+1} = \frac{1}{8}$

3. $25^x = 125^{2x-1}$

4. $\sqrt{5^x} = 125$

5. $3^x = 4^{x+1}$

6. $10^{x-1} = 23$

- To solve a logarithmic equation
  - Use the product or quotient rule to condense the log expression. (If there is more than one log.)
  - Solve for the log expression.
  - Rewrite the equation in exponential form.
  - Solve the new equation.
  - Verify the solution is within the restricted domain of each logarithmic function in the original equation.
    - Logarithms of numbers less than or equal to zero are undefined.

Example: Solve the equation and give the exact answer.

1. $\log_2(x+2) = 3$ $2^3 = x + 2$ $8 = x + 2$ $6 = x$ $\{6\}$	2. $\ln(2) + \ln(x) = 5$ $\ln(2x) = 5$ $\log_e(2x) = 5$ $e^5 = 2x$ $\frac{e^5}{2} = x$ $\left\{\frac{e^5}{2}\right\}$	3. $9\log_2(x+1) = 18$ $\log_2(x+1) = 2$ $2^2 = x + 1$ $4 = x + 1$ $3 = x$ $\{3\}$

Practice: Solve the equation and give the exact answer.

1. $\log(2x - 3) = 0$

2. $\log(x+2)+\log(x+5)=1$

3. $\log(3+x)=0.5$

4. $\log_2(3x)+\log_2(4)=4$

5. $\log(x)=2-\log(x)$

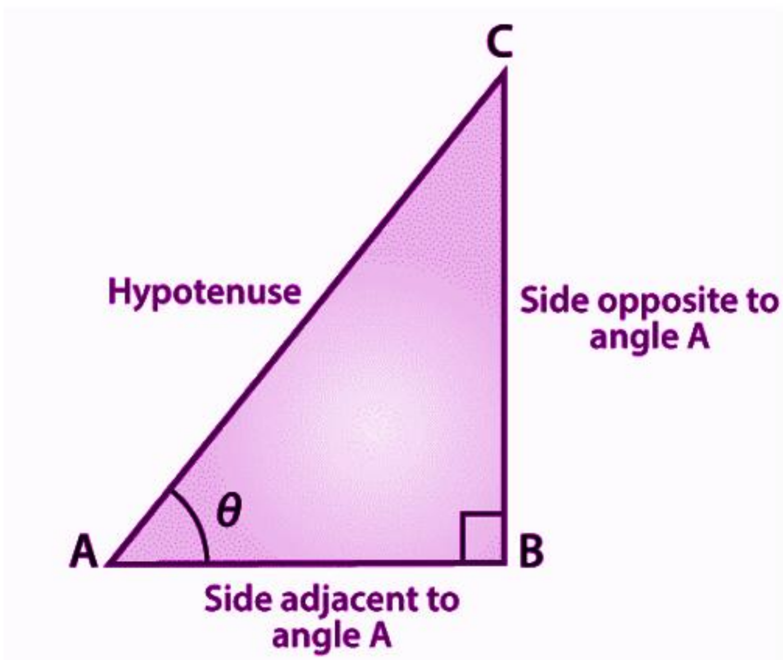
6. $ln^2\ (x)=\ln(x^2)$

7. $ln(x)+\ln(x-10)=\ln(3)$

THEME 4. Trigonometry

Trigonometry of the Right Triangle

The trigonometric ratios, sine, cosine, and tangent, of an angle θ in a right triangle are the ratios of the side lengths and denoted as *sin* θ*, cos* θ *and tan* θ, respectively and defined as follows:



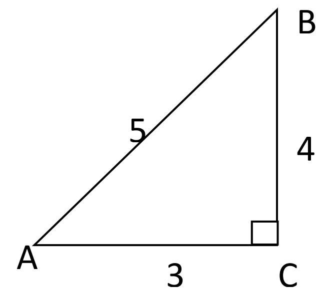
Remember: SOH-CAH-TOA

$$\sin\theta = \frac{\text{Opposite side}}{\text{Hypotenuse}}$$

$$\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$$

Tan θ = $\frac{\text{Opposite side}}{\text{Adjacent side}}$

1. Find the sine, cosine and tangent of the given angle.

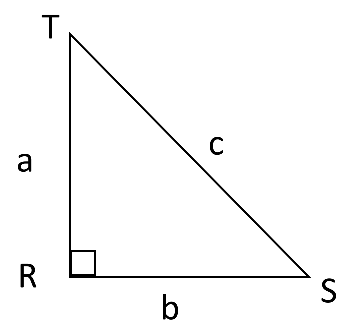


a) sin A = b) cos A =

c) tan A = d) sin B =

e) cos B = f) tan B =

2. Use the diagrams to complete the following:

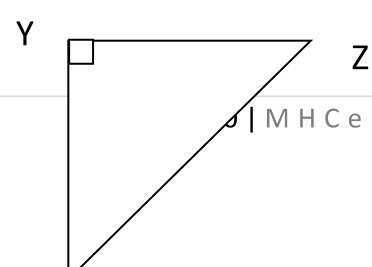


a) sin S = cos _____ = $\frac{b}{c}$

tan _____ = $\frac{a}{b}$ sin _____ = $\frac{b}{c}$

cos T = tan T =

b) sin X = cos X =



tan X = sin Z =

cos _____ = $\frac{YZ}{XZ}$ tan Z =

3 What is the value of $\tan\left(\text{Arc}\cos\frac{15}{17}\right)$?

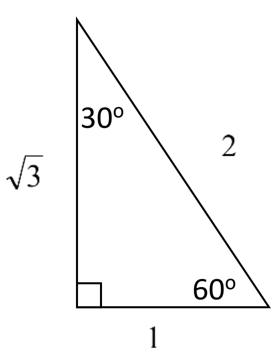
1) $\frac{8}{15}$

2) $\frac{8}{17}$

3) $\frac{15}{8}$

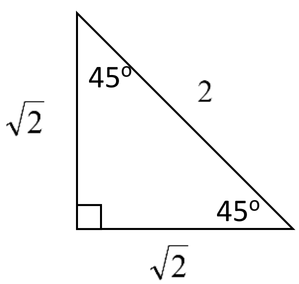
4) $\frac{17}{8}$

1-2 Special Right Triangles



Complete the table below.

	$30°$	$45°$	$60°$
sin			
cos			
tan			



Find the exact value of each of the following

1. $\cos 60^o + \sin 30^o - \tan 45^o$

2. $\sin 30^o + \cos 30^o + \tan 30^o$

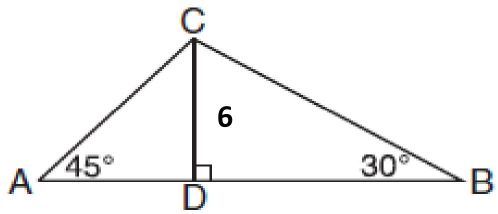
3. $(\cos 60^\circ)^2 + (\sin 30^\circ)^2$

4. $$\frac{1}{\cos 60^\circ} + \frac{\cos 45^\circ}{\sin 45^\circ} + \frac{1}{\sin 30^\circ}$$

5. $(\sin 30^\circ)^{-\frac{1}{2}}$

6. $(\cos 45^\circ)^2$

7. In the accompanying diagram, $\overline{CD}$ is an altitude of ΔABC. If CD = 6, $m\angle A = 45$ and $m\angle B = 30$, find the perimeter of ΔABC in simplest radical form.



8. Find the length of the altitude of an equilateral triangle whose side has a length of 10.

Radian Measure

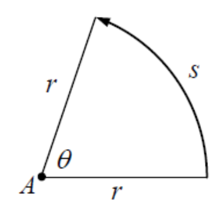
Just as distance can be measure in inches, feet, miles, centimeters and so on, rotations about a point can also be measured in different ways. Measuring one complete rotation in terms of 360° is somewhat arbitrary.

A common unit of angle measurement that is an alternative to degrees is called the radian. It is defined in terms of the arc length of a circle and the circle's radius.

THE DEFINITION OF A RADIAN

The radian angle, θ, created by a rotation about a point A using a radius of r and passing through an arc length of s is defined as

$$\theta = \frac{s}{r} \quad \text{or equivalently} \quad s = \theta \cdot r$$



Radians measure the total number of radii that have been traversed about the circumference of a circle in a given rotation. Based on the circumference formula of a circle, there will always be 2π radians in one full rotation.

Example: Find the arc length if the radius of a circle is 2 units and $\theta = 1$ radian.

Trigonometric Conversions

Relationship between Degrees and Radians:

Radians = degrees $\bullet \frac{\pi}{180°}$ **Degrees** = radians $\bullet \frac{180°}{\pi}$

Convert each of the following to degrees or radians. Express answers in terms of π when necessary.

1. $\theta = 90°$
2. $\theta = 225°$
3. $\theta = 120°$
4. $\theta = \frac{3\pi}{2}$
5. $\theta = \frac{5\pi}{6}$
6. $\theta = \frac{3\pi}{4}$

7. $\theta = \frac{2\pi}{3}$

8. $\theta = 30°$

9. $\theta = 45°$

10. $\theta = \pi$

11. $\theta = 60°$

12. $\theta = \frac{\pi}{4}$

13. $\theta = \frac{2\pi}{3}$

14. $\theta = 180°$

15. $\theta = 315°$

16. $\theta = 300°$

17. $\theta = 270°$

18. $\theta = \frac{\pi}{6}$

19. $\theta = -\frac{\pi}{2}$

20. $\theta = \frac{11\pi}{4}$

21. $\theta = 330°$

22. $\theta = \frac{\pi}{3}$

23. $\theta = 135°$

24. $\theta = -\frac{4\pi}{3}$

25. $\theta = \frac{\pi}{2}$

26. $\theta = 240°$

27. $\theta = 40°$

28. In a circle with a central angle of 60^{o}, find the radius if the arc length is 4 feet.

29. In a circle with a central angle of 45°, find the arc length if the diameter is 10 feet.

30. In a circle with a central angle of 30°, find the arc length if the diameter of the circle is four feet.

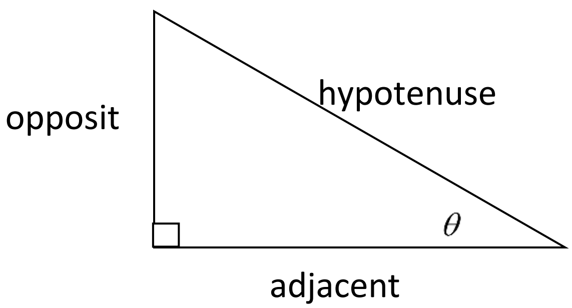
31. If a circle has an arc length of 10 feet and a radius of 2 feet, find the amount of degrees in the central angle to the nearest tenth.

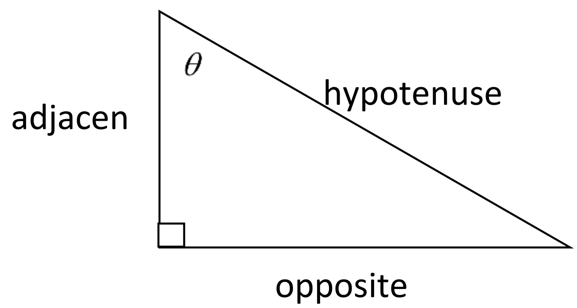
32. If a circle has an arc length of 6 feet and a radius of 3, find how many radians the central angle is?

Reciprocal Trigonometric Functions

In a right triangle, there are actually six possible trigonometric ratios, or functions.

A Greek letter (such as theta θ or phi φ) will be used to represent the angle.





$\sin\theta =$

$\csc\theta =$

$\cos\theta =$

$\sec\theta =$

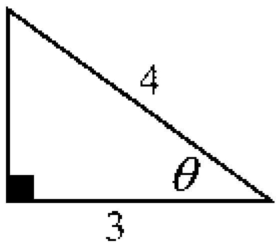
$\tan\theta =$

$\cot\theta =$

Notice that the three new ratios above are reciprocals of the ratios on the left. Applying a little algebra shows the connection between these functions.

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\frac{\text{opposite}}{\text{hypotenuse}}} = \frac{1}{\sin\theta}$$

1. Given the triangle below, express the exact value of the six trigonometric functions in relation to theta.



The following examples pertain to a right triangle in Quadrant I.

2. Given $\cos\theta = \frac{12}{13}$, find csc Θ, $\sec\theta$, tan Θ, sin Θ, cot Θ

3. Find tan Θ, csc Θ, $\sec\theta$ and $\cot\theta$, given $\sin\theta = \frac{2}{3}$ and $\cos\theta = \frac{\sqrt{5}}{3}$.

4. Determine the value of each of the following exactly in simplest form.

(a) $\cot(45°)$ (b) $\cot(60°)$ (c) $\csc(30°)$

(d) $\csc\left(\frac{\pi}{4}\right)$ (e) $\sec^2\left(\frac{\pi}{3}\right)$ (f) $\cot\left(\frac{\pi}{3}\right)$

(g) $\sec\left(\frac{\pi}{4}\right)$ (h) $\csc\left(\frac{\pi}{6}\right)$ (i) $\sec^2(30°)$

6. Find the exact value of each of the following:

(a) $\sin(30^\circ)$ (b) $\csc\left(\frac{\pi}{6}\right)$ (c) $\cos(90^\circ)$

(d) $\sec\left(\frac{\pi}{3}\right)$ (e) $\tan(45^\circ)$ (f) $\cot(\pi)$

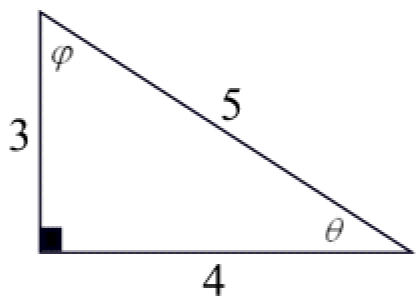
7. In simplest radical form, $\sec(135^\circ)$ is equal to

(1) $-\frac{\sqrt{2}}{3}$ (3) $-\frac{\sqrt{2}}{2}$

(2) $-\sqrt{2}$ (4) $-\frac{\sqrt{3}}{2}$

8. Which of the following is nearest to the value of $\cot(220°)$?

(1)	1.19	(3)	-2.74
(2)	3.17	(4)	-0.85



In the diagram at the left, the measures of the angles designated by φ and θ add to 90º. These angles are complementary angles.

$$\theta + \varphi = 90^\circ \begin{cases} \varphi = 90^\circ - \theta \\ \theta = 90^\circ - \varphi \end{cases}$$

The sine of an acute angle is equal to the cosine of its complement.

In this triangle, $\sin\theta = \frac{3}{5}$ and $\cos\varphi = \frac{3}{5}$.

The cosine of an acute angle is equal to the sine of its complement.

$$\sin\theta = \cos\varphi \begin{cases} \sin\theta = \cos(90^\circ - \theta) \\ \cos\varphi = \sin(90^\circ - \varphi) \end{cases}$$

There are three sets of COFUNCTION identities

sine and **cosine** are cofunctions

$$\sin\theta = \cos(90^\circ - \theta)$$
$$\cos\theta = \sin(90^\circ - \theta)$$

tangent and **cotangent** are cofunctions

$$\tan\theta = \cot(90^\circ - \theta)$$
$$\cot\theta = \tan(90^\circ - \theta)$$

secant and **cosecant** are cofunctions

$$\sec\theta = \csc(90^\circ - \theta)$$
$$\csc\theta = \sec(90^\circ - \theta)$$

Notice the connection of the letters **C** & **O**:

* sine and **co**sine are **co**functions
* tangent and **co**tangent are **co**functions
* secant and **co**secant are **co**functions
* **co**mplementary

1. Find a value of θ for which $\sin\theta = \cos 15°$.

2. Write the expression tan 265º as the function of an acute angle of measure less than 45º.

3. Complete each statement.

(a) $\sin 60° = \cos$________

(b) $\sin 45° = \cos$________

(c) $\cos 90° = \sin$________

(d) $\cos 30° = \sin$________

(e) $\cos° = \sin$________

(f) $\sin 0° = \cos$________

(g) $\sin\dfrac{\pi}{4} = \cos$________

(h) $\cos\dfrac{\pi}{6} = \sin$________

(i) $\cos\frac{\pi}{3} = \sin$________

4. Solve for one value of x: sin(2x) = cos(x)

5. If $\sin(x + 20) = \cos(2x + 10)$, find one value of x.

6. If $\tan(4x + 40) = \cot(2x - 10)$, find one value of x.

7. $\sec(2x) = \csc(x - 10)$

Identifying the amplitude, period and frequency given a trigonometric equation.

Graphing the sine curve

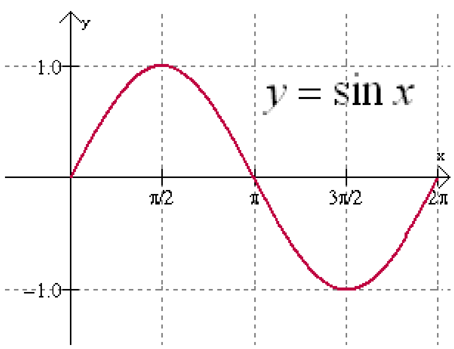
Amplitude

The distance from the axis of the curve to the maximum point on the curve is called the **amplitude**.

In a trigonometric equation **y = a sin(bx), | a | = amplitude**

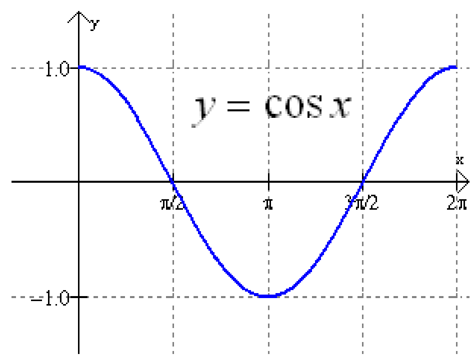
Basic Sine Curve: **y = sin x**

amplitude = ____



Basic Cosine Curve: **y = cos x**

amplitude = ____



Determine the amplitude of each trigonometric function. Examine the graph of each on the graphing calculator in radian mode.

1. y = 2 sin x
2. y = -2 sin x
3. y = ½ sin x
4. y = 3 sin x

5. y = 3 cos x
5. y = -3 cos x
6. y = 4 cos x
7. y = - cos x

Note:

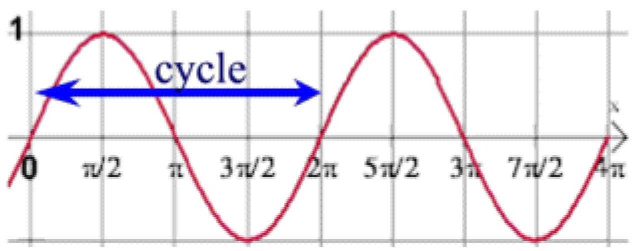
while vertical shifts alter the maximum and minimum values of a function, they do not alter the amplitude. Also horizontal shifts (phase shifts) do not affect the amplitude.

Period

A **periodic function** is an oscillating (wave-like) function which repeats a pattern of *y*-values at regular intervals. One complete repetition of the pattern is called a **cycle.**

The **period** of a function is the horizontal length of one complete cycle.

The **basic sine curve**, ***y* = sin *x*,** has a period of 2π the horizontal length of one complete cycle.



In a trigonometric equation **y = a sin(bx)** **Period** = $\frac{2\pi}{b}$

Determine the period for each trigonometric function.

1. y = sin 2x 2. y = cos 3x 3. y = sin ½ x4. y = cos ½ x

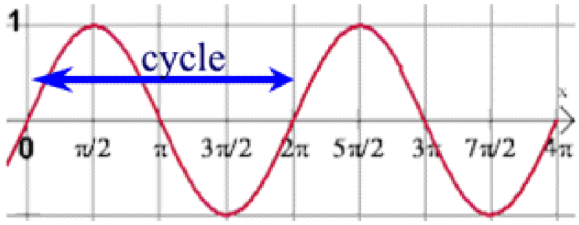
Frequency

The **frequency** of a trigonometric function is the number of cycles it completes in a given interval. This interval is generally 2π radians (or 360º) for the sine and cosine curves**.**

In terms of a formula: $frequency = \frac{1}{period}$

This sine curve, ***y* = sin *x*,** completes 1 cycle in the interval from 0 to 2π radians. Its frequency is 1 in the interval 2π .

I

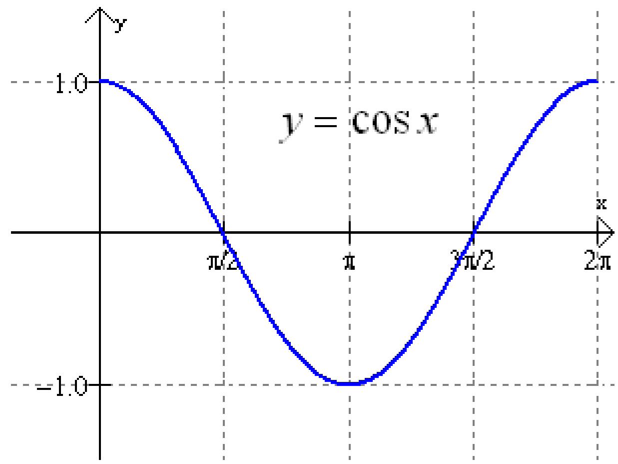


n a trigonometric equation **y = a sin(bx), b = frequency**

Determine the frequency of each trigonometric function.

1.   y = sin 2x   2.   y = cos 3x 3.   y = sin ½ x4.   y = cos ½ x

Graphing the cosine curve



Cosine Function: $y = \cos x$

called a "wave" because of its rolling wave-like appearance

amplitude: ________ period: __________

frequency: ____ cycle in 2π radians

domain: $\{x \mid x \in \mathbb{R}\}$ range: $\{y \mid -1 \le y \le 1\}$

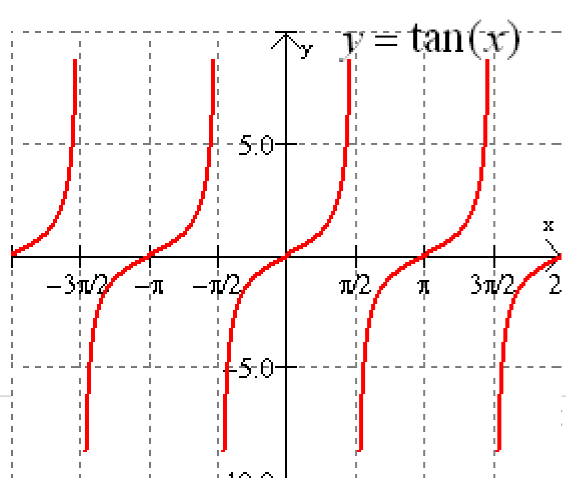
At *x* = 0, the y-value is equal to one

Did you notice that the cosine curve is really the exact same graph as the sine curve shifted 90º (or $\frac{\pi}{2}$ radians) to the left!

Graphing the tangent, secant, cosecant and cotangent curves

An **asymptote** is a line that is approached, but never reached by a curved graph. The tangent, cotangent, cosecant, and secant graphs have vertical asymptotes.

Tangent Function: $y = \tan(x)$



One cycle occurs between ______and _____.

There are vertical asymptotes at each end of the cycle.

The asymptote that occurs at $\frac{\pi}{2}$ repeats every π units.

period: π

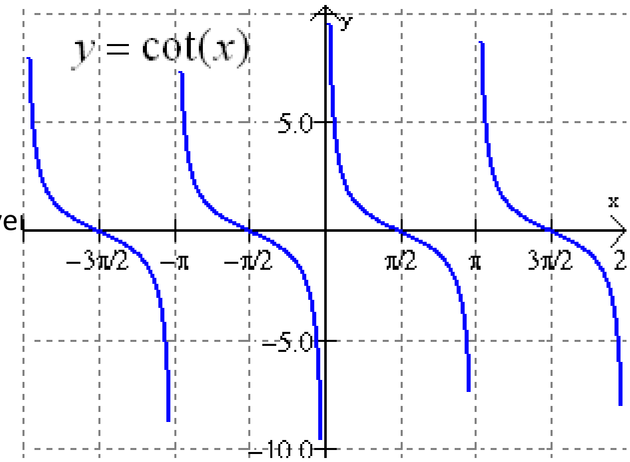
amplitude: none, graphs go on forever in vertical directions.

The graph does not STOP even though the plot may "appear" as if the graph stops as the y-values increase/decrease.

Graphs of Other Trigonometric Functions

Cotangent Function:

One cycle occurs between 0 and π.

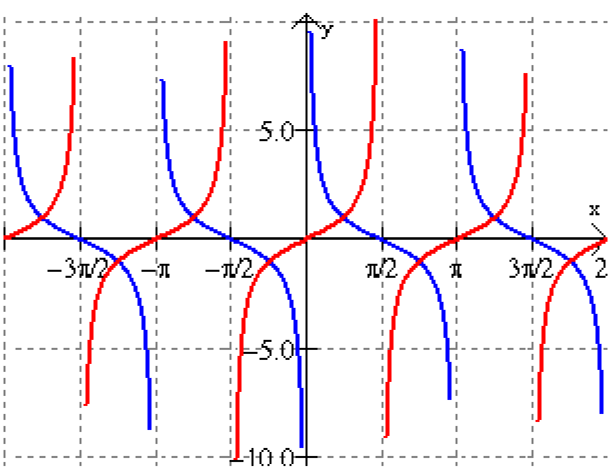


There are **vertical asymptotes** at each end of the cycle. The asymptote that occurs at π repeats eve
π units.

period: π

amplitude: none, graphs go on forever in vertical directions

$y = \cot(x)$; $y = \tan(x)$



The *x*-intercepts of the graph of y = tan(*x*) are the asymptotes of the graph of y = cot(*x*).

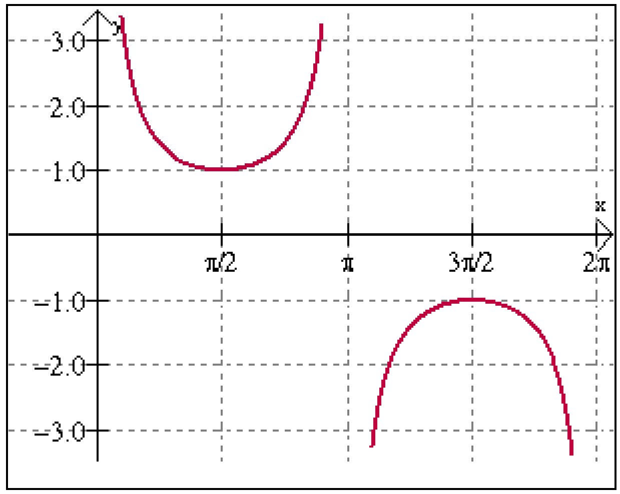
The asymptotes of the graph of y = tan(*x*) are the *x*-intercepts of the graph of y = cot(*x*).

The graphs of y = tan(*x*) and y = cot(*x*) have the same x-values for y-values of 1 and -1.

Note: The graphs of y = tan(*x*) and y = cot(*x*) "face" in opposite directions.

Cosecant Function: $y = \csc(x)$

$y = \csc(x)$

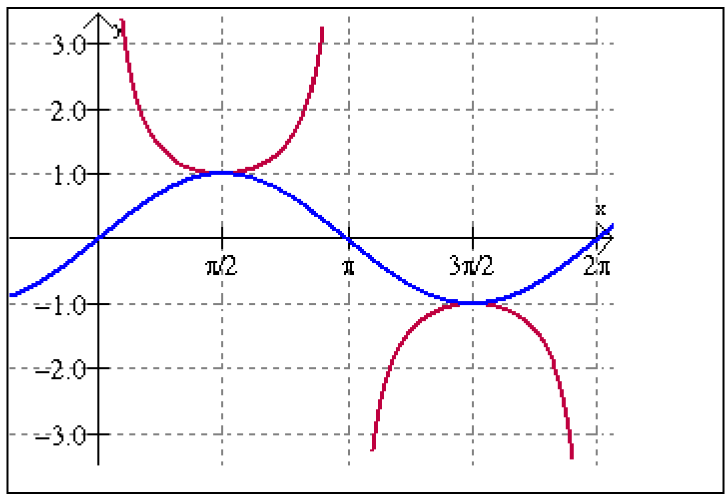


There are vertical asymptotes. The asymptote that occurs at π repeats every π units.

period: 2π

amplitude: none, graphs go on forever in vertical directions.

$y = \csc(x)$ $y = \sin(x)$



The maximum values of y = sin *x* are minimum values of the positive sections of y = csc *x*. The minimum values of y = sin *x* are the maximum values of the negative sections of y = csc *x.*

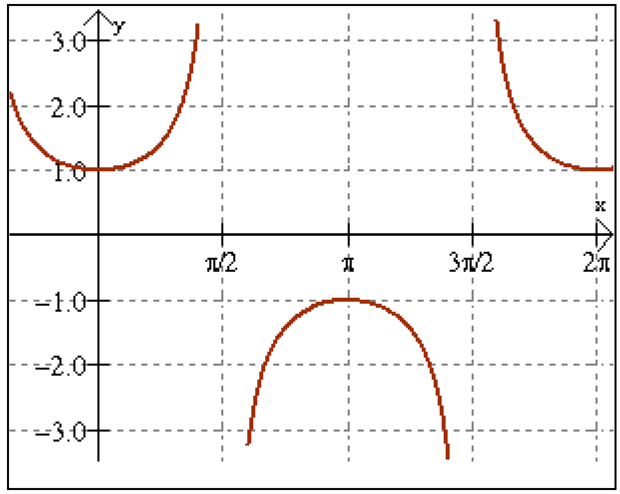
The *x*-intercepts of y = sin *x* are the asymptotes for y = csc *x.*

Note: the U shapes of the cosecant graph are tangent to its reciprocal function, sine, at sine's max and min locations.

Secant Function: $y = \sec(x)$

There are vertical asymptotes. The asymptote that occurs at $\frac{\pi}{2}$ repeats every π units.

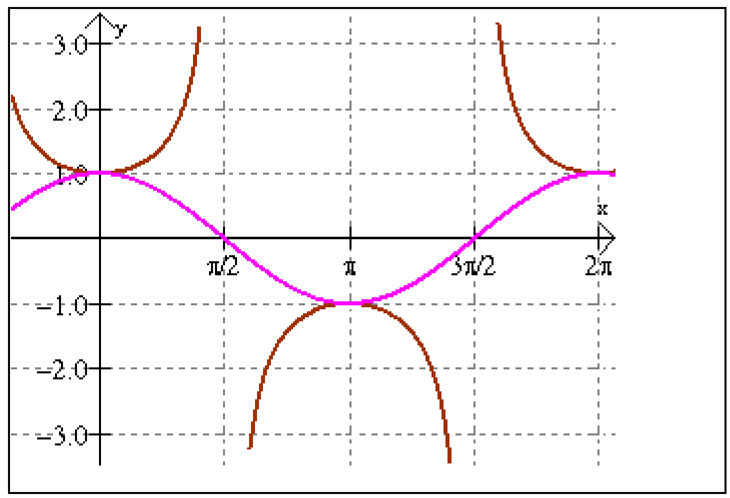
$y = \sec(x)$



period: 2π

amplitude: none, graphs go on forever in vertical directions.

$y = \sec(x)$ $y = \cos(x)$



The maximum values of y = cos *x* are minimum values of the positive sections of y = sec *x*. The minimum values of y = cos *x* are the maximum values of the negative sections of y = sec *x.*

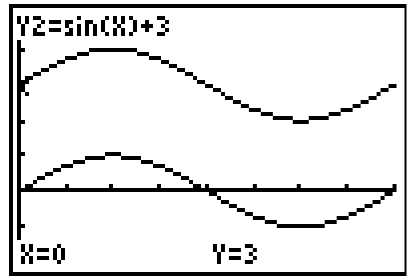
The *x*-intercepts of y = cos *x* are the asymptotes for y = sec *x.*

Note: the U shapes of the secant graph are tangent to its reciprocal function, cosine, at cosine's max and min locations.

Vertical and Horizontal Shifting of Sinusoidal Graphs

Graphs comprised of the sine or cosine function is known as sinusoidal. These graphs can be stretched vertically and horizontally and undergo other transformations as well.

Vertical Shifts

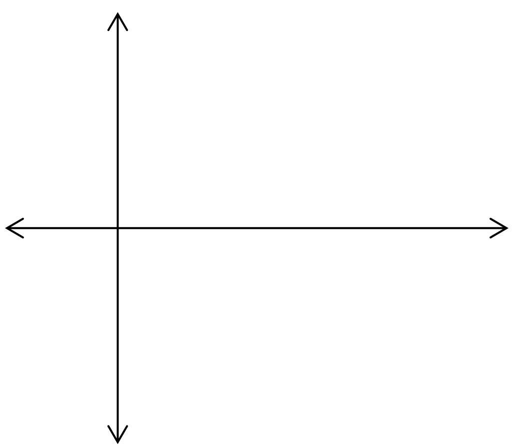


Y1 = sin(x)

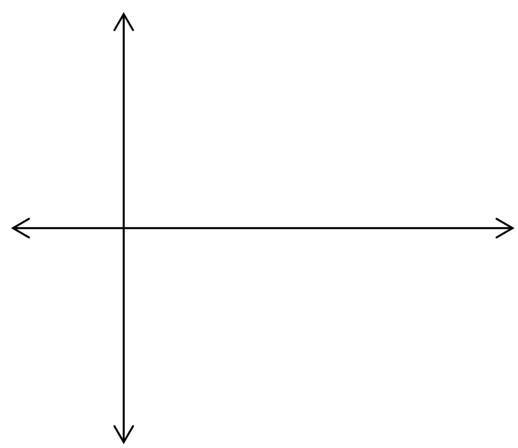
____

____

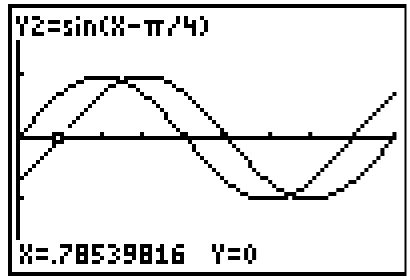
1. $y = 2\cos(x) + 1$



2. y = - cos (x) – 4

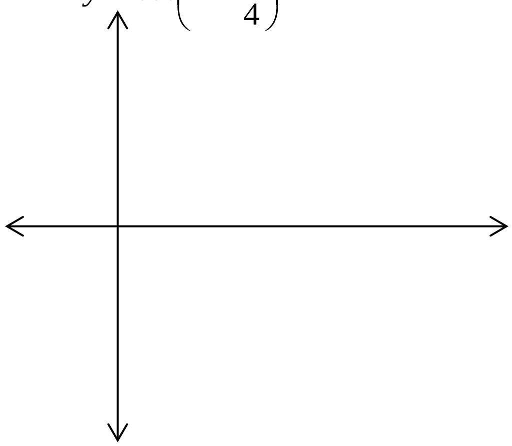


Horizontal Shifts

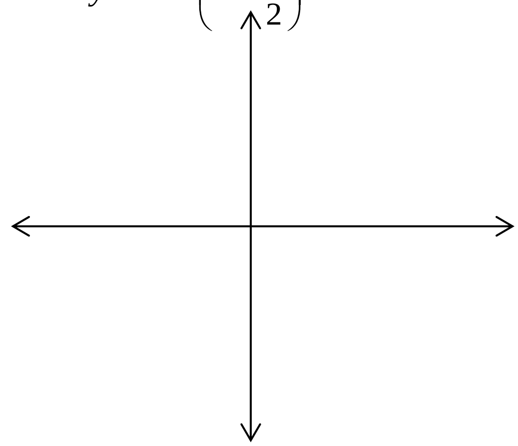


Y1 = sin(x)

1. $y = \cos\left(x - \frac{\pi}{4}\right)$

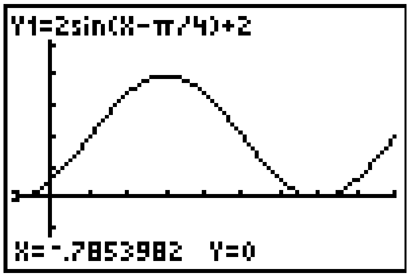


2. $y = 2\sin\left(x + \frac{\pi}{2}\right)$

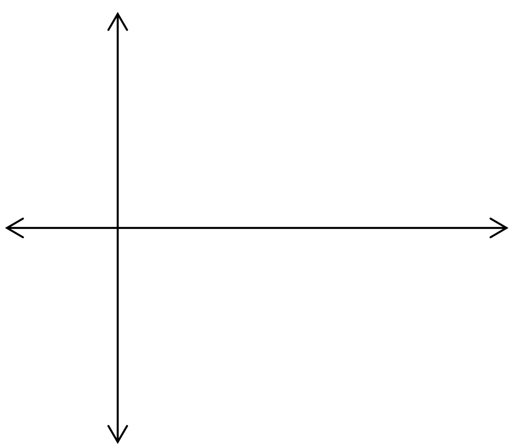


Sinusoidal Shifts:

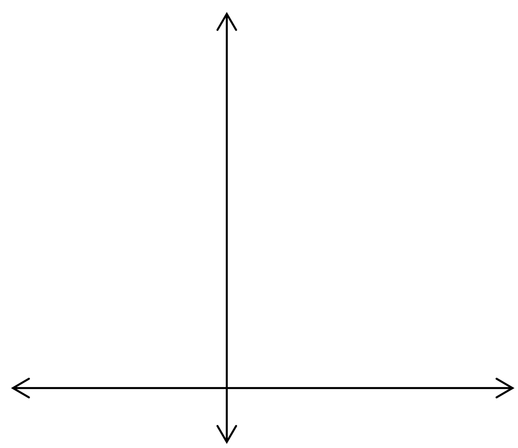
$$y = A\sin\left(B(x-C)\right) + D$$



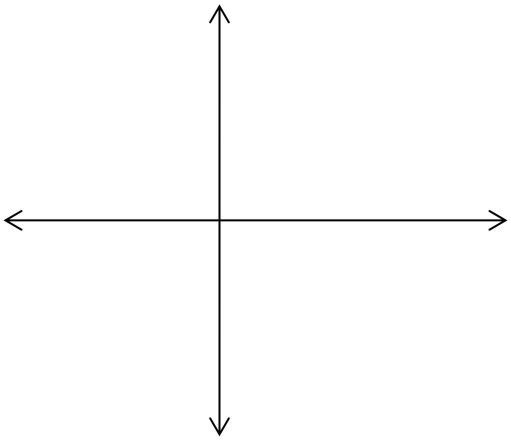
1. $y = 2\cos\left(x - \frac{\pi}{4}\right) + 1$



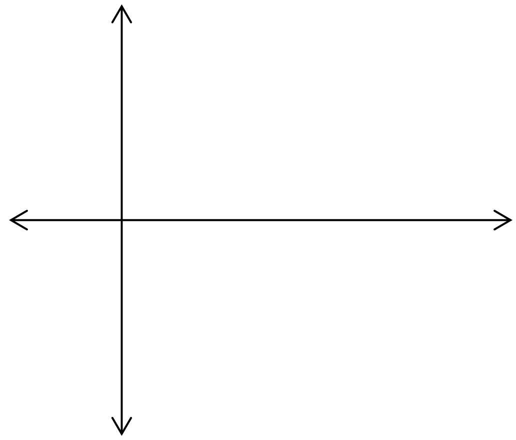
2. $y = 3\sin\left(x + \frac{\pi}{4}\right) - 1$



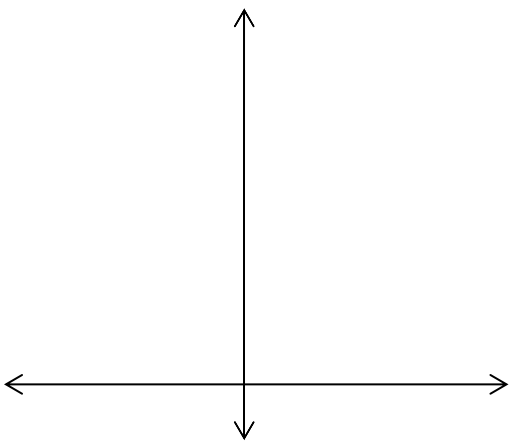
3. $y = -3\sin\left(x + \frac{\pi}{2}\right) + 2$



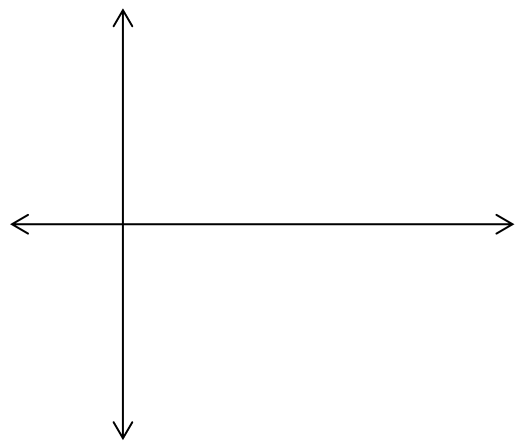
4. $y = 7\sin\left(x - \frac{\pi}{4}\right) - 3$



5. $y = -5\cos\left(x + \frac{\pi}{2}\right) + 2$

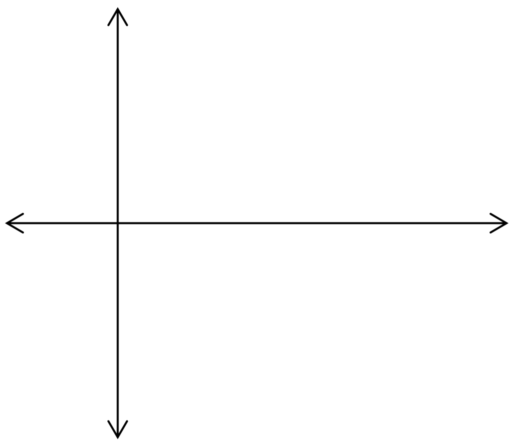


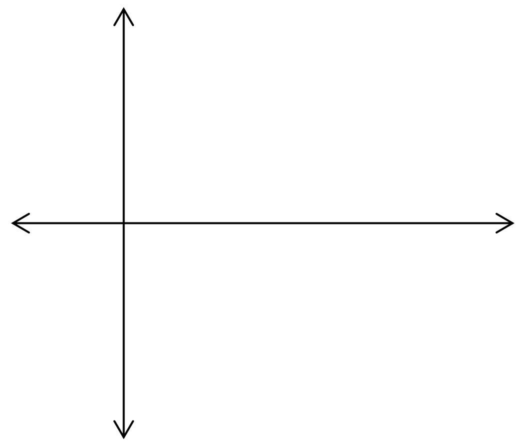
6. $y = 4\sin(x) + 2$



7. $y = -\sin\left(x - \frac{\pi}{4}\right) + 4$

8. y = 4 cos(x) + 2.5





Rotations & Terminology

Standard Position

An angle is in **standard position** if its vertex is located at the origin and one ray is on the positive *x*-axis. The ray on the *x*-axis is called the **initial side** and the other ray is called the **terminal side.**

If the terminal side of an angle lies **"on"** the axes (such as 0º, 90º, 180º, 270º, 360º), it is called a **quadrantal** angle.

The angle shown at the right is referred to as a Quadrant I angle since its terminal side lies in Quadrant I.

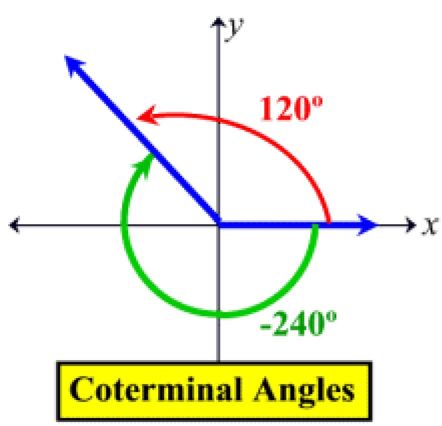
Coterminal Angles

The angle is measured by the amount of rotation from the initial side to the terminal side.

If measured in a **counterclockwise** direction the measurement is **positive**.

If measured in a **clockwise** direction the measurement is **negative.** (A negative associated with an angle's measure refers to its "direction" of measurement, clockwise.)

If two angles in standard position have the same terminal side, they are called **coterminal angles.**

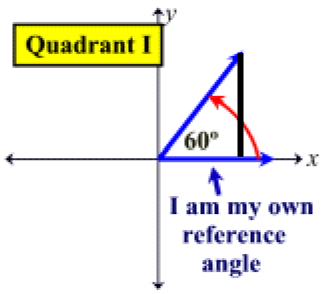


Example:

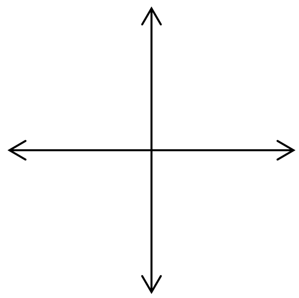
θ = -240 Find the (positive) coterminal angle.

Reference Angles

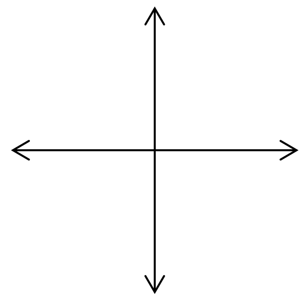
The reference angle is the **acute angle** formed by the **terminal side of the given angle and the *x*-axis.** Reference angles may appear in all four quadrants.



$\theta = 45°$



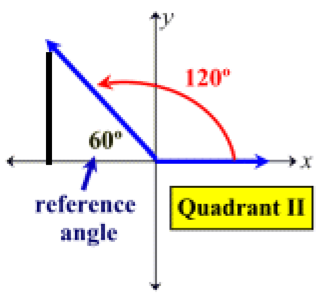
$\theta = -300°$

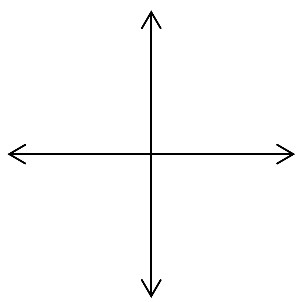


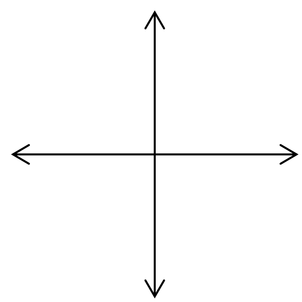
Reference angle = θ

$\theta = 135°$

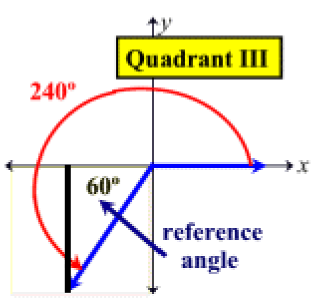
$\theta = -225°$



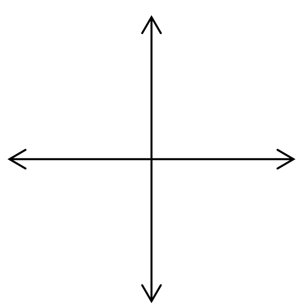




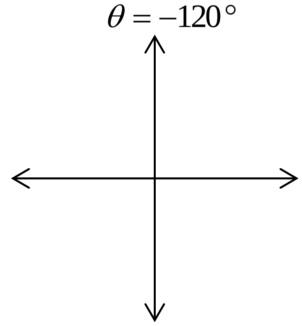
Reference angle = $180 - \theta$



$\theta = 210°$



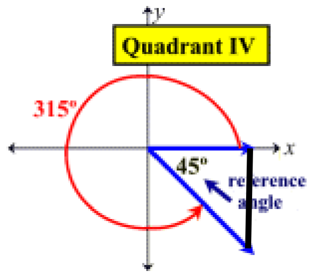
$\theta = -120°$

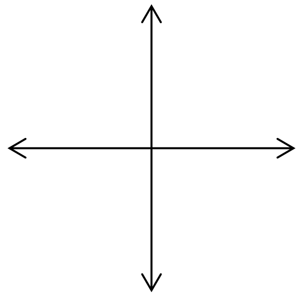


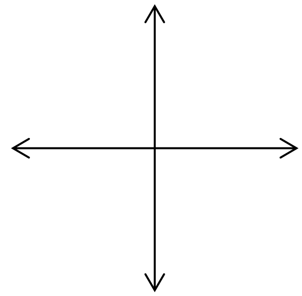
Reference angle = $\theta - 180$

$\theta = 300°$

$\theta = -45°$



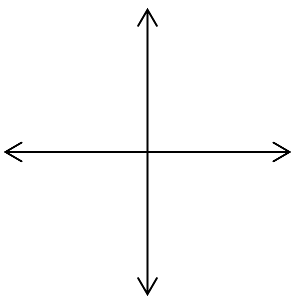


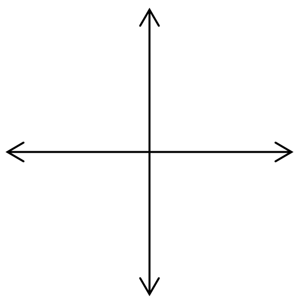


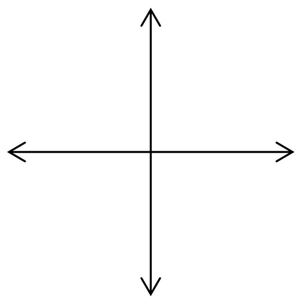
Reference angle = $360-\theta$

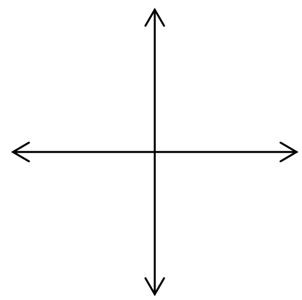
For each of the following angles, sketch the angle in standard position, then sketch its reference angle. Label the reference angle as α , and determine its measure.

(a) $\theta = 145^\circ$ (b) $\theta = 320^\circ$ (c) $\theta = 72^\circ$ (d) $\theta = -210^\circ$

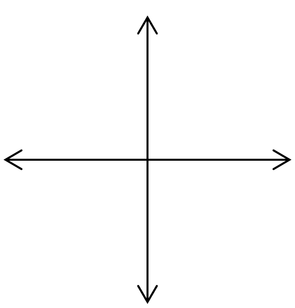


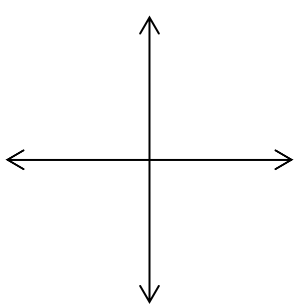


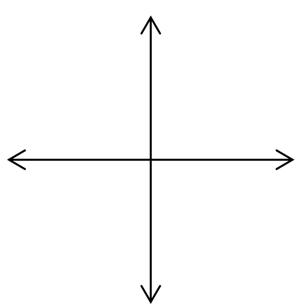


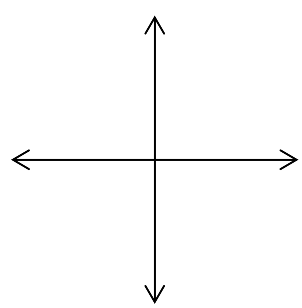


(e) $\theta = 250^\circ$ (f) $\theta = -310^\circ$ (g) $\theta = 460^\circ$ (h) $\theta = -400^\circ$





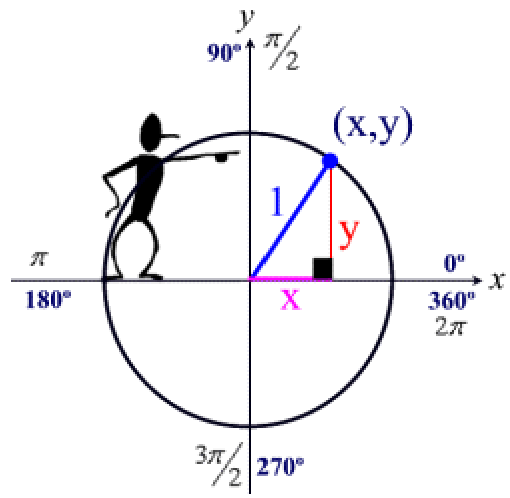




The Unit Circle

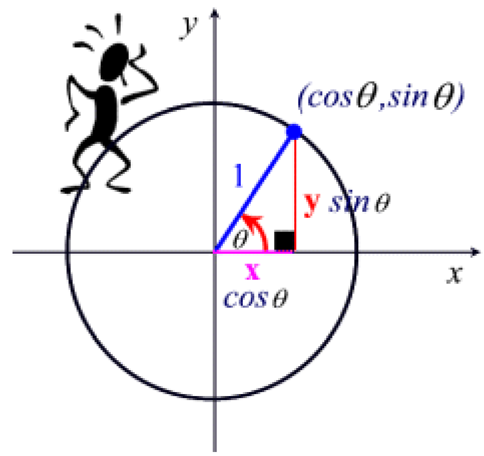
A **unit circle** is a circle with a radius of one (a unit radius). In trigonometry, the unit circle is centered at the origin.

For the point (*x, y*) in Quadrant I, the lengths *x* and *y* become the legs of a right triangle whose hypotenuse is 1. By the Pythagorean Theorem, we have $x^2 + y^2 = 1$.



If we examine angle θ (in standard position) in this unit circle, we can see that

$$\cos\theta = \frac{adjacent}{hypotenuse} =$$



$$\sin\theta = \frac{opposite}{hypotenuse} =$$

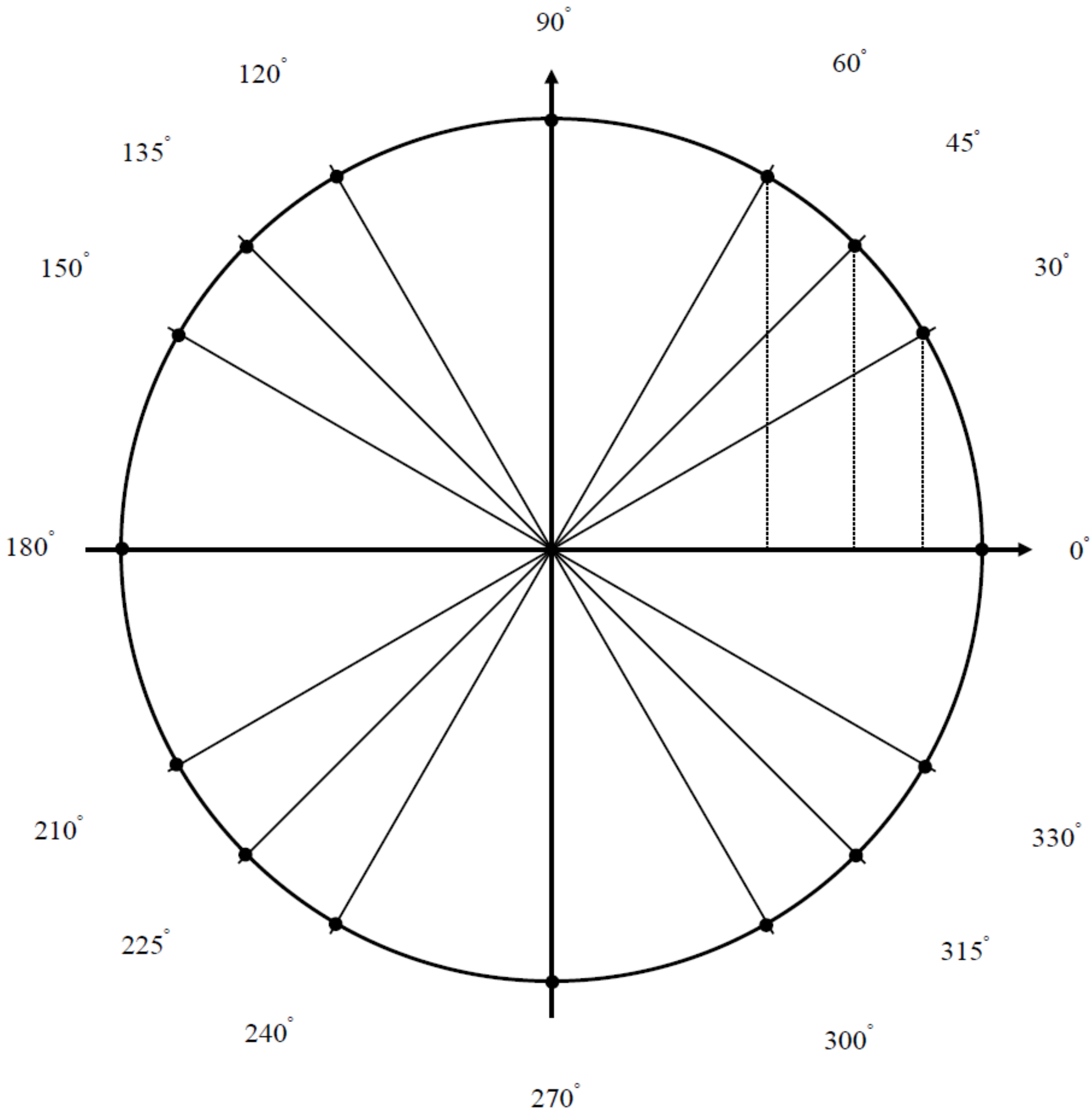
$$\tan\theta = \frac{opposite}{adjacent} = \frac{\sin\theta}{\cos\theta} =$$

which show us that in a unit circle, ___________________________ and ___________________________ also creating

$$(x, y) = (\cos\theta, \sin\theta)$$

Sine is represented by the vertical leg. Cosine is represented by the horizontal leg.

Note that $x^2 + y^2 = 1$ becomes $\cos^2 \theta + \sin^2 \theta = 1$



For each of the following angles drawn in standard position, give the coordinate pair from the unit circle.

(a) $-120°$ (b) $495°$ (c) $\frac{\pi}{3}$ (d) $\frac{3\pi}{2}$

1. Draw a rotation diagram for each of the following angles and then determine the ordered pair that lies on the unit circle for each angle.

   (a) $\theta = 330^{\circ}$ (b) $\theta = 135^{\circ}$ (c) $\theta = -270^{\circ}$

   (d) $\theta = -240^{\circ}$ (e) $\theta = 540^{\circ}$ (f) $\theta = -300^{\circ}$

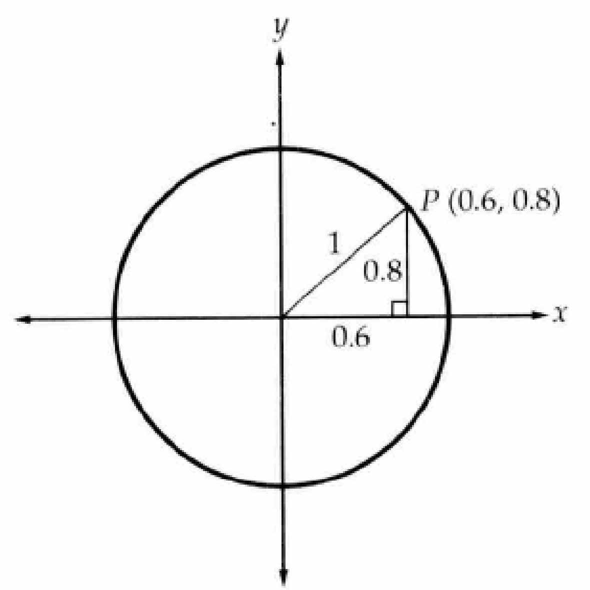
   (a) $\alpha = \frac{\quad}{3}$ (b) $\alpha = -\frac{\quad}{2}$ (c) $\alpha = \frac{\quad}{6}$

   (d) $\alpha = -\frac{\pi}{2}$ (e) $\alpha = \frac{3\pi}{4}$ (f) $\alpha = \frac{4\pi}{3}$

1-13 Definition of Sine, Cosine, Tangent on the Unit Circle

A line segment is positive when it is above the x-axis or to the right of the y-axis. It is negative when it is below the x-axis or left of the y-axis.

P is a point on a unit circle with coordinates (0.6, 0.8) as shown below.

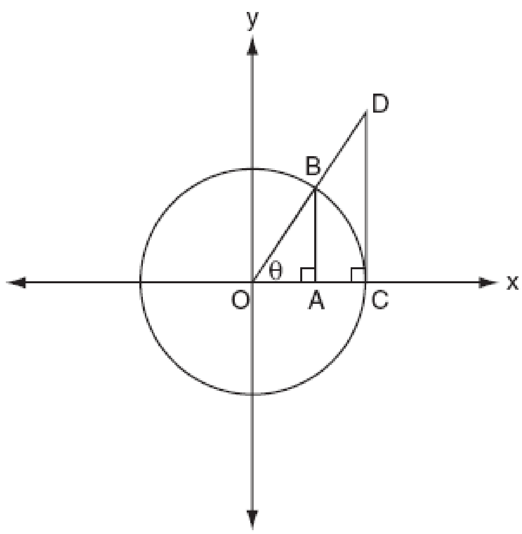


$\cos\theta$

$\sin\theta$

$\tan\theta$

Quadrant I

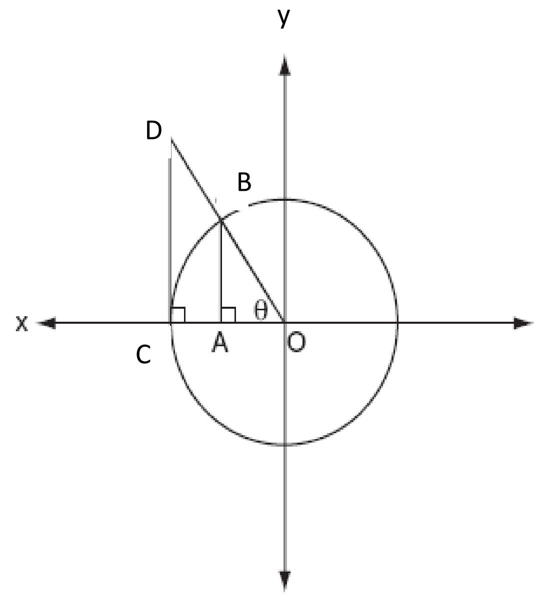


$\cos\theta =$

$\sin\theta =$

$\tan\theta =$

Quadrant II



$\cos\theta =$

$\sin\theta =$

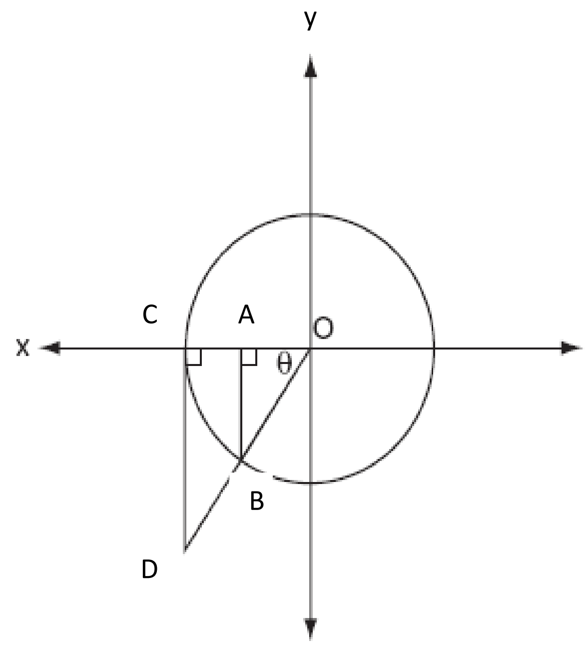
$\tan\theta =$

Quadrant III

$\cos\theta =$

$\sin\theta =$

$\tan\theta =$

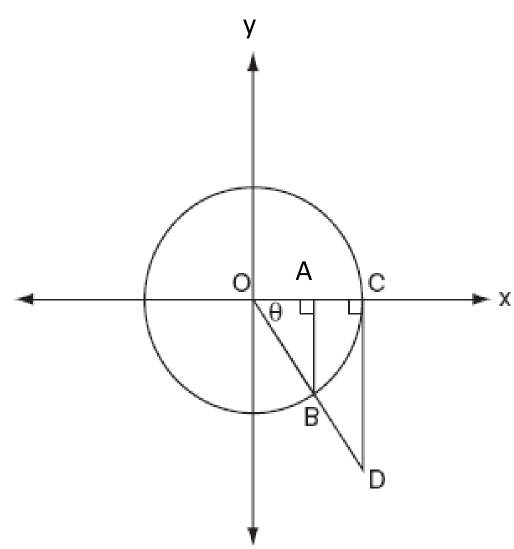


Quadrant IV

$\cos\theta =$

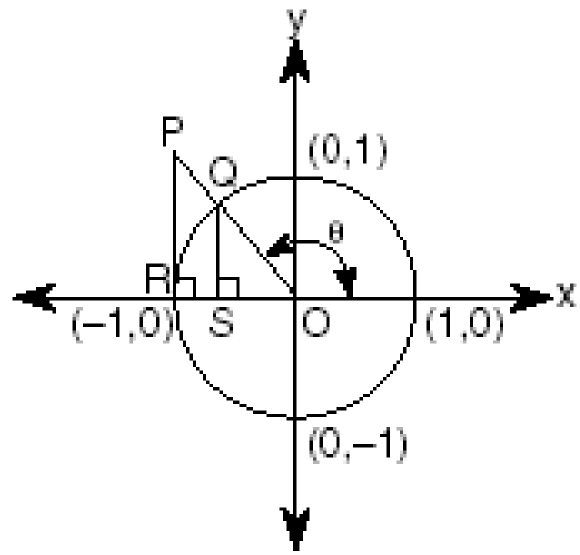
$\sin\theta =$

$\tan\theta =$



Example:

$\cos\theta =$



$\sin\theta =$

$\tan\theta =$

Trigonometric Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

$$\sin\left(A+B\right) = \sin A \cos B + \sin B \cos A$$
$$\sin\left(A-B\right) = \sin A \cos B - \sin B \cos A$$
$$\cos\left(A+B\right) = \cos A \cos B - \sin A \sin B$$
$$\cos\left(A-B\right) = \cos A \cos B + \sin A \sin B$$

$$\sin\left(2A\right) = 2\sin A \cos A$$
$$\cos\left(2A\right) = \cos^2 A - \sin^2 A$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

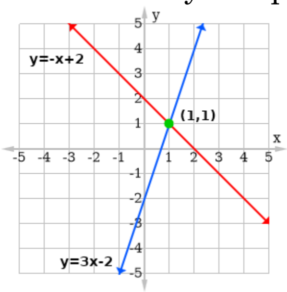
$$\sin 2A = 2\sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

THEME 5. Systems of Equations

Systems of Linear Equations in Two Variables

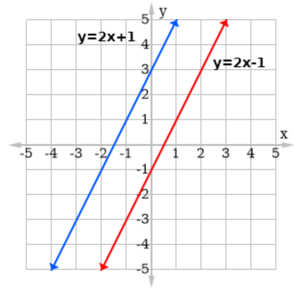
- There are three possibilities for the solution of a system of **linear** equations:

One Solution: The graphs interest at exactly one point.



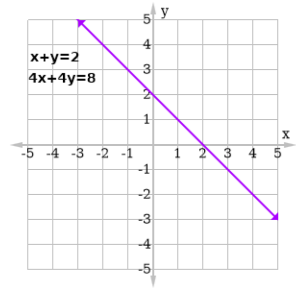
Solution: $\{(1,1)\}$

No Solution: The graphs do not intersect.



Solution: $\emptyset$

Infinitely Many: The graphs are the same line.



Solution: $\{(x, 2-x)\}$

- There are three methods to solve a system of linear equations:
  - **Graphing, Substitution and Elimination**

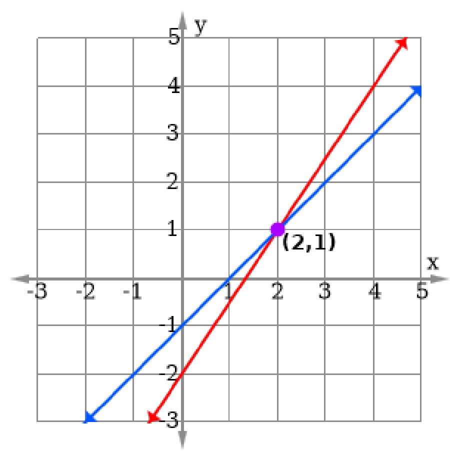
Example: Solve the system using graphing.

$$3x - 2y = 4$$
$$y = x - 1$$

Solve the top equation for y, then graph both equations to find the point of intersection:

$$y = \frac{3}{2}x - 2$$
$$y = x - 1$$

The solution is: $\{(2,1)\}$



Example: Solve the system using substitution and then elimination.

$$\begin{aligned} 2x + 7y &= -12 \\ x + 2y &= 0 \end{aligned}$$

Substitution	Elimination
Solve the bottom equation for x, then plug the expression into the top equation and solve for y. $x + 2y = 0$ $x = -2y$ $2(-2y) + 7y = -12$ $-4y + 7y = -12$ $3y = -12$ $y = -4$ $x + 2(-4) = 0$ $x - 8 = 0$ $x = 8$ Solution: $\{(8, -4)\}$	Multiply the bottom equation by -2, then add the two equations together to eliminate x. $2x + 7y = -12$ $-2(x + 2y = 0)$ $2x + 7y = -12$ $\underline{-2x - 4y = 0}$ $3y = -12$ $y = -4$ $x + 2(-4) = 0$ $x - 8 = 0$ $x = 8$ Solution: $\{(8, -4)\}$

Practice: Solve the system by the method indicated.

1. $\begin{aligned} x + 4y &= -1 \\ 2x - 5y &= 11 \end{aligned}$; Substitution

2. $\begin{aligned} x - y &= 3 \\ 7x - y &= -3 \end{aligned}$; Elimination

3. Two numbers have a sum of 69. The larger number is three less than twice the smaller number. **Set up a system** and solve to find the numbers.